Chapter 6 The Laplace Transform

6-1 Definition of the Laplace Transform

1. For a linear time-invariant (LTI) system with impulse response $h(t)$, the output $y(t)$ corresponding to the input of the form e^{st} is

$$y(t) = \underbrace{H(s)}_{\text{eigenvalue}} \cdot \underbrace{e^{st}}_{\text{eigenfunction}} \tag{6.1}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt \tag{6.2}$$

is referred to as the Laplace transform of $h(t)$. Replacing s by $j\omega$, we have

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt. \tag{6.3}$$

This is called the Fourier transform of $h(t)$.

2. The Laplace transform of a general signal $x(t)$:

$$\begin{cases} X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt \equiv \mathcal{L}\{x(t)\} \\ x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \end{cases} \tag{6.4}$$

$$X(s)\big|_{s=j\omega} = X(j\omega) = \mathcal{F}\{x(t)\} \tag{6.5}$$

3. The Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)$ after multiplication by a real exponential.

$$s = \sigma + j\omega$$

$$\Rightarrow \begin{aligned} X(s) &= X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \\ &= \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t} \right] e^{-j\omega t} dt = \mathcal{F}\left\{ x(t) e^{-\sigma t} \right\} \end{aligned} \tag{6.6}$$

Example 6.1: $x(t) = e^{-at} u(t)$, $X(j\omega)$ converges for $a > 0$ $\left(\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \right)$.

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{j\omega + a},\ a > 0$$

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} e^{-(s+a)t} dt$$

With $s=\sigma+j\omega$, we have

$$X(\sigma+j\omega)=\int_0^{\infty}e^{-(\sigma+a)t}\cdot e^{-j\omega t}dt=\mathcal{F}\left\{e^{-(\sigma+a)t}u(t)\right\}$$
$$=\frac{1}{j\omega+(\sigma+a)},\ \sigma+a>0,\text{ i.e., }\sigma>-a\text{ or }\mathrm{Re}\{s\}>-a$$
$$=\frac{1}{s+a},\ \mathrm{Re}\{s\}>-a$$
$$\Rightarrow X(s)=\frac{1}{s+a},\ \mathrm{Re}\{s\}>-a$$

Note:

- The Laplace transform converges for some values of $\mathrm{Re}\{s\}$, and not for the others.
- The existence of the Laplace transform does not imply the existence of the Fourier transform , e.g., $x(t)=e^{-at}u(t),\ a<0$.

■

Example 6.2: $x(t)=-e^{-at}u(-t)$

$$X(s)=-\int_{-\infty}^{\infty}e^{-at}e^{-st}u(-t)dt=-\int_{-\infty}^{0}e^{-(s+a)t}dt=\frac{1}{s+a}$$
$$\left(\begin{array}{l}=-\int_{-\infty}^{\infty}e^{-(\sigma+a)t}e^{-j\omega t}u(-t)dt=-\mathcal{F}\left\{e^{-(\sigma+a)t}u(-t)\right\}\\(\because t<0,\ \therefore \sigma+a<0\Rightarrow\sigma<-a)\end{array}\right)$$

$\Rightarrow X(s)$ exists only for $\mathrm{Re}\{s\}<-a$.

Note:

- In specifying the Laplace transform of a signal, both the algebraic expression and the range of values for which this expression is valid are required.
- The range of values for which the Laplace transform exists is referred to as the region of convergence (ROC) of the Laplace transform.

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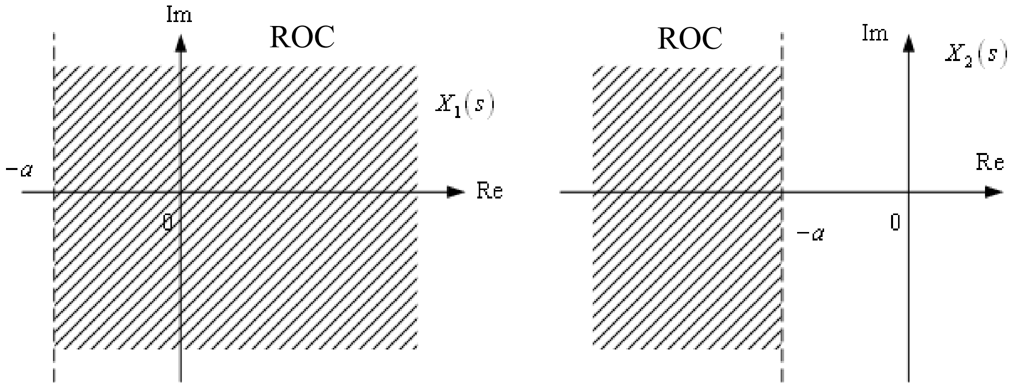
Example 6.3: The ROCs of $X_1(s)=\mathcal{L}\left\{e^{-at}u(t)\right\}=1/(s+a)$ in Example 6.1 and $X_2(s)=\mathcal{L}\left\{-e^{-at}u(-t)\right\}=1/(s+a)$ in Example 6.2 are illustrated as follows:



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Example 6.4: $x(t)=e^{-t}u(t)+e^{-2t}u(t)$

$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty}\left[e^{-t}u(t)+e^{-2t}u(t)\right]e^{-st}dt \\
&= \int_{-\infty}^{\infty} e^{-t}e^{-st}u(t)dt+\int_{-\infty}^{\infty} e^{-2t}e^{-st}u(t)dt \\
&= \frac{1}{s+1}+\frac{1}{s+2},\ \ \text{Re}\{s\}>-1
\end{aligned}
$$

$$
\left(
\begin{aligned}
&= \underbrace{\mathcal{L}\{e^{-t}u(t)\}}_{\frac{1}{s+1},\ \text{Re}\{s\}>-1}+\underbrace{\mathcal{L}\{e^{-2t}u(t)\}}_{\frac{1}{s+2},\ \text{Re}\{s\}>-2} \\
&\Rightarrow \mathcal{L}\{e^{-t}u(t)+e^{-2t}u(t)\} = \frac{1}{s+1}+\frac{1}{s+2},\ \ \text{Re}\{s\}>-1 \\
&\qquad = \frac{2s+3}{s^2+3s+2},\ \ \text{Re}\{s\}>-1 \\
&\qquad = \frac{N(s)}{D(s)} \begin{array}{l}\to \text{numerator polynomial} \\ \to \text{denominator polynomial}\end{array}
\end{aligned}
\right)
$$

Note:

- Whenever $x(t)$ is a linear combination of real or complex exponentials, $X(s)$ can be expressed by $X(s)=N(s)/D(s)$, i.e., $X(s)$ is rational.
- The roots of the numerator polynomial (denominator polynomial) are referred to as the zeros (poles) of $X(s)$ since for those values of s, $X(s)=0$ ($X(s)\to\infty$).
- $X(s)=N(s)/D(s)$

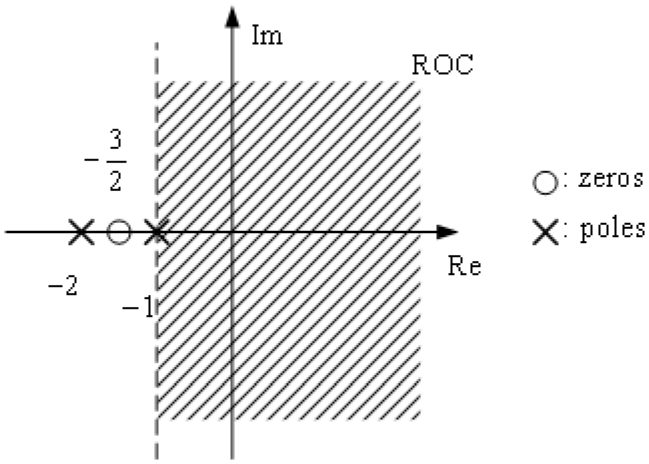
  [The order of $N(s)$] < [The order of $D(s)$]

  $\Rightarrow$ exist zeros at infinity $(s\to\infty,\ X(s)\to 0)$

  [The order of $N(s)$] > [The order of $D(s)$]

  $\Rightarrow$ exist poles at infinity $(s\to\infty,\ X(s)\to\infty)$
- The representation of $X(s)=N(s)/D(s)$ through its poles and zeros in the s-plane is referred to as the pole-zero diagram or the pole-zero plot. ■

