Chapter 4 The Continuous-Time Fourier Transform

4-1 The Fourier Transform of Aperiodic Continuous-Time Signals

1. Development of the Fourier Transform
   - Periodic signals $\rightarrow$ Fourier series
   - Aperiodic signals $\rightarrow$ Fourier transform

   Consider a periodic square wave shown in Fig. 4.1.

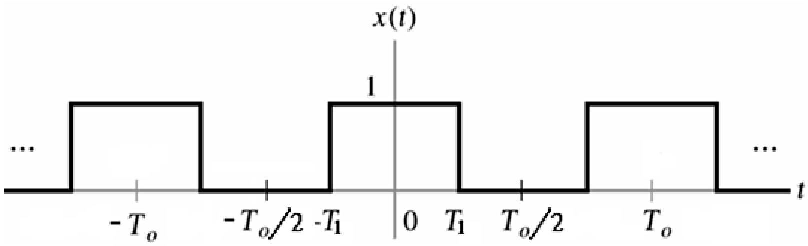


Figure 4.1 A periodic square wave.

The corresponding Fourier series representation can be expressed as follows:

$$
\begin{aligned}
&x(t)=\sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t} \qquad \omega_0 = 2\pi/T_0 \\
&a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T_0} \\
&T_0 a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega=k\omega_0} = 2T_1\cdot\left.\frac{\sin\omega T_1}{\omega T_1}\right|_{\omega=k\omega_0} = 2T_1\cdot \text{sinc}\left.\left(\frac{\omega T_1}{\pi}\right)\right|_{\omega=k\omega_0}
\end{aligned}
\tag{4.1}
$$

$\Rightarrow$ The function $\left(2\sin(\omega T_1)/\omega\right)$ represents the envelope of $T_0 a_k$, i.e., $T_0 a_k$ is a sampled value of $\left(2\sin(\omega T_1)/\omega\right)$, as shown in Fig. 4.2.

$\because$ The sampling interval is ω_0

$\therefore T_0 \uparrow \Rightarrow \omega_0 \downarrow \Rightarrow$ sampling spacing $\downarrow \Rightarrow$ Fourier series coefficients approach the envelope function

$T_0 \rightarrow \infty \Rightarrow x(t)$ is a rectangular pulse (aperiodic)

Note:

- We can think of an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large.
- Consider a general aperiodic signal $x(t)$ that is of finite duration. From this aperiodic signal, we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is of one period, as shown in Fig. 4.3.

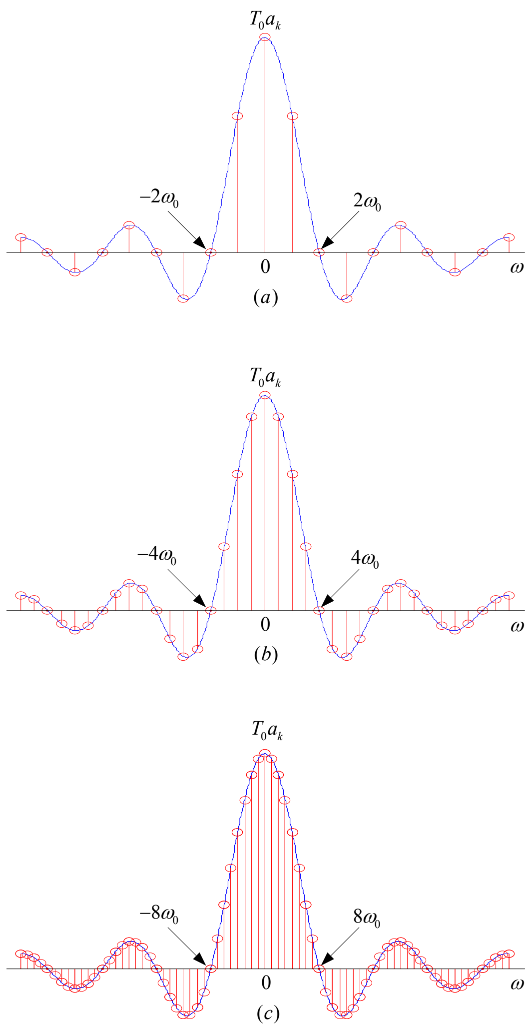


Figure 4.2 The Fourier coefficients and their envelope of the periodic square wave in Fig. 4.1 for several values of T_0 (with T_1 fixed): (a) $T_0 = 4T_1$; (b) $T_0 = 8T_1$; (c) $T_0 = 16T_1$.

