Chapter 5 The Discrete-Time Fourier Transform

5-1 The Fourier Transform of Aperiodic Discrete-Time Signals

1. Consider a general aperiodic sequence $x[n]$ which is of finite duration. From this aperiodic sequence, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is of one period.

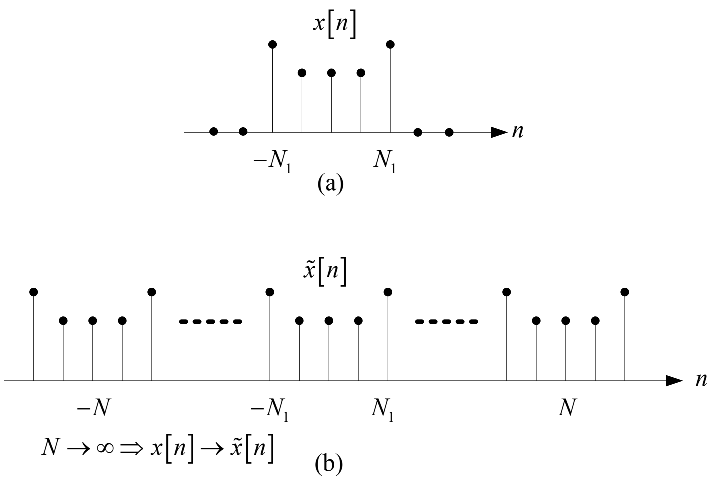


Figure 5.1 (a) A finite-duration signal $x[n]$; (b) a periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

The discrete-time Fourier series representation of $\tilde{x}[n]$ is

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{5.1}$$

$$a_k = \frac{1}{N} \sum_{k=<N>} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} \tag{5.2}$$

$$\because x[n] = \tilde{x}[n] \text{ for } |n| \le N_1 \tag{5.3}$$

$$\therefore a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n} \tag{5.4}$$

Defining the envelope of Na_k as $X\left(e^{j\Omega}\right)$, we have

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \tag{5.5}$$

$$Na_k = X\left(e^{j\Omega}\right)\Big|_{\Omega=\frac{2\pi k}{N}} \quad \left(\text{or } a_k = \frac{1}{N} X\left(e^{j\frac{2\pi}{N}k}\right)\right) \tag{5.6}$$

The coefficients a_k are proportional to equally spaced samples of the envelope function $X\left(e^{j\Omega}\right)$, where the sample spacing is equal to $\Omega_0 = 2\pi/N$.

$$\Rightarrow \tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X\left(e^{j(k\Omega_0)}\right) e^{jk\Omega_0 n} \tag{5.7}$$

$$\Rightarrow \tilde{x}[n] = \frac{1}{2\pi} \sum_{k=<N>} X\left(e^{j(k\Omega_0)}\right) e^{jk\Omega_0 n}\Omega_0 \tag{5.8}$$

As $N \to \infty$, $\tilde{x}[n] \to x[n]$. This means that the above equation becomes a representation of $x[n]$ and the summation operator becomes the integration with $\Omega_0 \to d\Omega$ and $k\Omega_0 \to \Omega$. Accordingly, we have the following discrete-time Fourier transform pair:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega \quad \text{(Synthesis Equation)} \tag{5.9}$$

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \quad \text{(Analysis Equation)} \tag{5.10}$$

$X\left(e^{j\Omega}\right)$ is called the discrete-time Fourier transform (or spectrum) of $x(n)$, and $x(n)$ is called the inverse Fourier transform of $X\left(e^{j\Omega}\right)$.

2. The synthesis equation can be interpreted as that $x[n]$ is a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes $X\left(e^{j\Omega}\right)\left(d\Omega/2\pi\right)$.

3. The convergence of the discrete-time Fourier transform is guaranteed if $x[n]$ is absolutely summable or if the sequence has finite energy, i.e.,

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty \tag{5.11}$$

or

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 < \infty. \tag{5.12}$$

4. The major differences between the continuous-time Fourier transform and the discrete-time Fourier transform:

   (1) The discrete-time Fourier transform is periodic of period 2π, and the continuous-time Fourier transform is aperiodic except for some special cases such as the periodic impulse train.

(2) The discrete-time Fourier transform has a finite interval of integration in the synthesis equation, while the continuous-time Fourier transform has an infinite interval of integration in the synthesis equation.

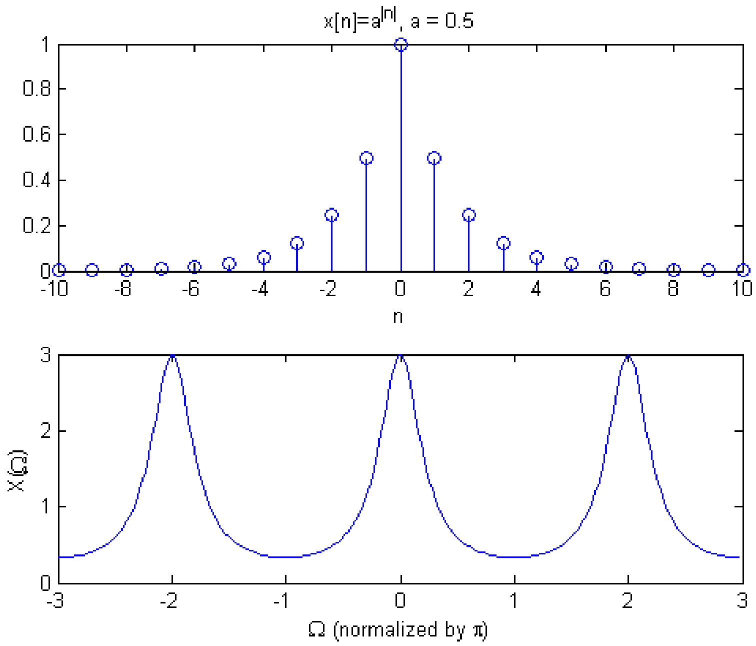
$$\begin{cases} x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \end{cases} \qquad \begin{cases} x[n] = \dfrac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega \\ X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \end{cases}$$

Example 5.1: $x[n] = a^{|n|},\ |a| < 1$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} + \sum_{n=-\infty}^{-1} a^{-n}e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} + \sum_{m=1}^{\infty} (ae^{j\Omega})^m \quad (m = -n)$$

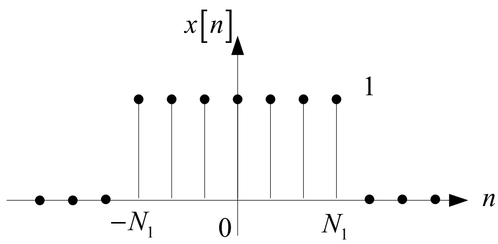
$$= \frac{1}{1-ae^{-j\Omega}} + \left(\frac{1}{1-ae^{j\Omega}} - 1\right) = \frac{1-a^2}{1-2a\cos\Omega + a^2}$$



■

Example 5.2:

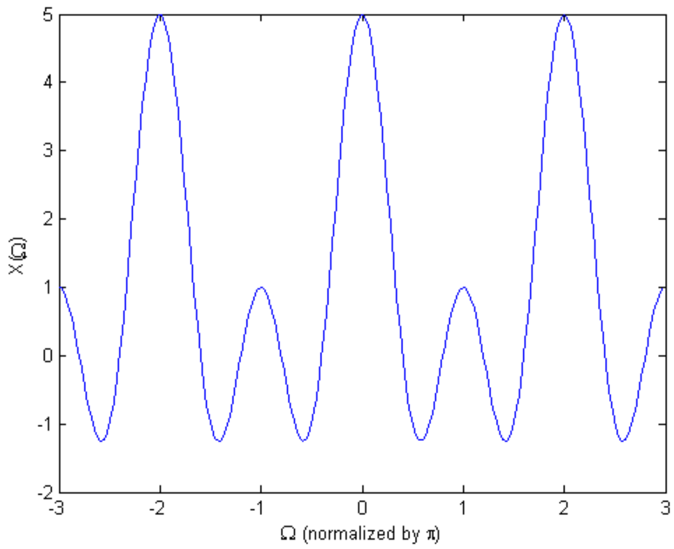
$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases} \quad \text{(rectangular pulse)}$$



$$X(e^{j\Omega}) = \sum_{n=-N_1}^{N_1} e^{-j\Omega n} = \sum_{m=0}^{2N_1} e^{-j\Omega(m-N_1)} \quad (m = n + N_1)$$

$$= e^{j\Omega N_1} \sum_{m=0}^{2N_1} e^{-j\Omega m} = e^{j\Omega N_1} \frac{1-e^{-j\Omega(2N_1+1)}}{1-e^{-j\Omega}}$$

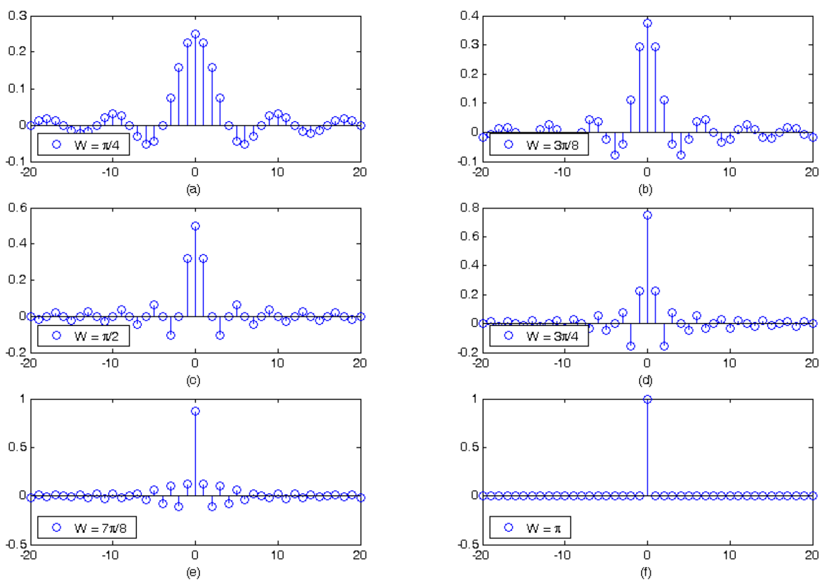
$$= e^{j\Omega N_1} \frac{e^{-j\Omega(2N_1+1)/2}}{e^{-j\Omega/2}} \left(\frac{e^{j\Omega(2N_1+1)/2} - e^{-j\Omega(2N_1+1)/2}}{e^{j\Omega/2} - e^{-j\Omega/2}} \right) = \frac{\sin(\Omega(2N_1+1)/2)}{\sin(\Omega/2)}$$



The discrete-time counterpart of the sinc function: periodic with period 2π. ■

Example 5.3: Let $x[n] = \delta[n]$. Then $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1$. The following figure shows the approximation of $x[n]$ by $\hat{x}[n]$ for different values of W:

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-W}^{W} e^{j\Omega n} d\Omega = \frac{1}{2\pi} \cdot \frac{1}{jn} e^{j\Omega n} \Big|_{-W}^{W} = \frac{1}{j2\pi n} \left(e^{jWn} - e^{-jWn} \right) = \frac{1}{\pi n} \sin(Wn)$$



As $W \to \pi$, $\hat{x}[n] \to x[n]$ with no convergence problems and no Gibbs phenomenon. ■

5-2 Periodic Signals and the Discrete-Time Fourier Transform

1. Fourier series coefficients as samples of the Fourier transform of one period

   Let $\tilde{x}[n]$ be a periodic signal with period N, and let $x[n]$ represent one period of $\tilde{x}[n]$, i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & M \le n \le M+N-1 \\ 0, & \text{otherwise} \end{cases}$$

   where M is an arbitrary integer. Then

$$Na_k = X\left(e^{jk\frac{2\pi}{N}}\right)$$

   with a_k being the discrete-time Fourier series coefficients of $\tilde{x}[n]$ and $X\left(e^{j\Omega}\right)$ being the discrete-time Fourier transform of $x[n]$.

   ⇒ Na_k for k=0, 1, 2, …, N-1 correspond to N samples of the Fourier transform of one period.

   When M is varied, $X\left(e^{j\Omega}\right)$ is changed. But the values of $X\left(e^{j\Omega}\right)$ at the sample frequencies $2\pi k/N$ for k=0, 1, 2, …, N–1 do not depend on M.

Example 5.4:

Let $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$. Then the corresponding discrete-time Fourier series coefficients can be calculated by

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}.$$

Let $x_1[n] = \delta[n]$ (i.e., M=0). Then the corresponding discrete-time Fourier transform is $X_1\left(e^{j\Omega}\right) = 1$.

Let $x_2[n] = \delta[n-N]$ (i.e., $0 < M < N$). Then the corresponding discrete-time Fourier transform is $X_2\left(e^{j\Omega}\right) = e^{-j\Omega N}$.

Clearly, $X_1\left(e^{j\Omega}\right) \ne X_2\left(e^{j\Omega}\right)$. However, at the set of sample frequencies $\Omega = 2\pi k/N$ for k=0, 1, 2, …, N–1, $X_1\left(e^{j\Omega}\right)$ and $X_2\left(e^{j\Omega}\right)$ are identical. ■

2. The discrete-time Fourier transform for periodic signals

Consider the signal

$$x[n] = e^{j\Omega_0 n}.$$

Can we compute the corresponding discrete-time Fourier transform? --- No.

$$\begin{aligned} X\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n}e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} e^{-j(\Omega-\Omega_0)n} = ? \end{aligned}$$

Let us consider the discrete-time Fourier transform

$$X\left(e^{j\Omega}\right) = \sum_{l=-\infty}^{\infty} 2\pi\delta\left(\Omega-\Omega_0-2\pi l\right). \tag{5.13}$$

Can we compute the corresponding inverse discrete-time Fourier transform? --- Yes.

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\Omega-\Omega_0-2\pi l\right)e^{j\Omega n}d\Omega \\ &= e^{j\Omega_0 n + j2\pi rn} = e^{j\Omega_0 n} \quad \left(\Omega = \Omega_0 + 2\pi r,\ \text{with } l = r\right) \end{aligned} \tag{5.14}$$

(Any interval of length 2π includes exactly one impulse in the summation.)