Example 6.5: $X(s)=\dfrac{2s+3}{s^2+3s+2}$



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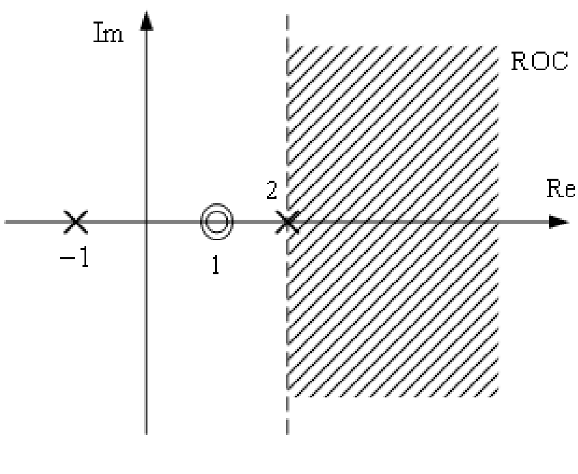
Example 6.6: $x(t)=\delta(t)-\dfrac{4}{3}e^{-t}u(t)+\dfrac{1}{3}e^{2t}u(t)$

$$\delta(t)\xleftrightarrow{\mathcal{L}}1,\ \text{ROC: entire s plane}$$
$$\frac{4}{3}e^{-t}u(t)\xleftrightarrow{\mathcal{L}}\frac{4}{3}\cdot\frac{1}{s+1},\ \ \text{Re}\{s\}>-1$$
$$\frac{1}{3}e^{2t}u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{3}\cdot\frac{1}{s-2},\ \ \text{Re}\{s\}>2$$
$$\left(\int_{-\infty}^{\infty}e^{2t}e^{-st}u(t)\,dt=\int_{-\infty}^{\infty}e^{(2-\sigma)t}e^{-j\omega t}u(t)\,dt,\ 2-\sigma<0\Rightarrow\sigma>2\Rightarrow\text{Re}\{s\}>2\right)$$

$$X(s)=1-\frac{4}{3}\cdot\frac{1}{s+1}+\frac{1}{3}\cdot\frac{1}{s-2},\ \ \text{Re}\{s\}>2$$
$$=\frac{(s-1)^2}{(s+1)(s-2)},\ \ \text{Re}\{s\}>2$$

We will refer to the order of pole or zero as the number of times it is repeated at a given location.



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6-2 Properties of the ROC for Laplace Transforms

1. The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

   Example 6.7: $X(s)$ converges only for $\text{Re}\{s\} > a$ (or $\text{Re}\{s\} < a$). The ROC depends only on the real part of s. ■

2. For rational Laplace transforms, the ROC does not contain any poles.

$$s = \text{pole} \ \Rightarrow X(s) \to \infty$$

3. If $x(t)$ is of finite duration and if there is at least one value of s for which the Laplace transform converges, then the ROC is the entire s-plane.

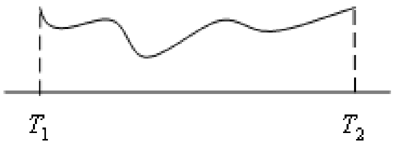

Figure 6.1 A finite-duration signal.

Proof:

Let $x(t)$ be zero outside the interval between T_1 and T_2. Then

$$X(s) = \int_{T_1}^{T_2} x(t) e^{-st} dt \ . \tag{6.7}$$

Assume that the line $\text{Re}\{s\} = \sigma_0$ is in the ROC. Then

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma_0 t} dt < \infty \ . \tag{6.8}$$

(1) For $\sigma_1 > \sigma_0$,

$$\begin{aligned} \int_{T_1}^{T_2} |x(t)| e^{-\sigma_1 t} dt &= \int_{T_1}^{T_2} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t} dt \\ &\le e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{T_2} |x(t)| e^{-\sigma_0 t} dt < \infty \end{aligned} \tag{6.9}$$

where the maximum value of $e^{-(\sigma_1 - \sigma_0)t}$ in the interval of integration is $e^{-(\sigma_1 - \sigma_0)T_1}$. This implies that the s-plane for $\text{Re}\{s\} > \sigma_0$ is in the ROC.

(2) For $\sigma_2 < \sigma_0$,

$$\begin{aligned} \int_{T_1}^{T_2} |x(t)| e^{-\sigma_2 t} dt &= \int_{T_1}^{T_2} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_2 - \sigma_0)t} dt \\ &\le e^{-(\sigma_2 - \sigma_0)T_2} \int_{T_1}^{T_2} |x(t)| e^{-\sigma_0 t} dt < \infty \ . \end{aligned} \tag{6.10}$$

This implies that the s-plane for $\text{Re}\{s\} > \sigma_0$ is in the ROC. From (1) and (2), we can see that the ROC of a finite-duration signal includes the entire s-plane.

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4. If $x(t)$ is right-sided and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\text{Re}\{s\} > \sigma_0$ will also be in the ROC.

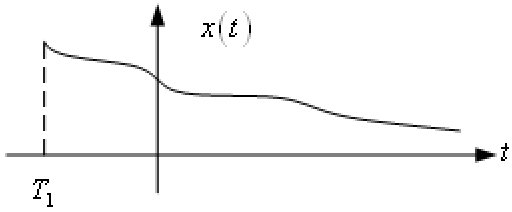


Figure 6.2 A right-sided signal.

Note: For some special signals such as $x(t) = e^{t^2}u(t)$, the corresponding Laplace transforms do not exist, i.e., there is no value of s for which the Laplace transforms converge.

Suppose that the Laplace transform of $x(t)$ converges for some value of σ, denoted by σ_0. Then

$$\int_{-\infty}^{\infty} |x(t)| e^{-\sigma_0 t} dt < \infty \tag{6.11}$$

$$\Rightarrow \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} dt < \infty . \tag{6.12}$$

For $\sigma_1 > \sigma_0$,

$$\int_{T_1}^{\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t} dt \le e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} dt < \infty \tag{6.13}$$

where the maximum value of $e^{-(\sigma_1 - \sigma_0)t}$ in the interval of integration is $e^{-(\sigma_1 - \sigma_0)T_1}$. So the ROC includes the s-plane right to the line $\text{Re}\{s\} = \sigma_0$. The ROC is shown as follows:

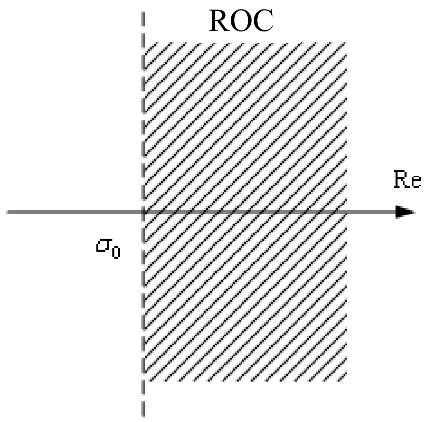


Figure 6.3 The ROC of a right-sided signal.

5. If $x(t)$ is left-sided and if the line $\mathrm{Re}\{s\}=\sigma_0$ is in the ROC, then all values of s for which $\mathrm{Re}\{s\}<\sigma_0$ will also be in the ROC. An example of this case is shown as follows:

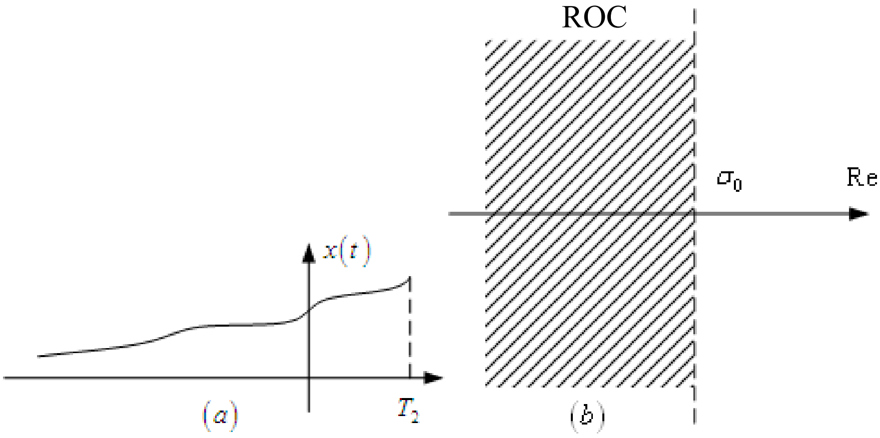


Figure 6.4 (a) A left-sided signal; (b) the ROC of a left-sided signal.