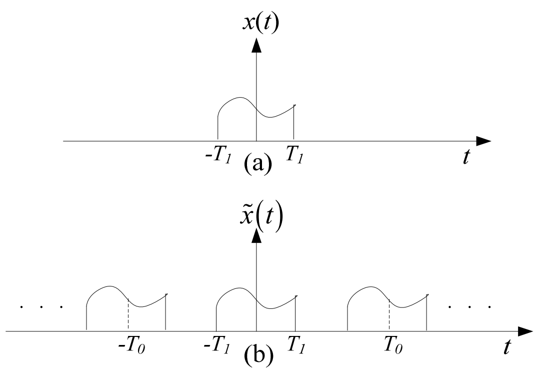


Figure 4.3 (a) An aperiodic signal $x(t)$; (b) a periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

As $T_0 \to \infty$, $\tilde{x}(t) \to x(t)$

The Fourier series representation of $\tilde{x}(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \tilde{x}(t)\cdot e^{-jk\omega_0 t}dt \qquad (4.2)$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cdot e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\cdot e^{-jk\omega_0 t}dt.$$

Defining the envelope of $T_0 a_k$ as $X(j\omega)$, we have

$$T_0 a_k = X(j\omega)\Big|_{\omega=k\omega_0} = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}dt\Big|_{\omega=k\omega_0} \qquad (4.3)$$

$$a_k = \frac{1}{T_0}X(j\omega)\Big|_{\omega=k\omega_0} \qquad (4.4)$$

$$\Rightarrow \tilde{x}(t) = \sum_{k=-\infty}^{\infty}\frac{1}{T_0}X(jk\omega_0)e^{jk\omega_0 t} \qquad (4.5)$$

$$\because T_0 = \frac{2\pi}{\omega_0} \qquad \therefore \frac{1}{T_0} = \frac{1}{2\pi}\omega_0 \qquad (4.6)$$

$$\Rightarrow \tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \qquad (4.7)$$

As $T_0 \to \infty$, $\tilde{x}(t) \to x(t)$, and the above equation becomes a representation of $x(t)$.

As $T_0 \to \infty$, $\omega_0 \to 0$, $\sum \to \int$, and $\omega_0 \to d\omega$. Accordingly, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad (4.8)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad (4.9)$$

This is called the Fourier transform pair. $X(j\omega)$ is the Fourier transform or Fourier integral of $x(t)$, and $x(t)$ is the inverse Fourier transform of $X(j\omega)$. (4.8) is referred to as a synthesis equation, and (4.9) is referred to as an analysis equation.

2. Convergence of the continuous-time Fourier transform

(1) If $x(t)$ is *square-integrable*, i.e., if

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

then the Fourier transform $X(j\omega)$ converges in the sense that the total error energy E of the difference of $x(t)$ and $\hat{x}(t)$ synthesized by (4.8) is zero.

(2) The following three Dirichlet conditions ensure that the Fourier transform of $x(t)$ converges in the sense of $E=0$, where $\hat{x}(t)$ is equal to $x(t)$ for any t except at a discontinuous point for which it is equal to the average value of the discontinuity.

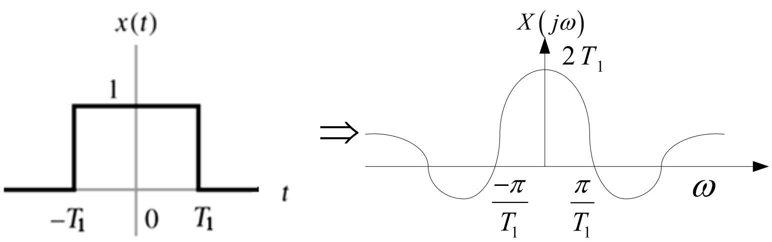
- $x(t)$ is *absolutely integrable*, i.e., $\int_{-\infty}^{+\infty} |x(t)| dt < \infty$.
- $x(t)$ has a finite number of maxima and minima within any finite interval.
- $x(t)$ has a finite number of finite discontinuities within any finite interval.