More generally, if $x[n]$ is the sum of an arbitrary set of complex exponentials, i.e.,

$$x[n] = b_1 e^{j\Omega_1 n} + b_2 e^{j\Omega_2 n} + \cdots + b_M e^{j\Omega_M n}. \tag{5.15}$$

Then

$$\begin{aligned} X\left(e^{j\Omega}\right) &= b_1\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\Omega-\Omega_1-2\pi l\right) + b_2\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\Omega-\Omega_2-2\pi l\right) \\ &+ \cdots + b_M\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\Omega-\Omega_M-2\pi l\right). \end{aligned} \tag{5.16}$$

Note:

- $e^{j\Omega_0 n}$ is periodic when $2\pi/\Omega_0 = m/N$ is a rational number or integer.
- $x[n] = b_1 e^{j\Omega_1 n} + b_2 e^{j\Omega_2 n} + \cdots + b_M e^{j\Omega_M n}$ is periodic only when all of the $2\pi/\Omega_i = m/N$ are rational numbers or integers.
- If $x[n]$ is a periodic sequence with period *N*, we can determine its Fourier series representation first and then compute the corresponding Fourier transform as follows:

$$x[n] = a_0 + a_1 e^{j\frac{2\pi}{N}n} + a_2 e^{j2(\frac{2\pi}{N})n} + \cdots + a_{N-1}e^{j(N-1)(\frac{2\pi}{N})n} \tag{5.17}$$

$$\begin{aligned} X\left(e^{j\Omega}\right) &= a_0\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\Omega-2\pi l\right) + a_1\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\Omega-\frac{2\pi}{N}-2\pi l\right) \\ &+ \cdots + a_{N-1}\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\Omega-(N-1)\frac{2\pi}{N}-2\pi l\right) \end{aligned} \tag{5.18}$$

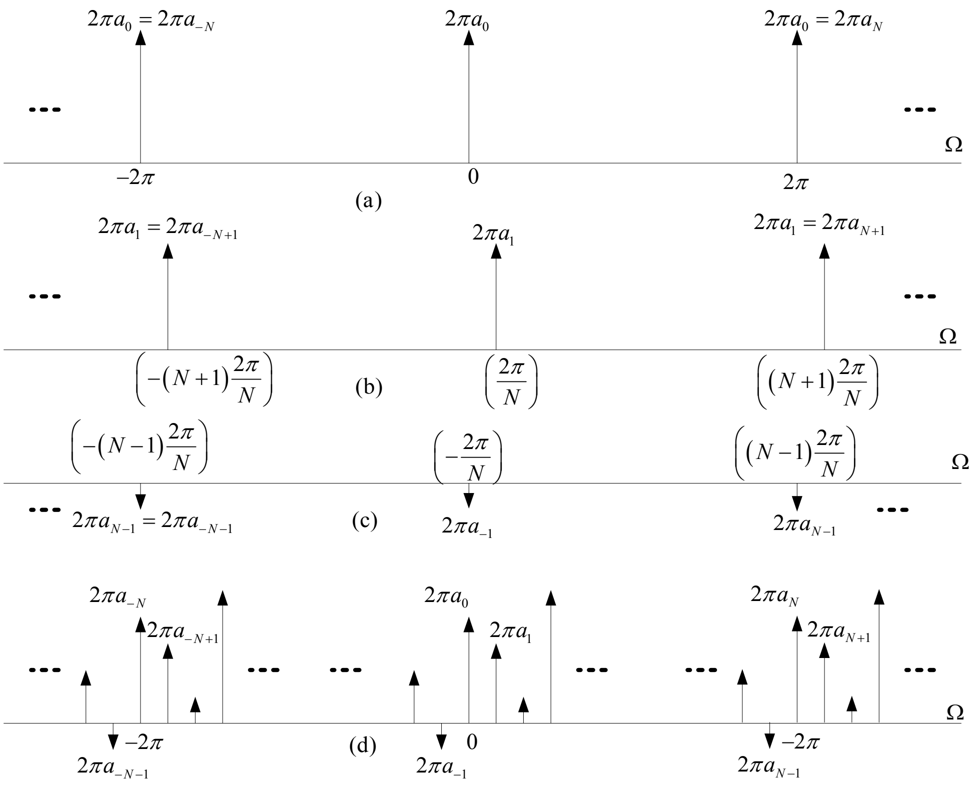


Figure 5.2 The Fourier transform of a discrete-time periodic signal: (a) the first summation on the right-hand side of (5.18); (b) the second summation on the right-hand side of (5.18); (c) the final summation on the right-hand side of (5.18); (d) the entire expression of (5.18).

3. The discrete Fourier transform

Let

$$x[n] = 0 \text{, outside the interval } 0 \le n \le N_1 - 1$$

$$\tilde{x}[n] = x[n],\ 0 \le n \le N - 1$$

where $\tilde{x}[n]$ is periodic with period N and $N \ge N_1$.

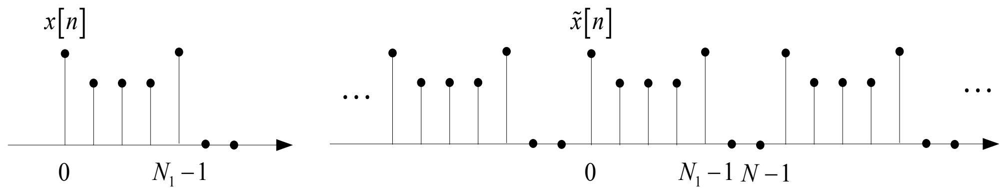


Figure 5.3 A nonperiodic signal $x[n]$ with finite duration and a periodic signal $\tilde{x}[n]$ (with period N) constructed to be equal to $x[n]$ over one period.

The Fourier series representation of $\tilde{x}[n]$ is

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$$

where

$$a_k = \frac{1}{N}\sum_{n=<N>} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}.$$

Let $X[k] = Na_k$. Then we can define the *N*-point discrete Fourier transform (DFT) of $x[n]$ as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \qquad k = 0,1,2,\ldots,N-1 \text{ --- DFT} \qquad (5.19)$$

with

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}n} \qquad n = 0,1,2,\ldots,N-1 \text{ --- Inverse DFT (IDFT)} \qquad (5.20)$$

Note:

- The original finite duration signal can be reconstructed from its DFT.
- The length of DFT is chosen approximately so that fast algorithms can easily be used for the computation. (Fast Fourier Transform algorithms) For example, a power of 2 ($2^m = N$) is often chosen as a transform length.