6. If $x(t)$ is two-sided and if the line $\mathrm{Re}\{s\}=\sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane which includes the line $\mathrm{Re}\{s\}=\sigma_0$.

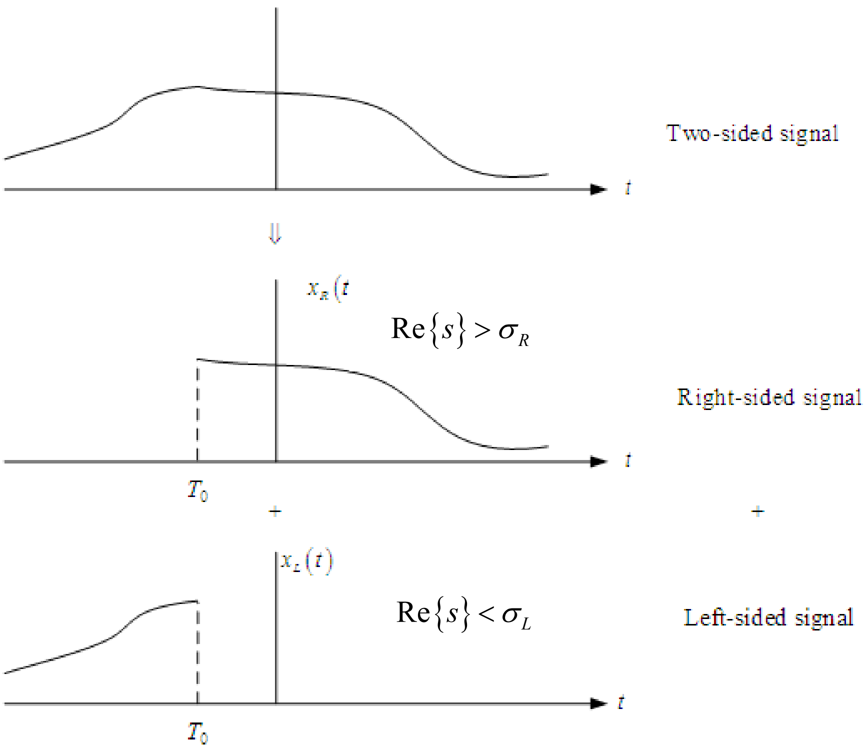


Figure 6.5 A two-sided signal divided into the sum of a right-sided signal and a left-sided signal.

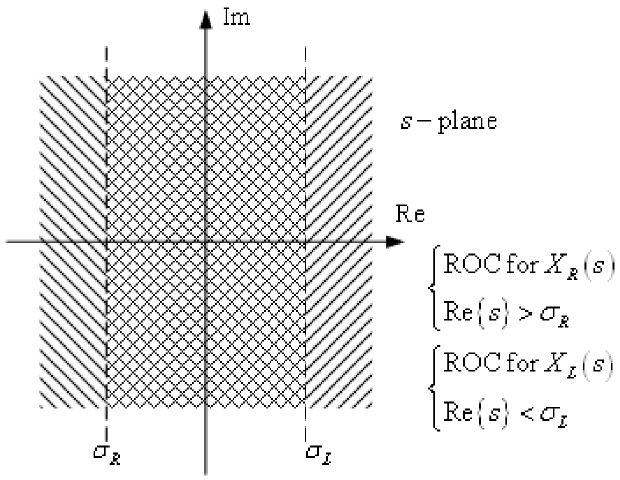


Figure 6.6 The ROCs for the Laplace transforms of $x_R(t)$ and $x_L(t)$, assuming that they overlap. The overlap of the two ROCs is the ROC for $x(t) = x_R(t) + x_L(t)$.

Note:

- σ_L must be greater than σ_R; otherwise, the Laplace transform of $x(t)$ does not exist.

Example 6.8: Consider a finite-duration sequence given by

$$x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & \text{otherwise.} \end{cases}$$

Find the Laplace transform $X(s)$ and its ROC.

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

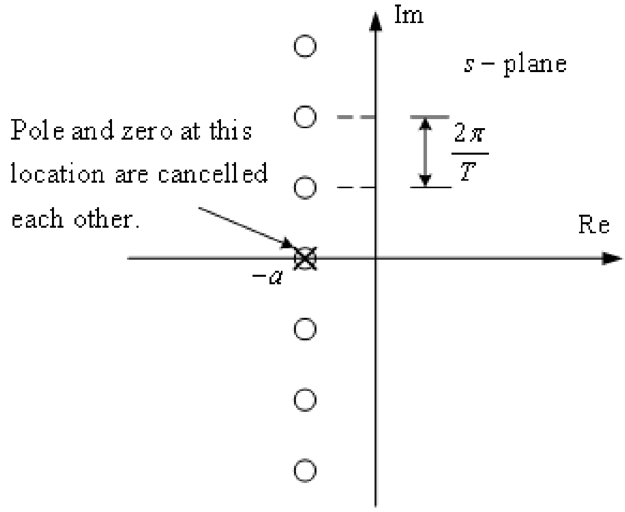
$$\left(\begin{array}{l} s = -a \Rightarrow \left. \begin{array}{r} s + a \to 0 \\ 1 - e^{-(s+a)T} \to 0 \end{array} \right\} \\ \therefore \lim_{s \to -a} X(s) = \lim_{s \to -a} \dfrac{\dfrac{d}{ds}\left[1 - e^{-(s+a)T}\right]}{\dfrac{d}{ds}(s+a)} = \lim_{s \to -a} T e^{-aT} e^{-sT} = T \end{array} \right)$$

$$\begin{cases} X(s) \text{ has no poles.} \\ X(s) \text{ has an infinite number of zeros.} \end{cases}$$

$$1 - e^{-(s+a)T} = 0 \Rightarrow (s+a)T = j2\pi k,\ k = 0,\ \pm 1,\ \pm 2, \ldots$$

$$\Rightarrow s = -a + j\frac{2\pi k}{T},\ k = 0,\ \pm 1,\ \pm 2, \ldots$$

The ROC is the entire s-plane and the pole-zero diagram is as follows:



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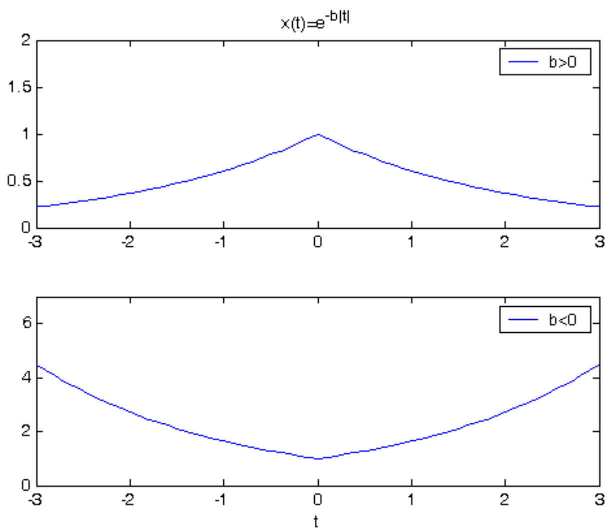
Example 6.9: $x(t)=e^{-b|t|}$

$$x(t)=\underbrace{e^{-bt}u(t)}_{\text{right-sided}}+\underbrace{e^{bt}u(-t)}_{\text{left-sided}}$$

$$e^{-bt}u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s+b},\quad \mathrm{Re}\{s\}>-b$$

$$e^{bt}u(-t)\xleftrightarrow{\mathcal{L}}\frac{-1}{s-b},\quad \mathrm{Re}\{s\}<+b$$

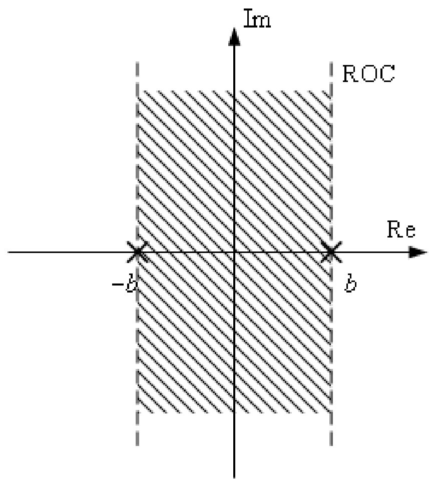


For $b<0$, there is no common region of convergence. $\Rightarrow$ $x(t)$ has no Laplace transform if $b<0$.