3. Examples of the continuous-time Fourier transform

Example 4.1: The Fourier transform of a rectangular pulse

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = 2 \cdot \frac{\sin \omega T_1}{\omega} = 2T_1 \cdot \frac{\sin \omega T_1}{\omega T_1} = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi}\right) \quad (4.10)$$

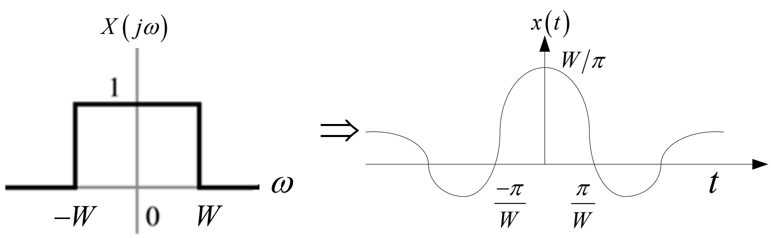


■

Example 4.2: The inverse Fourier transform of a rectangular pulse

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} d\omega = \frac{\sin Wt}{\omega \pi} = \frac{W}{\pi} \frac{\sin Wt}{Wt} = \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi}\right) \quad (4.11)$$



Note: Broader in the time domain → narrower in the frequency domain
Narrower in the time domain → narrower in the frequency domain ■

4-2 Periodic Signals and the Continuous-Time Fourier Transform

1. Fourier series coefficients as samples of the Fourier transform of one period

Consider a periodic signal $\tilde{x}(t)$ with fundamental period T_0 and the following Fourier series representation:

$$\tilde{x}(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\ ;\ \ a_k=\frac{1}{T_0}\int_{T_0}\tilde{x}(t)\cdot e^{-jk\omega_0 t}dt\ .$$

Let $x(t)$ be an aperiodic signal formed by one period of $\tilde{x}(t)$ as follows:

$$x(t)=\begin{cases}\tilde{x}(t) & , \quad -\frac{T_0}{2}\le t\le\frac{T_0}{2}\\ 0 & , \quad \text{otherwise.}\end{cases}$$

Then the Fourier transform of $x(t)$ is

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$$\Rightarrow a_k=\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\tilde{x}(t)\cdot e^{-jk\omega_0 t}dt$$

$$=\frac{1}{T_0}X(jk\omega_0)=\frac{1}{T_0}\int_{-\infty}^{\infty}x(t)\cdot e^{-jk\omega_0 t}dt$$

General statement:

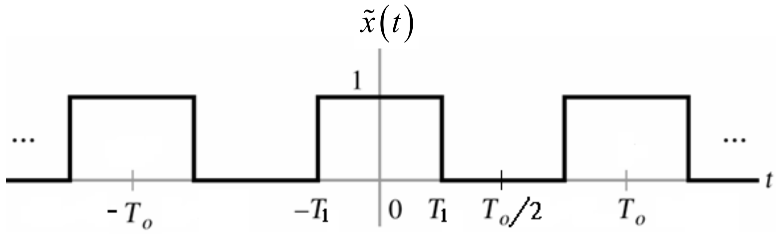
Let $x(t)=\begin{cases}\tilde{x}(t) & , \quad s\le t\le s+T_0\\ 0 & , \quad \text{otherwise.}\end{cases}$

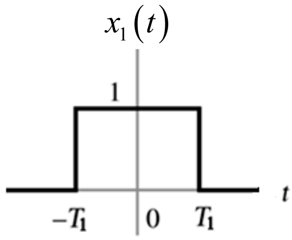
Then the Fourier series coefficients of $\tilde{x}(t)$ are given by

$$a_k=\frac{1}{T_0}X(jk\omega_0)$$

where $X(j\omega)$ is the Fourier transform of $x(t)$.

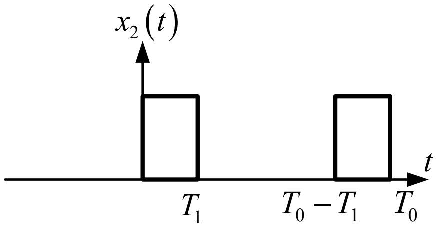
Example 4.3: Fourier transforms of different intervals for a periodic signal.





$$\Rightarrow X_1(j\omega) \underset{\text{sampling period}=\omega_o=2\pi/T_0}{\Rightarrow} \text{Fourier series coefficients of } \tilde{x}(t)$$

$$X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$



$$\Rightarrow X_2(j\omega) \underset{\text{sampling period}=\omega_o=2\pi/T_0}{\Rightarrow} \text{Fourier series coefficients of } \tilde{x}(t)$$

$$\begin{aligned} X_2(j\omega) &= \int_0^{T_1} e^{-j\omega t} dt + \int_{T_0-T_1}^{T_0} e^{-j\omega t} dt \\ &= \frac{2}{\omega}\sin\left(\frac{\omega T_1}{2}\right)\left[e^{-j\omega T_1/2} + e^{-j\omega(T_0-T_1)/2}\right] \end{aligned}$$

$X_1(j\omega) \neq X_2(j\omega)$, but $X_1(jk\omega_0) = X_2(jk\omega_0) = 2\sin(k\omega_0 T_1)/k\omega_0$. --- Only some sample points are the same.