5-3 Properties of the Discrete-Time Fourier Transform

1. Periodicity

   The discrete-time Fourier transform is always periodic in Ω with period 2π.

$$\begin{cases} X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \\ x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega \end{cases}$$

2. Linearity

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1\left(e^{j\Omega}\right)$$

$$x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2\left(e^{j\Omega}\right)$$

$$a_1 x_1[n] + b_2 x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} a_1 X_1\left(e^{j\Omega}\right) + b_2 X_2\left(e^{j\Omega}\right) \qquad (5.21)$$

3. Symmetry properties

If $x[n]$ is a real-valued sequence, then

(1) $X\left(e^{j\Omega}\right)=X^*\left(e^{j(-\Omega)}\right)$ (5.22)

(2) $\mathrm{Re}\left\{X\left(e^{j\Omega}\right)\right\}=\mathrm{Re}\left\{X\left(e^{j(-\Omega)}\right)\right\}$: even function (5.23)

(3) $\mathrm{Im}\left\{X\left(e^{j\Omega}\right)\right\}=-\mathrm{Im}\left\{X\left(e^{j(-\Omega)}\right)\right\}$: odd function (5.24)

(4) $\left|X\left(e^{j\Omega}\right)\right|=\left|X\left(e^{j(-\Omega)}\right)\right|$ (5.25)

(5) $\angle X\left(e^{j\Omega}\right)=-\angle X\left(e^{j(-\Omega)}\right)$ (5.26)

(6) $x_e[n]\overset{\mathcal{F}}{\longleftrightarrow}\mathrm{Re}\left\{X\left(e^{j\Omega}\right)\right\}$ (5.27)

(7) $x_o[n]\overset{\mathcal{F}}{\longleftrightarrow}j\,\mathrm{Im}\left\{X\left(e^{j\Omega}\right)\right\}$ (5.28)

4. Time shifting and frequency shifting

If $x[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j\Omega}\right)$, then

$$x[n-n_0]\overset{\mathcal{F}}{\longleftrightarrow}e^{-j\Omega n_0}X\left(e^{j\Omega}\right) \quad (5.29)$$

$$e^{j\Omega_0 n}x[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j(\Omega-\Omega_0)}\right) \quad (5.30)$$

$$\frac{1}{2\pi}\int_{2\pi}X\left(e^{j(\Omega-\Omega_0)}\right)e^{j\Omega n}d\Omega=\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\Omega'}\right)e^{j(\Omega'+\Omega_0)n}d\Omega'$$
$$=e^{j\Omega_0 n}\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\Omega'}\right)e^{j\Omega' n}d\Omega'=e^{j\Omega_0 n}x[n]$$

5. Differencing and Accumulation

$$x[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j\Omega}\right)$$

(1) $x[n]-x[n-1]\overset{\mathcal{F}}{\longleftrightarrow}\left(1-e^{-j\Omega}\right)X\left(e^{j\Omega}\right)$ (5.31)

(2) $y[n]=\sum_{m=-\infty}^{n}x[m]=x[n]*u[n]$

$$y[n]+c-y[n-1]-c=x[n]\Rightarrow Y\left(e^{j\Omega}\right)\left(1-e^{-j\Omega}\right)=X\left(e^{j\Omega}\right)$$

$$\Rightarrow\ Y\left(e^{j\Omega}\right)=\frac{1}{1-e^{-j\Omega}}X\left(e^{j\Omega}\right)+?\ \text{(dependent on } c\text{)}$$

$$\Rightarrow \sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\Omega}}X\left(e^{j\Omega}\right)+\underbrace{\pi X\left(e^{j\cdot 0}\right)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)}_{\text{This term reflects the dc or average value that can result from summation.}} \qquad (5.32)$$

Note:

- $1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$
- Average value (or dc value) is $\frac{1}{2}X\left(e^{j\cdot 0}\right)=\frac{1}{2}\sum_{m=-\infty}^{n}x[m]$.

Example 5.5:

$$x[n]=\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right)=1$$

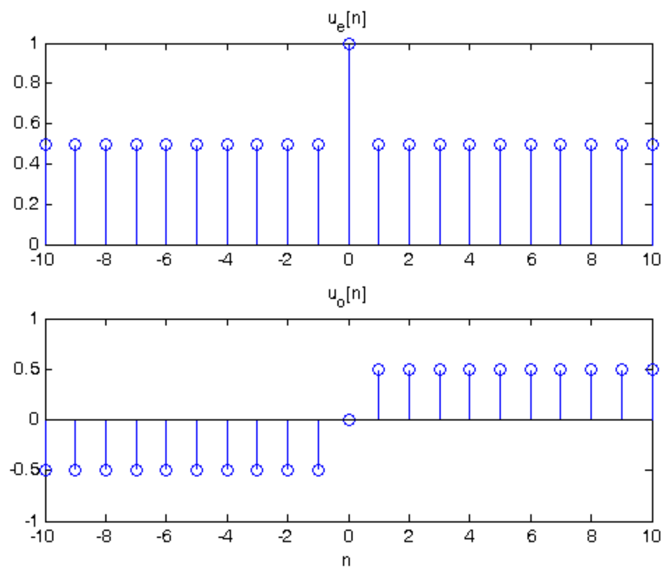
$$u[n]=\sum_{m=-\infty}^{n}\delta[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\Omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$$

$$\because u[n]+c-u[n-1]-c=\delta[n]$$

$$\therefore \mathcal{F}\{u[n]\}=\frac{1}{1-e^{-j\Omega}}+g\left(e^{j\Omega}\right)$$

where $g\left(e^{j\Omega}\right)$ accounts for the dc value of $u[n]$.

$$u[n]=\underbrace{\left(u[n]-\frac{1}{2}-\frac{1}{2}\delta[n]\right)}_{\text{odd part, } u_o[n]}+\underbrace{\frac{1}{2}+\frac{1}{2}\delta[n]}_{\text{even part, } u_e[n]}$$



$$\mathcal{F}\{u_o[n]\} = \mathcal{F}\{u[n]\} - \frac{1}{2}2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k) - \frac{1}{2}$$
$$= \frac{1}{1-e^{-j\Omega}} + g(e^{j\Omega}) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k) - \frac{1}{2}$$
$$= \left(\frac{1}{1-\cos\Omega + j\sin\Omega} - \frac{1}{2}\right) + g(e^{j\Omega}) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$$
$$= \left(\frac{1-\cos\Omega - j\sin\Omega}{2-2\cos\Omega} - \frac{1}{2}\right) + g(e^{j\Omega}) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$$
$$= \frac{-j\sin\Omega}{2-2\cos\Omega} + g(e^{j\Omega}) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$$

$\because \mathcal{F}\{u_o[n]\}$ is purely imaginary.

$\therefore g(e^{j\Omega}) = \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$

$\because \sum_{m=-\infty}^{n} x[m] = x[n] * u[n]$

$$\mathcal{F}\left\{\sum_{m=-\infty}^{n} x[m]\right\} = \mathcal{F}\{x[n]\}\cdot\mathcal{F}\{u[n]\} \quad \text{(convolution property)}$$
$$\therefore \quad = X(e^{j\Omega})\left[\frac{1}{1-e^{-j\Omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)\right]$$
$$= \frac{1}{1-e^{-j\Omega}}X(e^{j\Omega}) + \pi X(e^{j\cdot 0})\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$$

($X(e^{j\Omega})$ is periodic with period 2π.) ■

6. Time and frequency scaling

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\Omega})$$

(1) $x[-n] \xleftrightarrow{\mathcal{F}} X(e^{j(-\Omega)})$ (5.33)

$$\sum_{n=-\infty}^{\infty} x[-n]e^{-j\Omega n} = \sum_{m=-\infty}^{\infty} x[m]e^{j\Omega m} \qquad (m=-n)$$
$$= \sum_{m=-\infty}^{\infty} x[m]e^{-j(-\Omega)m}$$

(2) $x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}X\left(j\frac{\omega}{a}\right)$: continuous-time case

In the discrete-time case, the corresponding property is quite different. If a is an integer, $x[an]$ consists only of part of $x[n]$. What happens if a is not an integer?