For $b>0$,

$$x(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s+b}-\frac{1}{s-b}=\frac{-2b}{s^2-b^2},\quad -b<\mathrm{Re}\{s\}<b\,.$$

The pole-zero diagram with ROC is shown as follows:



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7. Summary for the ROC:

Finite-duration signal $\rightarrow \begin{cases} \text{entire } s\text{-plane} \\ \text{does not exist} \end{cases}$

Right-sided signal $\rightarrow \begin{cases} \text{right-half } s\text{-plane} \\ \text{does not exist} \end{cases}$

Left-sided signal $\rightarrow \begin{cases} \text{left-half } s\text{-plane} \\ \text{does not exist} \end{cases}$

Two-sided signal $\rightarrow \begin{cases} \text{a strip} \\ \text{does not exist} \end{cases}$

Note:

- The ROC is bounded by poles or extends to infinity.
- For a right-sided signal, the ROC is the region in the s-plane to the right of the rightmost pole.
- For a left-sided signal, the ROC is the region in the s-plane to the left of the leftmost pole.

6-3 The Inverse Laplace Transform

$$X(\sigma+j\omega)=\int_{-\infty}^{\infty}\left(x(t)e^{-\sigma t}\right)e^{-j\omega t}dt=\mathcal{F}\left\{x(t)e^{-\sigma t}\right\}$$

$$\Rightarrow x(t)e^{-\sigma t}=\mathcal{F}^{-1}\left\{X(\sigma+j\omega)\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\sigma+j\omega)e^{j\omega t}d\omega \qquad (6.14)$$

$$\Rightarrow x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\sigma+j\omega)e^{j\omega t}e^{\sigma t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\sigma+j\omega)e^{(\sigma+j\omega)t}d\omega \qquad (6.15)$$

If we change the variable of integration from ω to s and use the fact that σ is constant so that $ds=jd\omega$, we obtain

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds. \qquad (6.16)$$

where σ is any value in the ROC of $X(s)$. This is the basic inverse Laplace transform equation that requires the use of contour integration along a vertical line in the complex s-plane. For rational Laplace transforms, a much simpler technique using partial-fraction expansion is often used to determine the inverse transforms, instead of evaluating (6.16) directly. Once the original rational transform $X(s)$ has been expanded into a linear combination of lower-order rational transforms, we can infer the ROC for each of these terms from the overall ROC for $X(s)$ and then find the corresponding inverse transform.

Example 6.10:

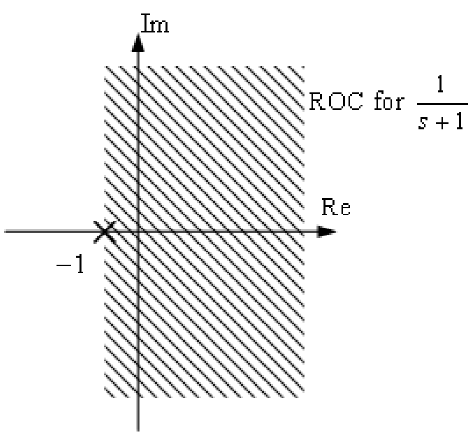
$$X(s)=\frac{1}{(s+1)(s+2)},\ \text{Re}\{s\}>-1$$

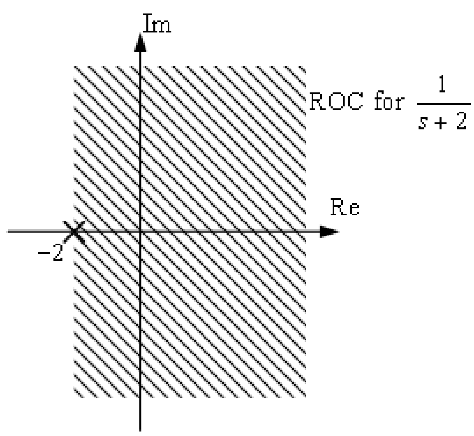
$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}=\frac{1}{s+1}-\frac{1}{s+2}$$

$$A=(s+1)X(s)\Big|_{s=-1}=1$$

$$B=(s+2)X(s)\Big|_{s=-2}=-1$$

Since the ROC for $X(s)$ is $\text{Re}\{s\}>-1$, the ROC for the individual terms in the partial fraction includes $\text{Re}\{s\}>-1$.





$$\Rightarrow \begin{cases} e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \dfrac{1}{s+1}, \ \mathrm{Re}\{s\} > -1 \ (\text{right-sided}) \\ e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \dfrac{1}{s+2}, \ \mathrm{Re}\{s\} > -2 \ (\text{right-sided}) \end{cases}$$

$$\Rightarrow \left(e^{-t} - e^{-2t}\right)u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)(s+2)}, \ \mathrm{Re}\{s\} > -1$$

■

Example 6.11:

$$X(s) = \frac{1}{(s+1)(s+2)}, \ \mathrm{Re}\{s\} < -2 \ (\text{left-sided})$$

$$= \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$x(t) = \left(-e^{-t} + e^{-2t}\right)u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)(s+2)}, \ \mathrm{Re}\{s\} < -2$$

$$\because e^{-bt}u(-t) \xleftrightarrow{\mathcal{L}} \frac{-1}{s-b}, \ \mathrm{Re}\{s\} < b$$

$$\therefore \begin{cases} e^{-t}u(-t) \xleftrightarrow{\mathcal{L}} \dfrac{-1}{s+1}, \ \mathrm{Re}\{s\} < -1 \\ e^{-2t}u(-t) \xleftrightarrow{\mathcal{L}} \dfrac{-1}{s+2}, \ \mathrm{Re}\{s\} < -2 \end{cases}$$

■

Example 6.12: $X(s) = \dfrac{1}{(s+1)(s+2)}, \ -2 < \mathrm{Re}\{s\} < -1$

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)(s+2)}, \ -2 < \mathrm{Re}\{s\} < -1$$

■

6-4 Properties of the Laplace Transform

1. Linearity

$$x_1(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s), \text{ with ROC} = R_1$$

$$x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_2(s), \text{ with ROC} = R_2$$

$$\Rightarrow ax_1(t) + bx_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} aX_1(s) + bX_2(s) \text{ with ROC containing } R_1 \cap R_2 \quad (6.17)$$

Note: The ROC of the Laplace transform for the combined signal could be larger than $R_1 \cap R_2$.

Example 6.13: $X_1(s) = \frac{1}{s+1}, \ \text{Re}\{s\} > -1$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \ \text{Re}\{s\} > -1$$

$$x(t) = x_1(t) - x_2(t)$$

$$X(s) = X_1(s) - X_2(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)}$$

$$= \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}, \ \text{Re}\{s\} > -2 \text{ (larger than } \text{Re}\{s\} > -1)$$

In the combination of $x_1(t)$ and $x_2(t)$, the pole at $s = -1$ is cancelled by a zero at $s = -1$. $\Rightarrow$ "pole-zero cancellation" ■

2. Time shifting

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \text{ with ROC} = R$$

$$\Rightarrow \begin{matrix} x(t-t_0) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s), \text{ with ROC} = R \\ \left(\int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt = e^{-st_0} X(s) \right) \end{matrix} \quad (6.18)$$

3. Shifting in the s-plane

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \text{ with ROC} = R$$

$$\Rightarrow \begin{matrix} e^{s_0 t} x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s-s_0), \text{ with ROC} = R + \text{Re}\{s_0\} \\ (\because \text{pole } s_p \to s_p + s_0) \end{matrix} \quad (6.19)$$