Note: The Fourier coefficients of a periodic signal can be obtained from samples of the Fourier transform of an aperiodic signal that equals the original periodic signal over any arbitrary interval of length T_0 and that is zero outside this interval. ■

2. The Fourier transform for periodic signals

Consider a signal $x(t)$ with Fourier transform $X(j\omega) = 2\pi\delta(\omega - \omega_0)$.

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

If $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$, then $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.

This is corresponding to the Fourier series representation of a periodic signal.

Note: If impulses are allowed in the continuous-time Fourier transform, we can define the following Fourier transform pairs:

$$e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0) \tag{4.12}$$

$$e^{j\cdot 0\cdot t} = 1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega) \tag{4.13}$$

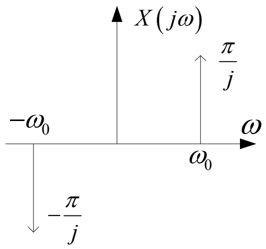
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \tag{4.14}$$

Periodic signal → Fourier series representation
→ Fourier transform

Example 4.4: The Fourier transform of a sinusoidal signal.

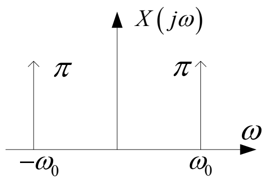
$$x(t) = \sin\omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$\Rightarrow \begin{aligned} X(j\omega) &= \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right] \\ &= \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0) \end{aligned}$$



$$x(t) = \cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\Rightarrow X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$



■

Example 4.5: The Fourier transform of a periodic impulse-train signal.

$$\begin{cases} x(t) = \sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},\ \omega_0 = \frac{2\pi}{T} \\ a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T} \end{cases} \tag{4.15}$$

$$\Rightarrow \begin{cases} x(t) = \sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\omega_0 t} \\ X(j\omega) = \sum_{k=-\infty}^{\infty}\frac{1}{T}2\pi\delta(\omega-k\omega_0) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right) \end{cases} \tag{4.16}$$

Note: Impulse train in the time domain $\stackrel{\mathcal{F}}{\longleftrightarrow}$ Impulse train in the frequency domain.■

4-3 Properties of the Continuous-Time Fourier Transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$
$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

Notations:

$$x(t)\stackrel{\mathcal{F}}{\longleftrightarrow}X(j\omega)$$

$$X(j\omega)=\mathcal{F}\{x(t)\}$$

$$x(t)=\mathcal{F}^{-1}\{X(j\omega)\}$$

1. Linearity

$$x_1(t)\stackrel{\mathcal{F}}{\longleftrightarrow}X_1(j\omega)$$
$$x_2(t)\stackrel{\mathcal{F}}{\longleftrightarrow}X_2(j\omega)$$

$$\Rightarrow ax_1(t)+bx_2(t)\stackrel{\mathcal{F}}{\longleftrightarrow}aX_1(j\omega)+bX_2(j\omega) \tag{4.17}$$

2. Symmetry Properties

If $x(t)$ is a real-valued function, then

$$X(-j\omega)=X^*(j\omega) \qquad * : \text{complex conjugate} \tag{4.18}$$

Proof:

$$X^*(j\omega)=\left[\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right]^*=\int_{-\infty}^{\infty}x^*(t)e^{j\omega t}dt$$
$$=\int_{-\infty}^{\infty}x(t)e^{j\omega t}dt=X(-j\omega)$$

Note:

- $X(j\omega)=\mathrm{Re}\{X(j\omega)\}+j\,\mathrm{Im}\{X(j\omega)\}$

  If $x(t)$ is real, then

$$\begin{aligned}\mathrm{Re}\{X(j\omega)\}&=\mathrm{Re}\{X(-j\omega)\}\cdots\cdots\text{even function}\\ \mathrm{Im}\{X(j\omega)\}&=-\mathrm{Im}\{X(-j\omega)\}\cdots\cdots\text{odd function}\end{aligned} \tag{4.19}$$