Let k be a positive integer, and define

$$x_{(k)}[n]=\begin{cases} x[n/k] & \text{, if } n \text{ is a multiple of } k \\ 0 & \text{, if } n \text{ is not a multiple of } k \end{cases}$$

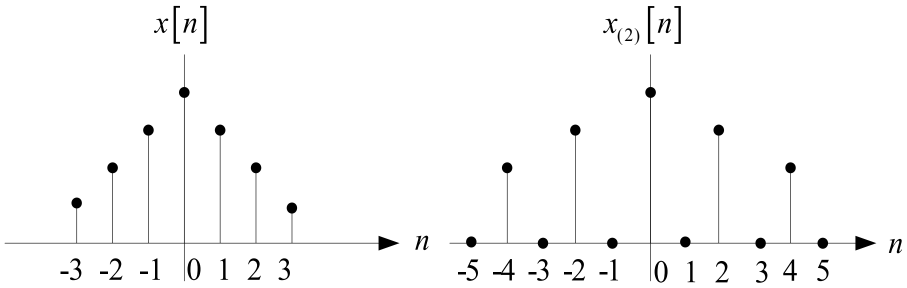


Figure 5.4 The signal $x_{(2)}[n]$ obtained from $x[n]$ by inserting one zero between successive values of the original signal.

$$\begin{aligned} X_{(k)}\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\Omega n} \\ &= \sum_{r=-\infty}^{\infty} x_{(k)}[rk]e^{-j\Omega rk} \qquad \left(x_{(k)}[n]\neq 0 \text{ when } n=rk\right) \\ &= \sum_{r=-\infty}^{\infty} x[r]e^{-j(k\Omega)r} = X\left(e^{j(k\Omega)}\right) \end{aligned}$$

$$x_{(k)}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \underbrace{X\left(e^{j(k\Omega)}\right)}_{\text{periodic with period } 2\pi/k} \tag{5.34}$$

7. Differentiation in frequency

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$\frac{dX\left(e^{j\Omega}\right)}{d\Omega} = -\sum_{n=-\infty}^{\infty} jnx[n]e^{-j\Omega n} \Rightarrow j\frac{dX\left(e^{j\Omega}\right)}{d\Omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\Omega n}$$

$$\Rightarrow nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{dX\left(e^{j\Omega}\right)}{d\Omega} \tag{5.35}$$

8. Parseval's relation

For aperiodic signals:

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right)$$

$$\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2=\frac{1}{2\pi}\int_{2\pi}\left|X\left(e^{j\Omega}\right)\right|^2 d\Omega \tag{5.36}$$

For periodic signals:

$$x[n]=\sum_{k=<N>}a_k e^{jk\frac{2\pi}{N}n}$$

$$\frac{1}{N}\sum_{n=<N>}\left|x[n]\right|^2=\sum_{k=<N>}\left|a_k\right|^2 \tag{5.37}$$

Proof:

(1) $\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2=\sum_{n=-\infty}^{\infty}x[n]x^*[n]=\sum_{n=-\infty}^{\infty}x[n]\frac{1}{2\pi}\int_{2\pi}X^*\left(e^{j\Omega}\right)e^{-j\Omega n}d\Omega$

$$=\frac{1}{2\pi}\int_{2\pi}X^*\left(e^{j\Omega}\right)\left(\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}\right)d\Omega$$

$$=\frac{1}{2\pi}\int_{2\pi}X^*\left(e^{j\Omega}\right)X\left(e^{j\Omega}\right)d\Omega=\frac{1}{2\pi}\int_{2\pi}\left|X\left(e^{j\Omega}\right)\right|^2 d\Omega$$

(2) $\frac{1}{N}\sum_{n=<N>}\left|x[n]\right|^2=\frac{1}{N}\sum_{n=<N>}x[n]x^*[n]$

$$=\frac{1}{N}\sum_{n=<N>}x[n]\sum_{k=<N>}a_k^* e^{-jk\frac{2\pi}{N}n}$$

$$=\sum_{k=<N>}a_k^*\left(\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk\frac{2\pi}{N}n}\right)=\sum_{k=<N>}a_k a_k^*$$

$$=\sum_{k=<N>}\left|a_k\right|^2$$

■

9. Convolution property

If $y[n]=x[n]*h[n]$, then

$$Y\left(e^{j\Omega}\right)=X\left(e^{j\Omega}\right)H\left(e^{j\Omega}\right) \tag{5.38}$$

where $X\left(e^{j\Omega}\right)=\mathcal{F}\left\{x[n]\right\}$, $H\left(e^{j\Omega}\right)=\mathcal{F}\left\{h[n]\right\}$, and $Y\left(e^{j\Omega}\right)=\mathcal{F}\left\{y[n]\right\}$.

Proof:

$$y[n]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

$$
\begin{aligned}
Y\left(e^{j\Omega}\right) &= \mathcal{F}\left\{y[n]\right\} = \sum_{n=-\infty}^{\infty} y[n] e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] h[n-m] e^{-j\Omega n} \\
&= \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} h[n-m] e^{-j\Omega n} \\
&= \sum_{m=-\infty}^{\infty} x[m] H\left(e^{j\Omega}\right) e^{-j\Omega m} = H\left(e^{j\Omega}\right) \sum_{m=-\infty}^{\infty} x[m] e^{-j\Omega m} \\
&= H\left(e^{j\Omega}\right) X\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right) H\left(e^{j\Omega}\right)
\end{aligned}
$$

(1) Periodic convolution

Consider the periodic convolution of two sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ which are periodic with the same period *N*. The periodic convolution of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ is defined as

$$
\begin{aligned}
\tilde{y}[n] &= \tilde{x}_1[n] \circledast \tilde{x}_2[n] \\
&= \sum_{m=<N>} \tilde{x}_1[m] \tilde{x}_2[n-m]
\end{aligned}
$$

where $\tilde{y}[n]$ is also periodic with period *N*.