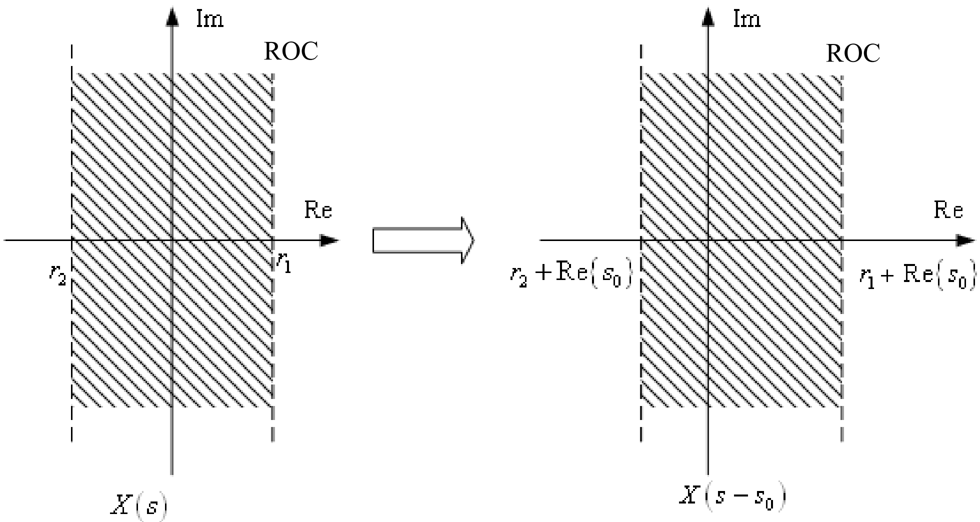


Figure 6.7 The effect on the ROC of shifting in the *s*-domain.

4. Time scaling

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \text{ with ROC} = R$$

$$\Rightarrow \begin{array}{l} x(at) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{|a|} X\left(\dfrac{s}{a}\right), \text{ with ROC} = aR \\ (\because \text{pole } s_p \to as_p) \end{array} \tag{6.20}$$

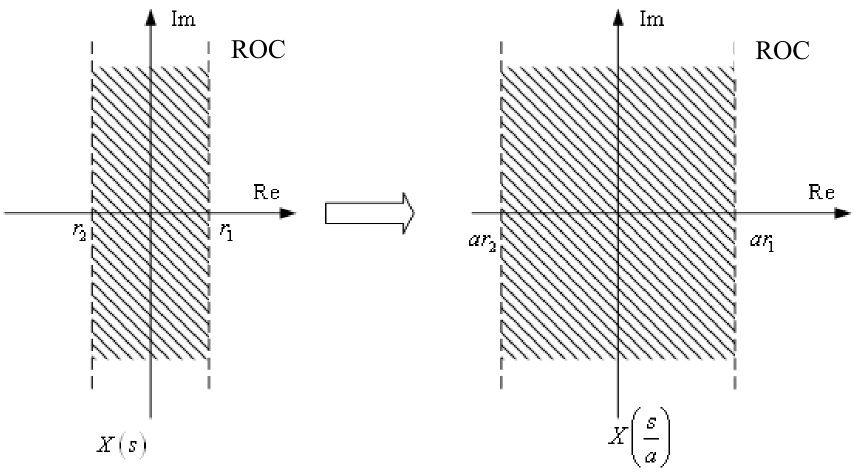


Figure 6.8 The effect on the ROC of time scaling.

$a > 0$,

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} dt = \int_{-\infty}^{\infty} x(t') e^{-\frac{s}{a}t'} \frac{1}{a} dt'$$
$$= \frac{1}{a} X\left(\frac{s}{a}\right) = \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

$a < 0$,

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} dt = \int_{\infty}^{-\infty} x(t') e^{-\frac{s}{a}t'} \frac{1}{a} dt'$$
$$= -\frac{1}{a} \int_{-\infty}^{\infty} x(t') e^{-\frac{s}{a}t'} dt' = \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

5. Convolution property

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s), \text{ with ROC} = R_1$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s), \text{ with ROC} = R_2$$

$$\Rightarrow x(t) = x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = X_1(s)X_2(s), \text{ with ROC containing } R_1 \cap R_2 \quad (6.21)$$

Note: The ROC of $X(s)$ may be larger than $R_1 \cap R_2$ if pole-zero cancellation occurs in the product.

Example 6.14:

$$X_1(s) = \frac{s+1}{s+2}, \ \text{Re}\{s\} > -2$$

$$X_2(s) = \frac{s+2}{s+1}, \ \text{Re}\{s\} > -1$$

Then $X(s) = X_1(s)X_2(s) = 1$, with ROC being the entire s-plane. ■

6. Differentiation in the time domain

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \text{ with ROC} = R$$

$$\Rightarrow \frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \text{ with ROC containing } R \quad (6.22)$$

$$\left[x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \Rightarrow \frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds \right]$$

Example 6.15: $x(t) = \frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right)$

$$e^{-3t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+3}, \text{ with ROC } \text{Re}\{s\} > -3$$

$$e^{-3(t-2)}u(t-2) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+3}e^{-2s}, \text{ with ROC } \text{Re}\{s\} > -3$$

$$x(t) = \frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \frac{s^2}{s+3}e^{-2s}, \text{ with ROC } \text{Re}\{s\} > -3$$

■

Example 6.16: $X(s)=\dfrac{2s^3-9s^2+4s+10}{s^2-3s-4}$, with $\mathrm{Re}\{s\}<-1$

$$\begin{array}{r} 2s-3 \\ s^2-3s-4\overline{\big)\,2s^3-9s^2+4s+10} \\ \underline{2s^3-6s^2-8s\quad\quad} \\ -3s^2+12s+10 \\ \underline{-3s^2+\ 9s+12} \\ 3s-2 \end{array}$$

$$X(s)=2s-3+\frac{1}{s+1}+\frac{2}{s-4},\ \text{with } \mathrm{Re}\{s\}<-1$$

$$x(t)=2\delta^{(1)}(t)-3\delta(t)-e^{-t}u(-t)-2e^{4t}u(-t)$$

Note: The ROC of $sX(s)$ includes the ROC of $X(s)$ and may be larger if $X(s)$ has a first order pole at $s=0$ which is cancelled by the multiplication by s. ■

7. Differentiation in the s-domain

$$x(t)\overset{\mathcal{L}}{\longleftrightarrow}X(s),\ \text{with ROC}=R$$

$$\Rightarrow -tx(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{dX(s)}{ds},\ \text{with ROC}=R \tag{6.23}$$

$$\left[\begin{array}{l} X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt \\ \dfrac{dX(s)}{ds}=\int_{-\infty}^{\infty}\left[-tx(t)\right]e^{-st}dt \end{array}\right]$$

8. Integration in the time domain

$$x(t)\overset{\mathcal{L}}{\longleftrightarrow}X(s),\ \text{with ROC}=R$$

$$\Rightarrow\int_{-\infty}^{t}x(\tau)d\tau\overset{\mathcal{L}}{\longleftrightarrow}X(s)/s,\ \text{with ROC containing } R\cap\{\mathrm{Re}\{s\}>0\} \tag{6.24}$$

Note:

$$\int_{-\infty}^{t}x(\tau)d\tau=u(t)*x(t)$$

$$u(t)\overset{\mathcal{L}}{\longleftrightarrow}s^{-1},\ \text{with ROC}=\mathrm{Re}\{s\}>0$$

$$\left(\text{From } e^{-at}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+a},\ \text{with ROC}=\mathrm{Re}\{s\}>-a\right)$$

$$\int_{-\infty}^{t}x(\tau)d\tau\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s}X(s),\ \text{with ROC containing } R\cap\{\mathrm{Re}\{s\}>0\}$$

9. The initial and final value theorems

Consider a signal $x(t)=0$ for $t<0$ and $x(t)$ contains no impulses or higher-order singularities at the origin. Then

$$x\left(0^{+}\right)=\lim_{s\to\infty} sX(s)\cdots\cdots\cdots\text{The Initial Value Theorem} \tag{6.25}$$

$$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)\cdots\cdots\cdots\text{The Final Value Theorem} \tag{6.26}$$

Proof:

Expanding $x(t)$ as a Taylor series at $t=0^{+}$, we obatin

$$x(t)=\left[x(0^{+})+x^{(1)}(0^{+})t+\cdots+x^{(n)}(0^{+})\frac{t^{n}}{n!}+\cdots\right]u(t)$$

where $x^{(n)}(0^{+})$ denotes the nth derivative of $x(t)$ evaluated at $t=0^{+}$.