- $X(j\omega) = |X(j\omega)| e^{j\theta(j\omega)}$ …… polar form

  If $x(t)$ is real, then

$$\begin{aligned} |X(j\omega)| &= |X(-j\omega)| \quad \cdots\cdots \text{ even function} \\ \theta(j\omega) &= -\theta(-j\omega) \quad \cdots\cdots \text{ odd function} \end{aligned} \tag{4.20}$$

- If $x(t)$ is both real and even, then $X(j\omega)$ is also both real and even.

$$\begin{aligned} X(-j\omega) &= \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(-\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \qquad (\because x(t) = x(-t)) \\ &= X(j\omega) \quad \cdots\cdots \text{ even} \end{aligned} \tag{4.21}$$

  The symmetry property in (4.18) $\Rightarrow X^*(j\omega) = X(-j\omega) = X(j\omega)$ …… real

- If $x(t)$ is both real and odd, then $X(j\omega)$ is both pure imaginary and odd.

- A real function $x(t)$ can always be expressed as

$$x(t) = \underset{\text{even part}}{x_e(t)} + \underset{\text{odd part}}{x_o(t)} \tag{4.22}$$

$$\mathcal{F}\{x(t)\} = \underbrace{\mathcal{F}\{x_e(t)\}}_{\mathrm{Re}\{X(j\omega)\}} + \underbrace{\mathcal{F}\{x_o(t)\}}_{j\,\mathrm{Im}\{X(j\omega)\}} \tag{4.23}$$

3. Time Shifting

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega) \tag{4.24}$$

Proof:

$$\begin{aligned} \mathcal{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(\sigma) e^{-j\omega(\sigma+t_0)} d\sigma \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\sigma) e^{-j\omega\sigma} d\sigma \\ &= e^{-j\omega t_0} X(j\omega) \end{aligned}$$

Time shifting only introduces a phase shift but leaves the magnitude unchanged.

4. Differentiation and Integration

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

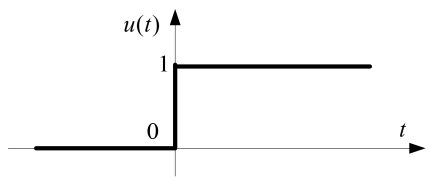
$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) \tag{4.25}$$

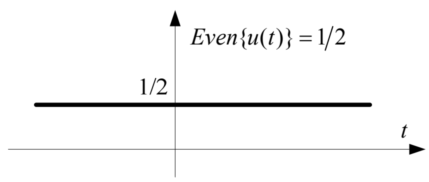
$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega) \tag{4.26}$$

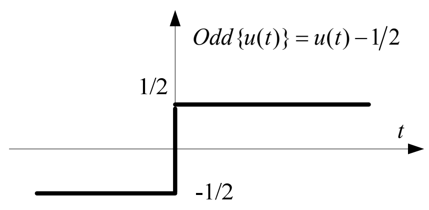
$X(0)$: reflects the dc or average value resulting from the integration.

$$\begin{cases} x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ \dfrac{dx(t)}{dt} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega \end{cases}$$

Example 4.6: Determine the Fourier transform of the unit step function $u(t)$.







$$u(t) = \frac{1}{2} + \underbrace{\left[u(t) - \frac{1}{2}\right]}_{v(t)}$$

$\because v'(t) = u'(t) = \delta(t)$

$\therefore \mathcal{F}\{\delta(t)\} = \mathcal{F}\left\{\frac{dv(t)}{dt}\right\} = j\omega V(j\omega)$

$\because \mathcal{F}\{\delta(t)\} = 1 \qquad \therefore V(j\omega) = \frac{1}{j\omega}$

$Even\{u(t)\} = \frac{1}{2} \Rightarrow \mathcal{F}\left\{\frac{1}{2}\right\} = \pi\delta(\omega)$

$\Rightarrow \boxed{\mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega)}$ --- agrees with the integration property in (4.26).