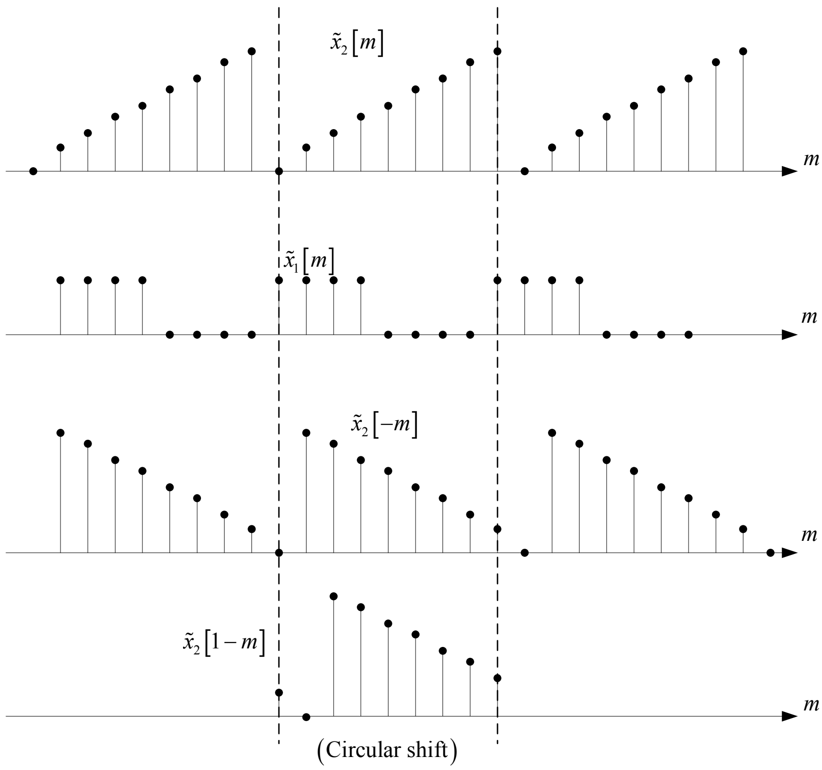


Figure 5.5 The procedure for computing the periodic convolution of two periodic sequences.

For periodic convolution, the counterpart of the convolution property can be expressed in terms of the Fourier series coefficients. Let

$$\tilde{x}_1[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} \qquad (\Omega_0 = 2\pi/N)$$

$$\tilde{x}_2[n] = \sum_{k=<N>} b_k e^{jk\Omega_0 n}$$

$$\tilde{y}[n] = \sum_{k=<N>} c_k e^{jk\Omega_0 n}.$$

Then

$$c_k = N a_k b_k \tag{5.39}$$

Proof:

$$\tilde{y}[n] = \sum_{m=<N>} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{n=<N>} \tilde{y}[n]e^{-jk\Omega_0 n} \\ &= \frac{1}{N}\sum_{n=<N>}\sum_{m=<N>} \tilde{x}_1[m]\tilde{x}_2[n-m]e^{-jk\Omega_0 n} \\ &= \sum_{m=<N>} \tilde{x}_1[m]\frac{1}{N}\sum_{n=<N>} \tilde{x}_2[n-m]e^{-jk\Omega_0 n} \\ &= \sum_{m=<N>} \tilde{x}_1[m]\frac{1}{N}\sum_{n'=<N>} \tilde{x}_2[n']e^{-jk\Omega_0 (n'+m)} \\ &= \sum_{m=<N>} \tilde{x}_1[m]e^{-jk\Omega_0 m} b_k = N a_k b_k \end{aligned}$$

■

(2) Let $x_1[n]$ and $x_2[n]$ be two finite-duration sequences, and suppose that

$x_1[n] = 0$, outside the interval $0 \le n \le N_1 - 1$

$x_2[n] = 0$, outside the interval $0 \le n \le N_2 - 1$

Let $y[n] = x_1[n] * x_2[n]$ (aperiodic convolution). Then we can find

$y[n] = 0$, outside the interval $0 \le n \le N_1 + N_2 - 2$.

Choose $N \ge N_1 + N_2 - 1$ and define signals $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ that are periodic with period N and such that

$$\tilde{x}_1[n] = x_1[n], \quad 0 \le n \le N-1$$

$$\tilde{x}_2[n] = x_2[n], \quad 0 \le n \le N-1.$$

Let $\tilde{y}[n]=\tilde{x}_1[n]\circledast\tilde{x}_2[n]$ (periodic convolution), then we obtain $y[n]=\tilde{y}[n]$, $0\le n\le N-1$.

$\Rightarrow$ The periodic convolution $\tilde{y}[n]$ equals the aperiodic convolution $y[n]$ over one period.

An algorithm for the calculation of the aperiodic convolution of $x_1[n]$ and $x_2[n]$:

(a) Calculate the DFTs $\tilde{X}_1(k)$ and $\tilde{X}_2(k)$ of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

(b) Multiply these DFTs together to obtain the DFT of $y[n]$:

$$\tilde{Y}(k)=\tilde{X}_1(k)\cdot\tilde{X}_2(k)$$

(c) Calculate the inverse DFT of $\tilde{Y}(k)$. The result is the desired convolution $\tilde{y}[n]$.

$$\begin{cases}\tilde{X}_1(k)=Na_k=\sum_{n=0}^{N-1}x_1[n]e^{-jk\frac{2\pi}{N}n},\ k=0,1,2,\ldots,N-1\\ \tilde{X}_2(k)=Nb_k=\sum_{n=0}^{N-1}x_2[n]e^{-jk\frac{2\pi}{N}n},\ k=0,1,2,\ldots,N-1\\ \tilde{Y}(k)=Nc_k=N^2a_kb_k=\tilde{X}_1(k)\cdot\tilde{X}_2(k),\ k=0,1,2,\ldots,N-1\\ \tilde{y}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{Y}(k)e^{jk\frac{2\pi}{N}n},\ n=0,1,2,\ldots,N-1\end{cases}$$

Example 5.6:

$$h[n]=\alpha^n u[n]\xleftrightarrow{\mathcal{F}}H(e^{j\Omega})=\frac{1}{1-\alpha e^{-j\Omega}}$$

$$x[n]=\beta^n u[n]\xleftrightarrow{\mathcal{F}}X(e^{j\Omega})=\frac{1}{1-\beta e^{-j\Omega}}$$

$$Y(e^{j\Omega})=H(e^{j\Omega})X(e^{j\Omega})=\frac{1}{(1-\alpha e^{-j\Omega})(1-\beta e^{-j\Omega})}$$

If $\alpha\neq\beta$,

$$Y(e^{j\Omega})=\frac{A}{1-\alpha e^{-j\Omega}}+\frac{B}{1-\beta e^{-j\Omega}}$$

$$A=\frac{\alpha}{\alpha-\beta},\ B=\frac{\beta}{\alpha-\beta}$$

$$\Rightarrow y[n]=\frac{\alpha}{\alpha-\beta}\alpha^n u[n]+\frac{\beta}{\alpha-\beta}\beta^n u[n]$$

If $\alpha = \beta$,

$$Y\left(e^{j\Omega}\right) = \left(\frac{1}{1-\alpha e^{-j\Omega}}\right)^2 = \frac{j}{\alpha}e^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$\alpha^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\Omega}}$$

$$n\alpha^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$(n+1)\alpha^{n+1}u[n+1] \stackrel{\mathcal{F}}{\longleftrightarrow} je^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$\left(\text{time shifting property, } x[n-n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\Omega n_0}X\left(e^{j\Omega}\right)\right)$$

$$\begin{aligned} y[n] &= \frac{1}{\alpha}(n+1)\alpha^{n+1}u[n+1] \\ &= (n+1)\alpha^n u[n+1] \\ &= (n+1)\alpha^n u[n] \ \ (\because n=-1,\ n+1=0) \end{aligned}$$

■

Example 5.7:

$$\text{Let } x_1[n] = x_2[n] = \begin{cases} 1, 0 \le n \le N-1 \\ 0, \ \text{otherwise} \end{cases}.$$