$$u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s}$$

$$tu(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s^{2}}$$

$$\vdots$$

$$\frac{t^{n}}{n!}u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s^{n+1}}$$

$$\Rightarrow \mathcal{L}\{x(t)\}=\frac{1}{s}x\left(0^{+}\right)+\frac{1}{s^{2}}x^{(1)}\left(0^{+}\right)+\cdots+\frac{1}{s^{n}}x^{(n)}\left(0^{+}\right)+\cdots=X(s)$$

$$\Rightarrow sX(s)=x\left(0^{+}\right)+\frac{1}{s}x^{(1)}\left(0^{+}\right)+\cdots+\frac{1}{s^{n-1}}x^{(n)}\left(0^{+}\right)+\cdots$$

$$\Rightarrow \lim_{s\to\infty} sX(s)=x\left(0^{+}\right)\cdots\cdots\cdots\text{The Initial Value Theorem}$$

Let us consider the limit of the integral $\int_{0^{+}}^{\infty}\frac{dx(t)}{dt}e^{-st}dt$ as s approach 0. Then we have

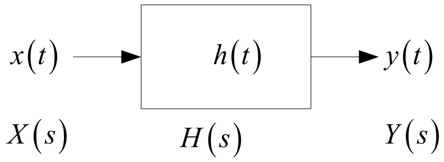
$$\lim_{s\to 0}\int_{0^{+}}^{\infty}\frac{dx(t)}{dt}e^{-st}dt=\int_{0^{+}}^{\infty}\frac{dx(t)}{dt}dt=x(t)\Big|_{0^{+}}^{\infty}=\lim_{t\to\infty}x(t)-x\left(0^{+}\right).$$

Also, it can be checked that the following equations hold:

$$\begin{aligned}\lim_{s\to 0}\int_{0^{+}}^{\infty}\frac{dx(t)}{dt}e^{-st}dt&=\lim_{s\to 0}\left\{\left[x(t)e^{-st}\right]_{0^{+}}^{\infty}-\int_{0^{+}}^{\infty}x(t)\frac{de^{-st}}{dt}dt\right\}\\&=\lim_{s\to 0}\left\{\left[x(t)e^{-st}\right]_{0^{+}}^{\infty}-\int_{0^{+}}^{\infty}x(t)(-s)e^{-st}dt\right\}\\&=\lim_{s\to 0}\left[-x\left(0^{+}\right)+sX(s)\right]=-x\left(0^{+}\right)+\lim_{s\to 0}sX(s)\end{aligned}$$

$$\Rightarrow \lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)\cdots\cdots\cdots\text{The Final Value Theorem} \qquad \blacksquare$$

6-5 Analysis and Characterization of LTI Systems Using the Laplace Transform



$$Y(s) = H(s)X(s)$$

Figure 6.8 An LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$.

$H(s)$: the system function or transfer function

With $s = j\omega$, $H(j\omega)$ is called the frequency response of the LTI system.

For a causal system, $h(t) = 0$ for $t < 0$ (Fig. 6.9).

$\Rightarrow$ $h(t)$ is a right-sided signal.

$\Rightarrow$ The ROC is the entire region in the s-plane to the right of the rightmost pole.

Note:

- Anticausal system $h(t)$
  $\Rightarrow$ Its ROC is the region in the s-plane to the left of the leftmost pole.
- An ROC to the right of the rightmost pole does not guarantee that the system is causal, only that the impulse response is right-sided.
- The Fourier transform of the impulse response for a stable LTI system exists.
  $\Rightarrow$ For a stable system, the ROC of $H(s)$ must include the $j\omega$-axis (Fig. 6.10).
- For a causal and stable LTI system with a rational system function, all poles must lie in the left half of the s-plane.

  causal $\Rightarrow$ The ROC must be to the right of the rightmost pole.

  stable $\Rightarrow$ The ROC must include the $j\omega$-axis.

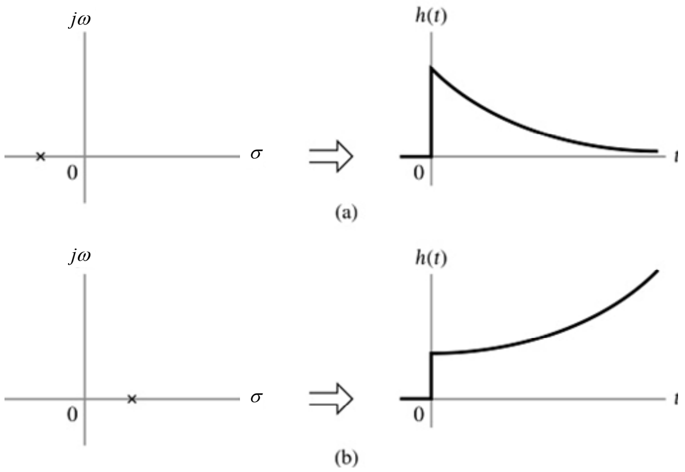


Figure 6.9 The relationship between the locations of poles and the impulse response in a causal system. (a) A pole in the left half of the s-plane corresponds to an exponentially decaying impulse response. (b) A pole in the right half of the s-plane corresponds to an exponentially increasing impulse response; the system is unstable in this case.

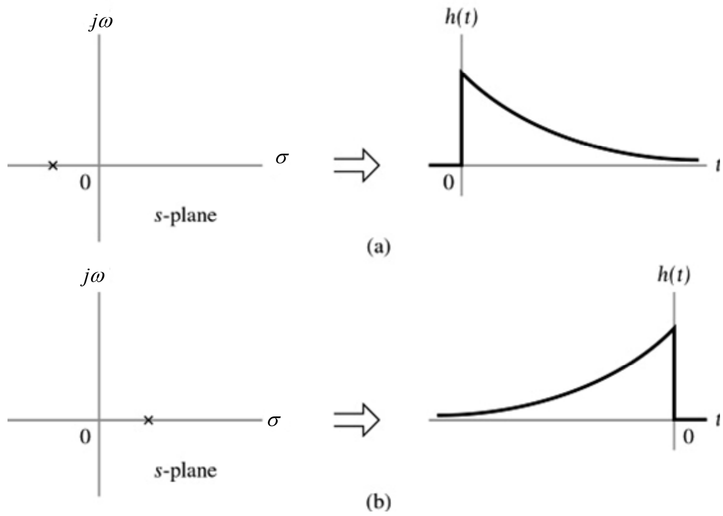


Figure 6.10 The relationship between the locations of poles and the impulse response in a stable system. (a) A pole in the left half of the *s*-plane corresponds to a right-sided impulse response. (b) A pole in the right half of the *s*-plane corresponds to an left-sided impulse response; the system is noncausal in this case.

Example 6.17: $h(t)=e^{-t}u(t)\Rightarrow H(s)=\dfrac{1}{s+1},\ \text{Re}\{s\}>-1$

$\Rightarrow$ causal and stable ■

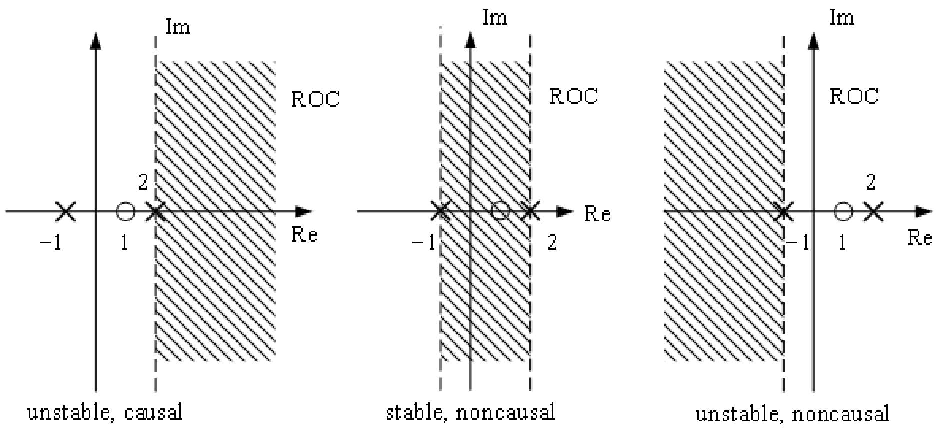
Example 6.18: $H(s)=\dfrac{e^{s}}{s+1},\ \text{Re}\{s\}>-1\ \Rightarrow\ e^{-t}u(t)\xleftrightarrow{\mathcal{L}}\dfrac{1}{s+1},\ \text{Re}\{s\}>-1$