Note:

- $\delta(t) = \frac{du(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1 \qquad (\because \omega\pi\delta(\omega) = 0)$ ■

5. Time and Frequency Scaling

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right) \tag{4.27}$$

Proof:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \qquad (\tau = at)$$

$$= \begin{cases} \frac{1}{-a}\int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau & (a < 0) \\ \frac{1}{a}\int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau & (a > 0) \end{cases}$$

6. Duality

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(j\omega) = f(\omega) \qquad f(u) = \int_{-\infty}^{\infty} g(v) e^{-juv} dv \tag{4.28}$$

$$f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(j\omega) = 2\pi g(-\omega) \qquad g(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u) e^{juv} du \tag{4.29}$$

Proof:

$$\begin{cases} f(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \\ f(\omega) = \mathcal{F}\{g(t)\} \end{cases}$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = 2\pi \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{j(-\omega)t} dt \right]$$

From (4.29), we have $\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{j(-\omega)t} dt = g(-\omega)$

$$\therefore F(j\omega) = 2\pi g(-\omega) \quad \Rightarrow f(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi g(-\omega)$$

Example 4.7: Compare the relationship between a rectangular pulse and a sinc function with the duality property.

rectangular in the time domain $\rightarrow$ sinc in the frequency domain
sinc in the time domain $\rightarrow$ rectangular in the frequency domain

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} X_1(j\omega) = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi} \right)$$

$$x_2(t) = \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi} \right) \overset{\mathcal{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

Letting $W = T_1$, we can see that the rectangular pulse and the sinc function satisfy the duality property. ■

Example 4.8: Calculate the Fourier transform of $x(t)$ given below using the duality property.

$$x(t) = \frac{2}{t^2 + 1}$$

Let $f(u) = \frac{2}{u^2 + 1}$

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} f(\omega) = \frac{2}{\omega^2 + 1}$$

$$g(t) = e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} f(\omega) = \frac{2}{\omega^2 + 1}$$

$$x(t) = f(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi g(-\omega) = 2\pi e^{-|\omega|}$$

$$\therefore \mathcal{F}\{x(t)\} = 2\pi e^{-|\omega|}$$ ■

Note: Other Duality Properties

- $-jtx(t) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{dX(j\omega)}{d\omega}$ $\qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ (4.30)

  $\dfrac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$ $\qquad \dfrac{dX(j\omega)}{d\omega} = \int_{-\infty}^{\infty} -jtx(t)e^{-j\omega t}dt$ (4.31)

- $e^{j\omega_0 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega-\omega_0))$ (4.32)

  $x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega)$ (4.33)

- $-\dfrac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} X(\eta)d\eta$ (4.34)

  $\int_{-\infty}^{t} x(t)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ (4.35)

7. Parseval's Relation

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \tag{4.36}$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\underbrace{\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]}_{X(j\omega)}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega
\end{aligned}$$

Note:

- The total energy in the signal $x(t)$ may be determined either by computing the energy per unit time and integrating over all time or by computing the energy per unit frequency and integrating over all frequencies.
- For periodic signals,

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 . \tag{4.37}$$

8. Convolution Property

$$y(t)=h(t)*x(t)\overset{\mathcal{F}}{\longleftrightarrow}Y(j\omega)=H(j\omega)X(j\omega) \tag{4.38}$$

Proof:

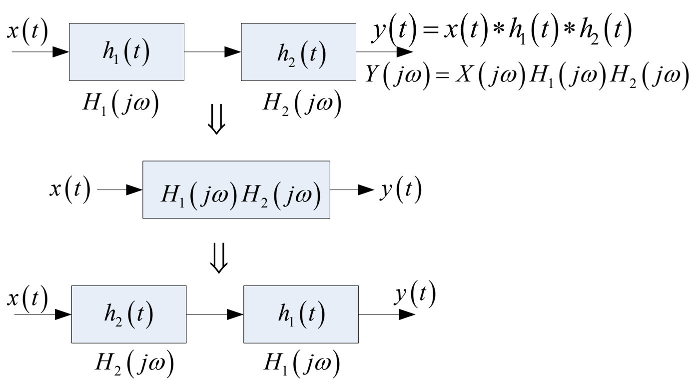
$$\begin{aligned}
y(t)&=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau \\
Y(j\omega)=\mathcal{F}\{y(t)\}&=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt \\
&=\int_{-\infty}^{\infty}x(\tau)\left[\int_{-\infty}^{\infty}h(t-\tau)e^{-j\omega t}dt\right]d\tau \\
&=\int_{-\infty}^{\infty}x(\tau)\cdot e^{-j\omega\tau}H(j\omega)d\tau \\
&=H(j\omega)\int_{-\infty}^{\infty}x(\tau)e^{-j\omega\tau}d\tau \\
&=H(j\omega)X(j\omega)
\end{aligned}$$

- $x(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega=\lim_{\omega_0\to 0}\dfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(jk\omega_0)\underbrace{e^{jk\omega_0 t}}_{\text{eigenfunction}}\omega_0$

$$\underbrace{H(jk\omega_0)}_{\text{eigenvalue}}=\int_{-\infty}^{\infty}h(t)e^{-jk\omega_0 t}dt$$

$$\begin{aligned}
\Rightarrow y(t)&=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}H(jk\omega_0)X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)X(j\omega)e^{j\omega t}d\omega \\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j\omega)e^{j\omega t}d\omega
\end{aligned}$$

- $H(j\omega)$: The Fourier transform of the system impulse response or the frequency response of the system.