(i) Find $\tilde{y}_1[n] = \tilde{x}_1[n] \circledast \tilde{x}_2[n]$ via DFT: $\tilde{x}_1[n] = \tilde{x}_2[n]$ is periodic with period *N*. $\tilde{x}_1[n]$ is equal to $\tilde{x}_2[n]$ for $0 \le n \le N-1$.

$$\tilde{X}_1(k) = \tilde{X}_2(k) = \sum_{n=0}^{N-1} e^{-jk\frac{2\pi}{N}n} = \begin{cases} N, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{Y}_1(k) = \tilde{X}_1(k)\tilde{X}_2(k) = \begin{cases} N^2, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{y}_1[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{Y}_1(k)e^{jk\frac{2\pi}{N}n} = N,\ 0 \le n \le N-1$$

(ii) Find $y_2[n] = x_1[n] * x_2[n]$ via DFT: Since $2N > (N+N-1)$, we use 2*N*-point DFT and IDFT for calculating $y_2[n]$ as follows:

$$\tilde{X}_1(k) = \tilde{X}_2(k) = \sum_{n=0}^{2N-1} e^{-jk\frac{2\pi}{2N}n},\ k = 0,1,2,\ldots,2N-1$$

$$\tilde{Y}_2(k) = \tilde{X}_1(k)\tilde{X}_2(k),\ k = 0,1,2,\ldots,2N-1$$

$$y_2[n] = \frac{1}{2N}\sum_{k=0}^{2N-1}\tilde{Y}_2(k)e^{jk\frac{2\pi}{2N}n},\ 0 \le n \le 2N-1$$

■

10. Modulation property

$$y[n] = x_1[n]x_2[n]$$

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1\left(e^{j\Omega}\right)$$

$$x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2\left(e^{j\Omega}\right)$$

$$y[n] \overset{\mathcal{F}}{\longleftrightarrow} Y\left(e^{j\Omega}\right)$$

$$\Rightarrow Y\left(e^{j\Omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right) X_2\left(e^{j(\Omega-\theta)}\right) d\theta = \frac{1}{2\pi} X_1\left(e^{j\Omega}\right) \circledast X_2 \qquad (5.40)$$

Proof:

$$Y\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\Omega n}$$

$$\because x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)e^{j\theta n}d\theta$$

$$\therefore Y\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)e^{j\theta n}d\theta\right\}e^{-j\Omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)\left(\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\Omega-\theta)n}\right)d\theta$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)X_2\left(e^{j(\Omega-\theta)}\right)d\theta$$

■

11. Duality between the discrete-time Fourier transform and the continuous-time Fourier series

(1) The Fourier series representation of a periodic continuous-time signal with fundamental period T_0:

$$\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \text{ fundamental frequency } \omega_0 = \dfrac{2\pi}{T_0} \\ a_k = \dfrac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt \end{cases}$$

(2) The Fourier transform pair of an aperiodic discrete-time signal:

$$\begin{cases} x[n] = \dfrac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega \\ X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \quad \text{(with fundamental period } 2\pi) \end{cases}.$$

$$\Rightarrow \begin{cases} x[-n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega \\ X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[-n]e^{j\Omega n} \quad \text{(with fundamental period } 2\pi\text{)} \end{cases} \tag{5.41}$$

Note that $X\left(e^{j\Omega}\right)$ is a periodic function of a continuous variable Ω (rather than *t*) with fundamental period 2π (corresponding to T_0). So the discrete-time Fourier transform expression can be regarded as a Fourier series representation with $\omega_0 = 1$ (corresponding to $2\pi / T_0$) and Fourier series coefficients $x[-n]$.

(3) The Fourier series coefficients of $X\left(e^{j\Omega}\right)$ is the original sequence $x[n]$ reversed in order.

$$x[n] \xleftrightarrow{\text{Fourier Transform}} X\left(e^{j\Omega}\right) \xleftrightarrow{\text{Fourier Series}} x[-k] \tag{5.42}$$

5-4 The Frequency Response of Systems Characterized by Linear Constant–Coefficient Difference Equations

1. Calculation of the frequency and impulse responses

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Assume that the Fourier transforms of $x[n]$, $y[n]$, and the system impulse response $h[n]$ all exist.

$$x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j\Omega}\right)$$

$$y[n] \xleftrightarrow{\mathcal{F}} Y\left(e^{j\Omega}\right)$$

$$h[n] \xleftrightarrow{\mathcal{F}} H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)}$$

$$\Rightarrow \sum_{k=0}^{N} a_k e^{-jk\Omega} Y\left(e^{j\Omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\Omega} X\left(e^{j\Omega}\right)$$

$$\Rightarrow H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\Omega}}{\sum_{k=0}^{N} a_k e^{-jk\Omega}}$$

Example 5.8: $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$

$$Y\left(e^{j\Omega}\right) - \frac{3}{4}e^{-j\Omega}Y\left(e^{j\Omega}\right) + \frac{1}{8}e^{-j2\Omega}Y\left(e^{j\Omega}\right) = 2X\left(e^{j\Omega}\right)$$

$$Y\left(e^{j\Omega}\right)\left(1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}\right) = 2X\left(e^{j\Omega}\right)$$

$$\Rightarrow H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}}$$

$$=\frac{2}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}=\frac{4}{1-\frac{1}{2}e^{-j\Omega}}-\frac{2}{1-\frac{1}{4}e^{-j\Omega}}$$

$$\Rightarrow h[n]=4\left(\frac{1}{2}\right)^n u[n]-2\left(\frac{1}{4}\right)^n u[n]$$

Note:

- $H\left(e^{j\Omega}\right)=\frac{1}{1-ae^{-j\Omega}}=1+ae^{-j\Omega}+a^2e^{-j2\Omega}+\cdots,\ |a|<1$

  $h[n]=\delta[n]+a\delta[n-1]+a^2\delta[n-2]+\cdots=a^nu[n]$

- $a^nu[n]\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{1-ae^{-j\Omega}},\ |a|<1$

■

Example 5.9: $H\left(e^{j\Omega}\right)=\frac{2}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}$

$$x[n]=\left(\frac{1}{4}\right)^n u[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j\Omega}\right)=\frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

$$Y\left(e^{j\Omega}\right)=H\left(e^{j\Omega}\right)X\left(e^{j\Omega}\right)=\frac{2}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2\left(1-\frac{1}{2}e^{-j\Omega}\right)}$$

$$=\frac{B_{11}}{\left(1-\frac{1}{4}e^{-j\Omega}\right)}+\frac{B_{12}}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2}+\frac{B_{21}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)}$$

$$Y(v)=\frac{2}{\left(1-\frac{1}{4}v\right)^2\left(1-\frac{1}{2}v\right)},\ v_1=4,\ v_2=2$$

$$B_{12}=\left(1-\frac{1}{4}v\right)^2Y(v)\bigg|_{v=v_1=4}=\frac{2}{1-\frac{1}{2}v}\bigg|_{v=v_1=4}=-2$$

$$B_{11}=-v_1\frac{d}{dv}\left[\left(1-\frac{1}{4}v\right)^2Y(v)\right]\bigg|_{v=v_1=4}=-4\frac{d}{dv}\left(\frac{2}{1-\frac{1}{2}v}\right)\bigg|_{v=4}=4\frac{2\cdot\frac{1}{2}}{\left(1-\frac{1}{2}v\right)^2}\bigg|_{v=4}=-4$$