$\Rightarrow e^{-(t+1)}u(t+1)\xleftrightarrow{\mathcal{L}}\dfrac{e^{s}}{s+1},\ \text{Re}\{s\}>-1$

$\Rightarrow h(t)=e^{-(t+1)}u(t+1)$, zero for $t<-1$ but not for $t<0$

$\Rightarrow$ The system is stable but not causal. ■

Example 6.19: Possible ROCs of $H(s)=\dfrac{s-1}{(s+1)(s-2)}$:



■

Example 6.20: Determine the impulse response with stability and causality constraints for the following system function:

$$H(s)=\frac{2}{s+3}+\frac{1}{s-2}.$$

If the system is stable, then the pole at $s=-3$ contributes a right-sided term to the impulse response, while the pole at $s=2$ contributes a left-sided term. Accordingly, the corresponding impulse response is

$$h(t)=2e^{-3t}u(t)-e^{2t}u(-t).$$

If the system is causal, then both poles must contribute right-sided terms to the impulse response and the corresponding impulse response is

$$h(t)=2e^{-3t}u(t)+e^{2t}u(t).$$ ■

1. System characterized by linear constant-coefficient differential equations

$$\sum_{k=0}^{N}a_k\frac{d^k y(t)}{dt^k}=\sum_{k=0}^{M}b_k\frac{d^k x(t)}{dt^k}$$

$$\updownarrow \mathcal{L} \qquad\qquad \updownarrow \mathcal{L}$$

$$\left\{\sum_{k=0}^{N}a_k s^k\right\}Y(s)=\left\{\sum_{k=0}^{M}b_k s^k\right\}X(s)$$

$$\Rightarrow H(s)=\frac{Y(s)}{X(s)}=\frac{\left\{\sum_{k=0}^{M}b_k s^k\right\}}{\left\{\sum_{k=0}^{N}a_k s^k\right\}}$$

The system function has zeros and poles respectively at the solutions of

$$\sum_{k=0}^{M}b_k s^k=0 \text{ and } \sum_{k=0}^{N}a_k s^k=0.$$

Note: With additional information such as stability or causality of the system, the ROC can be inferred and the corresponding impulse response can be obtained.

Example 6.21: Determine the impulse response of the system described by

$$\frac{dy(t)}{dt}+3y(t)=x(t).$$

$$\Rightarrow sY(s)+3Y(s)=X(s)\Rightarrow H(s)=\frac{1}{s+3}$$

If the system is causal, the ROC is $\operatorname{Re}\{s\}>-3$, and the corresponding impulse response is

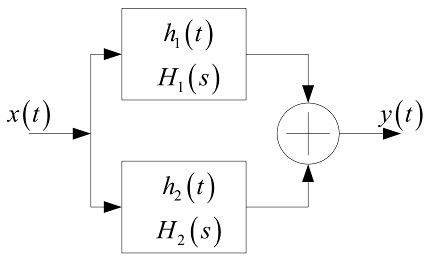
$$h(t)=e^{-3t}u(t).$$

If the system is noncausal, then the ROC is $\operatorname{Re}\{s\}<-3$, and the corresponding impulse response is

$$h(t)=-e^{-3t}u(-t).$$ ■

2. System function for interconnections of LTI systems

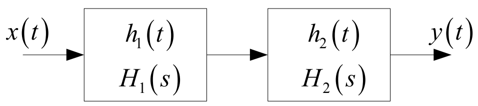
(1) Parallel interconnection



$$h(t) = h_1(t) + h_2(t) \Rightarrow H(s) = H_1(s) + H_2(s)$$

Figure 6.11 Parallel interconnection of two LTI systems.

(2) Cascade interconnection



$$h(t) = h_1(t) * h_2(t) \Rightarrow H(s) = H_1(s)H_2(s)$$

Figure 6.12 Cascade interconnection of two LTI systems.

(3) Feedback interconnection

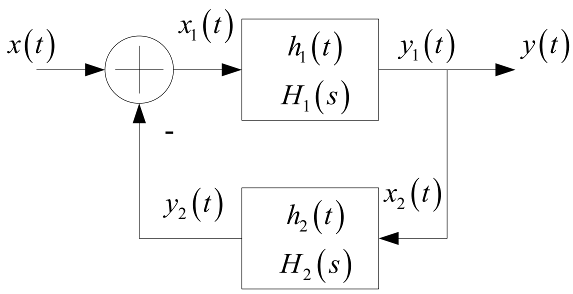


Figure 6.13 Feedback interconnection of two LTI systems.

$$Y_2(s) = H_2(s)X_2(s) = H_2(s)Y_1(s) = H_2(s)Y(s)$$

$$\begin{aligned} Y(s) &= H_1(s)X_1(s) = H_1(s)\left[X(s) - Y_2(s)\right] \\ &= H_1(s)X(s) - H_1(s)H_2(s)Y(s) \end{aligned}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

3. Design of a Butterworth filter with frequency response $B(j\omega)$

$$\left|B(j\omega)\right|^2 = \frac{1}{1+(\omega/\omega_c)^{2N}} = \frac{1}{1+(j\omega/j\omega_c)^{2N}} \tag{6.27}$$

$$\left|B(j\omega)\right|^2 = B(j\omega)B^*(j\omega) \tag{6.28}$$

Restricting the impulse response of the Butterworth filter to be real, we have

$$B^*(j\omega) = B(-j\omega) \tag{6.29}$$

$$B(j\omega)B(-j\omega) = 1\Big/\left[1+(j\omega/j\omega_c)^{2N}\right] \tag{6.30}$$

$$\because B(s)\big|_{s=j\omega} = B(j\omega) \tag{6.31}$$

$$\therefore B(s)B(-s) = 1\Big/\left[1+(s/j\omega_c)^{2N}\right] \tag{6.32}$$

The poles of $B(s)B(-s)$ are the solutions of

$$1+(s/j\omega_c)^{2N} = 0 \tag{6.33}$$

$$\Rightarrow s_p = (-1)^{1/2N}(j\omega_c) \tag{6.34}$$

$$\Rightarrow \left|s_p\right| = \omega_c,\ \angle s_p = \frac{\pi(2k+1)}{2N}+\frac{\pi}{2},\ k \text{ is an integer} \tag{6.35}$$

$$\Rightarrow s_p = \omega_c e^{j\left[\frac{\pi(2k+1)}{2N}+\frac{\pi}{2}\right]} \tag{6.36}$$

The pole locations of $B(s)B(-s)$ for N = 1, 2, 3, and 6 are shown as follows:

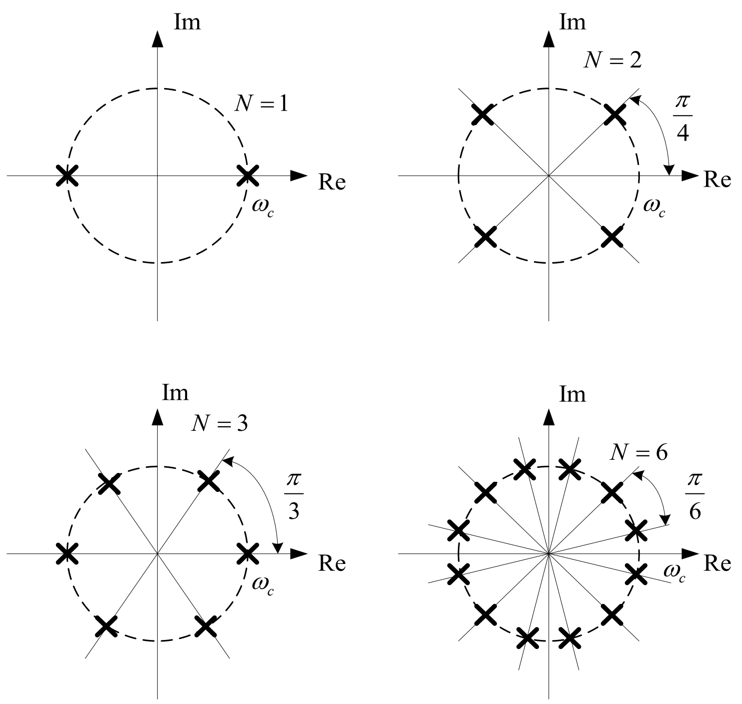


Figure 6.14 The pole locations of $B(s)B(-s)$ for N = 1, 2, 3, and 6.