- Periodic Convolution: [(periodic signal)∗(periodic signal)]

  Consider two periodic signals $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ with common period T_0. The periodic convolution of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ is defined as

$$\tilde{y}(t) = \int_{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau \,.$$

  Let

$$\tilde{x}_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \rightarrow \quad T_0 a_k = X_1(jk\omega_0),\ \ \underbrace{X_1(j\omega)}_{\text{envelope}}$$

$$\tilde{x}_2(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \quad \rightarrow \quad T_0 b_k = X_2(jk\omega_0),\ \ \underbrace{X_2(j\omega)}_{\text{envelope}}.$$

$$\tilde{y}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \rightarrow \quad T_0 c_k = Y(jk\omega_0),\ \ \underbrace{Y(j\omega)}_{\text{envelope}}$$

  Then $c_k = T_0 a_k b_k$.

Example 4.9:

$$h(t) = e^{-at}u(t),\ \ a > 0$$
$$x(t) = e^{-bt}u(t),\ \ b > 0 \qquad\qquad y(t) = h(t)*x(t) = ?$$

Answer:

$$H(j\omega) = \frac{1}{a+j\omega} \qquad\qquad X(j\omega) = \frac{1}{b+j\omega}$$

(i) If $a \neq b$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{a+j\omega}\cdot\frac{1}{b+j\omega} = \frac{A}{a+j\omega} + \frac{B}{b+j\omega}$$

$$\Rightarrow A(b+j\omega) + B(a+j\omega) = 1$$

$$\Rightarrow \left.\begin{array}{r} A+B=0 \\ Ab+Ba=1 \end{array}\right\} \Rightarrow \begin{array}{l} A = 1/(b-a) \\ B = -1/(b-a) \end{array}$$

$$\Rightarrow Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$$

$$\Rightarrow y(t) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right]$$

(ii) If $a=b$

$$\left.\begin{aligned} &Y(j\omega)=\frac{1}{(a+j\omega)^2}=j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right] \\ &\because e^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{a+j\omega} \\ &\Rightarrow te^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]=\frac{1}{(a+j\omega)^2} \end{aligned}\right\}\Rightarrow y(t)=te^{-at}u(t)$$ ■

Example 4.10:

$$h(t)=e^{-t}u(t)$$

$$x(t)=\sum_{k=-3}^{3}a_k e^{jk2\pi t} \qquad y(t)=h(t)*x(t)=?$$

Answer:

$$H(j\omega)=\frac{1}{1+j\omega};\ X(j\omega)=\sum_{k=-3}^{3}a_k 2\pi\delta(\omega-2\pi k)$$

$$\Rightarrow Y(j\omega)=\sum_{k=-3}^{3}2\pi a_k\cdot\frac{1}{1+j\omega}\delta(\omega-2\pi k)$$

$$=\sum_{k=-3}^{3}\frac{2\pi a_k}{1+j2\pi k}\delta(\omega-2\pi k)$$ ■

$$\Rightarrow y(t)=\sum_{k=-3}^{3}\frac{a_k}{1+j2\pi k}e^{j2\pi kt}$$

Example 4.11:

$$x(t)=e^{-t}u(t)-e^{-1}e^{-(t-1)}u(t-1)$$

$$y(t)=x(t)*h(t)=e^{-t}u(t) \qquad h(t)=?$$

Answer:

$$X(j\omega)=\frac{1}{1+j\omega}-\frac{e^{-1}e^{-j\omega}}{1+j\omega}=\frac{1-e^{-(1+j\omega)}}{1+j\omega};\ Y(j\omega)=\frac{1}{1+j\omega}$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{1-e^{-(1+j\omega)}}$$

$$\because \left|e^{-(1+j\omega)}\right|<1$$

$$\therefore \frac{1}{1-e^{-(1+j\omega)}}=1+e^{-1}e^{-j\omega}+e^{-2}e^{-2j\omega}+e^{-3}e^{-3j\omega}+\cdots\cdots$$

$$\Rightarrow h(t)=\delta(t)+e^{-1}\delta(t-1)+e^{-2}\delta(t-2)+e^{-3}\delta(t-3)+\cdots\cdots$$ ■

※ $\delta(t)\overset{\mathcal{F}}{\longleftrightarrow}1$

$\delta(t-t_0)\overset{\mathcal{F}}{\longleftrightarrow}e^{-j\omega t_0}$

9. Modulation Property

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(j\omega)$$
$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(j\omega)$$

$$r(t) = s(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right] \tag{4.39}$$

Note:

- Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other.
  $\Rightarrow$ The multiplication of two signals is often referred to as amplitude modulation.