$$\left(\left(1-\frac{1}{4}v\right)^2 Y(v) = B_{11}\left(1-\frac{1}{4}v\right) \xrightarrow{\frac{d}{dv}} \frac{d}{dv}\left(1-\frac{1}{4}v\right)^2 Y(v) = -\frac{1}{4}B_{11} = -\frac{1}{v_1}B_{11}\right)$$

$$B_{21} = \left(1-\frac{1}{2}v\right)Y(v)\Big|_{v=v_2=2} = \frac{2}{\left(1-\frac{1}{4}v\right)^2}\Bigg|_{v=2} = 8$$

$$\Rightarrow Y\left(e^{j\Omega}\right) = \frac{-4}{\left(1-\frac{1}{4}e^{-j\Omega}\right)} + \frac{-2}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2} + \frac{8}{\left(1-\frac{1}{2}e^{-j\Omega}\right)}$$

$$\Rightarrow y[n] = -4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n] + 8\left(\frac{1}{2}\right)^n u[n]$$

Note:

- $Z\left(e^{j\Omega}\right) = \frac{1}{\left(1-\alpha e^{-j\Omega}\right)^2} = \frac{j}{\alpha}e^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$

$$\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\Omega}}$$

$$n\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$(n+1)\alpha^{(n+1)}u[n+1] \overset{\mathcal{F}}{\longleftrightarrow} je^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$(n+1)\alpha^n u[n+1] \overset{\mathcal{F}}{\longleftrightarrow} \frac{j}{\alpha}e^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right) = \frac{1}{\left(1-\alpha e^{-j\Omega}\right)^2}$$

$$\Rightarrow (n+1)\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{\left(1-\alpha e^{-j\Omega}\right)^2} \quad \left(\because (n+1)\alpha^n u[n+1] = 0 \text{ when } n=-1\right)$$

■

2. Cascade- and parallel-form structures

(1) Cascade-form structure

$$H\left(e^{j\Omega}\right) = \frac{b_0\prod_{k=1}^{N}\left(1+\mu_k e^{-j\Omega}\right)}{a_0\prod_{k=1}^{M}\left(1+\eta_k e^{-j\Omega}\right)} \tag{5.43}$$

where μ_k and η_k may be complex, and then appear in complex-conjugate pairs. For simplicity, we assume $M = N$. Multiplying out $\left(1+\mu_k e^{-j\Omega}\right)\left(1+\mu_k^* e^{-j\Omega}\right)$ and $\left(1+\eta_k e^{-j\Omega}\right)\left(1+\eta_k^* e^{-j\Omega}\right)$, we obtain

$$1+\left(\mu_k+\mu_k^*\right)e^{-j\Omega}+\left|\mu_k\right|^2 e^{-j2\Omega}=1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-j2\Omega} \tag{5.44}$$

and

$$1+\left(\eta_k+\eta_k^*\right)e^{-j\Omega}+\left|\eta_k\right|^2 e^{-j2\Omega}=1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega} \tag{5.45}$$

Thus, we have

$$H\left(e^{j\Omega}\right)=\frac{b_0}{a_0}\cdot\frac{\prod_{k=1}^{P}\left(1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-j2\Omega}\right)\prod_{k=1}^{N-2P}\left(1+\mu_k e^{-j\Omega}\right)}{\prod_{k=1}^{Q}\left(1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}\right)\prod_{k=1}^{N-2Q}\left(1+\eta_k e^{-j\Omega}\right)} \tag{5.46}$$

where all the coefficients are real.

Note:

- The frequency response of any LTI system described by a linear constant coefficient difference equation can be written as a product of first- and second-order terms.
- The LTI system can be realized as a cascade of first- and second-order LTI subsystems.

(a) Realization of a second-order LTI subsystem

$$H_2\left(e^{j\Omega}\right)=\frac{1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-j2\Omega}}{1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}}=\frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} \tag{5.47}$$

$$Y\left(e^{j\Omega}\right)\left[1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}\right]=X\left(e^{j\Omega}\right)\left[1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-j2\Omega}\right] \tag{5.48}$$

$$y[n]+\alpha_{1k}y[n-1]+\alpha_{2k}y[n-2]=x[n]+\beta_{1k}x[n-1]+\beta_{2k}x[n-2] \tag{5.49}$$

$$y[n]=-\alpha_{1k}y[n-1]-\alpha_{2k}y[n-2]+\underbrace{x[n]+\beta_{1k}x[n-1]+\beta_{2k}x[n-2]}_{w[n]} \tag{5.50}$$

This difference equation can be realized using the direct form II structure as follows:

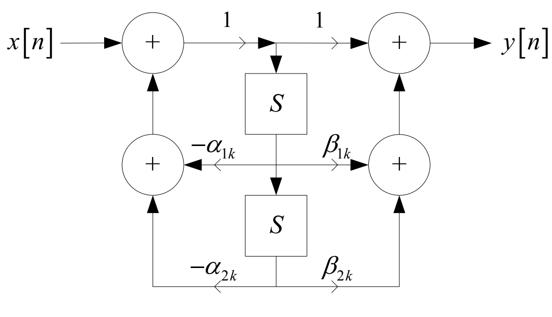


Figure 5.6 Direct form II realization of a second-order LTI subsystem.

(b) The first-order subsystems can also be realized using the second-order structure with β_{2k} and α_{2k} equal to zero.

(2) Parallel-form structure

$$H\left(e^{j\Omega}\right)=\frac{b_N}{a_N}+\sum_{k=1}^{N}\frac{A_k}{1+\eta_k e^{-j\Omega}} \tag{5.51}$$

Adding the pairs involving complex conjugate η_k's, we obtain

$$H\left(e^{j\Omega}\right)=\frac{b_N}{a_N}+\sum_{k=1}^{Q}\frac{r_{0k}+r_{1k}e^{-j\Omega}}{1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}}+\sum_{k=1}^{N-2Q}\frac{A_k}{1+\eta_k e^{-j\Omega}} \tag{5.52}$$

where all the coefficients are real. According to this equation, we can realize the LTI system using a parallel interconnection of first- and second-order LTI subsystems. Each second-order subsystem can be realized using the direct form II structure as follows:

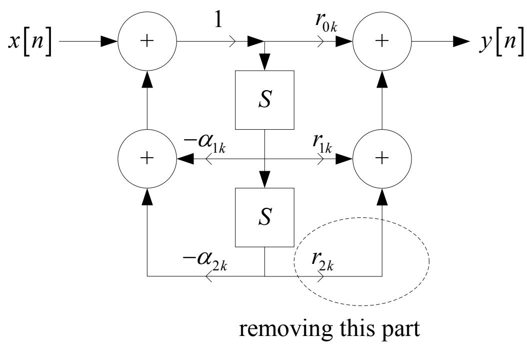


Figure 5.7 Direct form II realization of each second-order LTI subsystem in (5.52).

References

[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, Signals and Systems, 2nd Ed., Pearson Education Limited, 2014 (or Prentice-Hall, 1997).