(1) The poles of $B(s)B(-s)$ occurs in pairs, so that if there is a pole at $s = s_p$, then there is also a pole at $s = -s_p$.

(2) To construct $B(s)$, we choose one pole from each pair of poles.

(3) If we restrict the system to be stable and causal, then the poles of $B(s)$ should be in the left-half plane.

(4) $B^2(s)\big|_{s=0} = 1$

$$N = 1:\ B(s) = \frac{\omega_c}{s + \omega_c} \tag{6.37}$$

$$N = 2:\ B(s) = \frac{\omega_c^2}{\left(s + \omega_c e^{j\frac{\pi}{4}}\right)\left(s + \omega_c e^{-j\frac{\pi}{4}}\right)} = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} \tag{6.38}$$

$$N = 3:\ B(s) = \frac{\omega_c^3}{(s + \omega_c)\left(s + \omega_c e^{j\frac{\pi}{3}}\right)\left(s + \omega_c e^{-j\frac{\pi}{3}}\right)} = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3} \tag{6.39}$$

From these equations, we have

$$\frac{\omega_c}{s + \omega_c} = \frac{Y(s)}{X(s)} \Rightarrow \omega_c X(s) = sY(s) + \omega_c Y(s) \tag{6.40}$$

$$\frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} = \frac{Y(s)}{X(s)} \Rightarrow \omega_c^2 X(s) = s^2 Y(s) + \sqrt{2}\omega_c sY(s) + \omega_c^2 Y(s) \tag{6.41}$$

$$\begin{aligned} &\frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3} = \frac{Y(s)}{X(s)} \\ &\Rightarrow \omega_c^3 X(s) = s^3 Y(s) + 2\omega_c s^2 Y(s) + 2\omega_c^2 sY(s) + \omega_c^3 Y(s) \end{aligned} \tag{6.42}$$

Accordingly, the corresponding differential equations for these three cases are as follows:

$$N = 1:\ \frac{dy(t)}{dt} + \omega_c y(t) = \omega_c x(t) \tag{6.43}$$

$$N = 2:\ \frac{d^2 y(t)}{dt^2} + \sqrt{2}\omega_c \frac{dy(t)}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t) \tag{6.44}$$

$$N = 3:\ \frac{d^3 y(t)}{dt^3} + 2\omega_c \frac{d^2 y(t)}{dt^2} + 2\omega_c^2 \frac{dy(t)}{dt} + \omega_c^3 y(t) = \omega_c^3 x(t) \tag{6.45}$$

6-6 The Unilateral Laplace Transform

1. The unilateral Laplace transform of $x(t)$ is defined as

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t) e^{-st} dt . \tag{6.46}$$

It is different from the bilateral Laplace transform given by

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt .$$

When $x(t)=0$ for $t < 0$, the unilateral and bilateral Laplace transforms are identical.

Note: The ROC for the unilateral Laplace transform is always a right-half plane, since it can be regarded as the bilateral Laplace transform of a causal signal.

Example 6.22:

$$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$$
$$\mathcal{X}(s) = \frac{1}{(s+a)^n}, \ \operatorname{Re}\{s\} > -a$$

■

Example 6.23:

$$x(t) = e^{-a(t+1)} u(t+1)$$
$$X(s) = \frac{e^s}{s+a}, \ \operatorname{Re}\{s\} > -a$$
$$\mathcal{X}(s) = \int_{0^-}^{\infty} e^{-a(t+1)} u(t+1) e^{-st} dt = \int_{0^-}^{\infty} e^{-a} e^{-(s+a)t} dt = e^{-a} \int_{\infty}^{\infty} \left\{ e^{-at} u(t) \right\} e^{-st} dt$$
$$= \frac{e^{-a}}{s+a}, \ \operatorname{Re}\{s\} > -a$$

The unilateral and bilateral Laplace transforms are distinctly different. ■

2. Most of the properties of the unilateral Laplace transform are the same as for the bilateral Laplace transform.

3. The differentiation property of the unilateral Laplace transform

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = x(t) e^{-st} \Big|_{0^-}^{\infty} + s \int_{0^-}^{\infty} x(t) e^{-st} dt = s\mathcal{X}(s) - x(0^-) \tag{6.47}$$

(Integration by parts)

where $\mathcal{X}(s)$ is the unilateral Laplace transform of $x(t)$.

Similarly

$$\begin{aligned}\int_{0^-}^{\infty}\frac{d^2x(t)}{dt^2}e^{-st}dt &= s\int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}dt - x^{(1)}\left(0^-\right)\\ &= s\left[s\mathcal{X}(s)-x\left(0^-\right)\right]-x^{(1)}\left(0^-\right)\\ &= s^2\mathcal{X}(s)-sx\left(0^-\right)-x^{(1)}\left(0^-\right)\ .\end{aligned} \tag{6.48}$$

The general form for the differentiation property is

$$\frac{d^n}{dt^n}x(t)\xleftrightarrow{\mathcal{L}_u}\begin{cases} s^n\mathcal{X}(s)-\left.\frac{d^{n-1}}{dt^{n-1}}x(t)\right|_{t=0^-}-s\left.\frac{d^{n-2}}{dt^{n-2}}x(t)\right|_{t=0^-} \\ -\cdots-s^{n-2}\left.\frac{d}{dt}x(t)\right|_{t=0^-}-s^{n-1}x\left(0^-\right)\end{cases} \tag{6.49}$$

where the subscript u in $\mathcal{L}_u$ denotes the unilateral transform.

Example 6.24: $x(t)=e^{at}u(t)$

Apply the product rule for differentiation to obtain the derivative of $x(t)$, $t>0^-$:

$$\frac{d}{dt}x(t)=\frac{d}{dt}e^{at}u(t)=ae^{at}u(t)+\delta(t)\xleftrightarrow{\mathcal{L}}\frac{a}{s-a}+1=\frac{s}{s-a}$$

Using Eq. (6.50), we have

$$\frac{d}{dt}x(t)\xleftrightarrow{\mathcal{L}_u}s\frac{1}{s-a}+0=\frac{s}{s-a}\ .$$

The results are identical, since $x(t)$ is a causal signal.

■

4. The integration property of the unilateral Laplace transform

$$\int_{-\infty}^{t}x(\tau)d\tau\xleftrightarrow{\mathcal{L}_u}\frac{\int_{-\infty}^{0^-}x(\tau)d\tau}{s}+\frac{\mathcal{X}(s)}{s} \tag{6.50}$$

Proof: Let $y(t)=\int_{-\infty}^{t}x(\tau)d\tau$. Then

$$\frac{d}{dt}y(t)=x(t)$$

$$s\mathcal{Y}(s)-y\left(0^-\right)=\mathcal{X}(s)$$

$$\mathcal{Y}(s)=\frac{\mathcal{X}(s)}{s}+\frac{\int_{-\infty}^{0^-}x(\tau)d\tau}{s}$$

■

5. A primary use of the unilateral Laplace transform is in obtaining the solution of linear constant-coefficient differential equations with nonzero initial conditions.

Example 6.25: $\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t),\ y(0^-)=3,\ \frac{dy(0^-)}{dt}=-5$

Let $x(t)=2u(t)$. Then we obtain

$$s^2\mathcal{Y}(s)-sy(0^-)-y^{(1)}(0^-)+3s\mathcal{Y}(s)-3y(0^-)+2\mathcal{Y}(s)=\frac{2}{s}$$

$$\mathcal{Y}(s)=\frac{3s+4}{(s+1)(s+2)}+\frac{2}{s(s+1)(s+2)}$$

where $\mathcal{Y}(s)$ is the unilateral Laplace transform of $y(t)$. Thus, we have

$$\mathcal{Y}(s)=\frac{1}{s}-\frac{1}{s+1}+\frac{3}{s+2}\Rightarrow y(t)=\left[1-e^{-t}+3e^{-2t}\right]u(t).$$

■

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