Proof:

$$r(t) = s(t)p(t)$$
$$R(j\omega) = \int_{-\infty}^{\infty} s(t)p(t)e^{-j\omega t}dt$$
$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(jv)e^{jvt}dv$$
$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(t)\left[\int_{-\infty}^{\infty} P(jv)e^{jvt}dv\right]e^{-j\omega t}dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(jv)\underbrace{\left[\int_{-\infty}^{\infty} s(t)e^{-j(\omega-v)t}dt\right]}_{S(j(\omega-v))}dv$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(jv)S(j(\omega-v))dv$$
$$= \frac{1}{2\pi}P(j\omega) * S(j\omega)$$

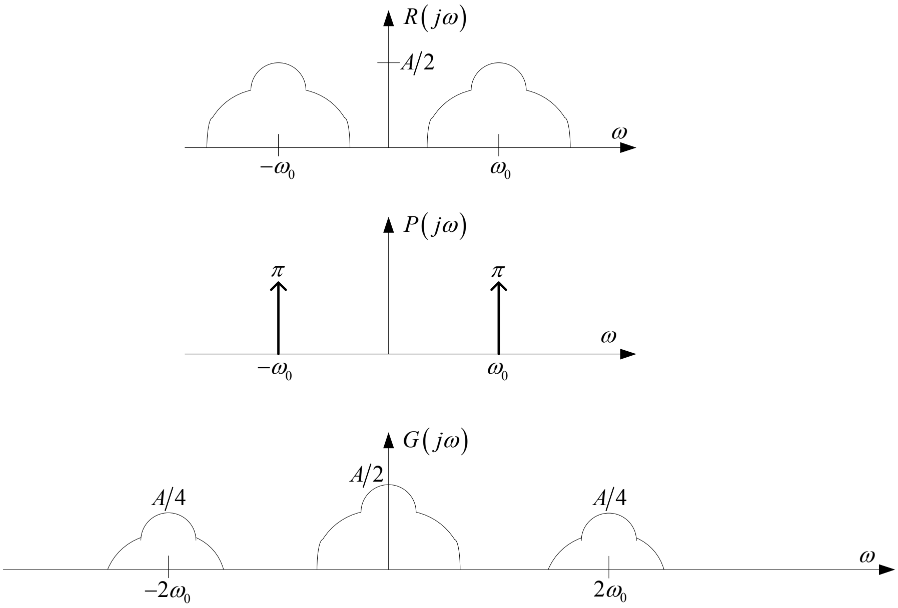
■

Example 4.12: Given $R(j\omega)$ and $p(t)$, calculate $G(j\omega) = \frac{1}{2\pi}\left[R(j\omega) * P(j\omega)\right]$.

$$g(t) = r(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} G(j\omega) = \frac{1}{2\pi}\left[R(j\omega) * P(j\omega)\right]$$

$$p(t) = \cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$\Rightarrow P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

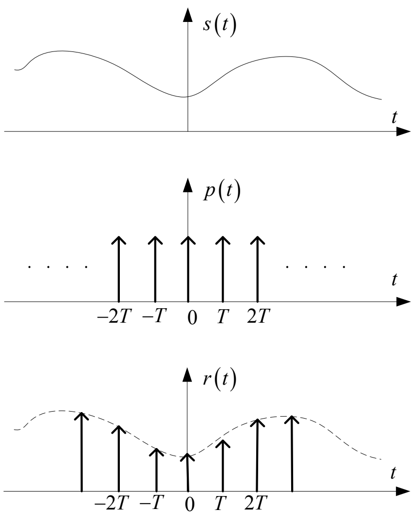


■

Example 4.13: Given $s(t)$ and $p(t)$, calculate $R(j\omega)=\frac{1}{2\pi}\left[S(j\omega)*P(j\omega)\right]$.

$$r(t)=s(t)\,p(t)$$

$$p(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)\Rightarrow P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$$

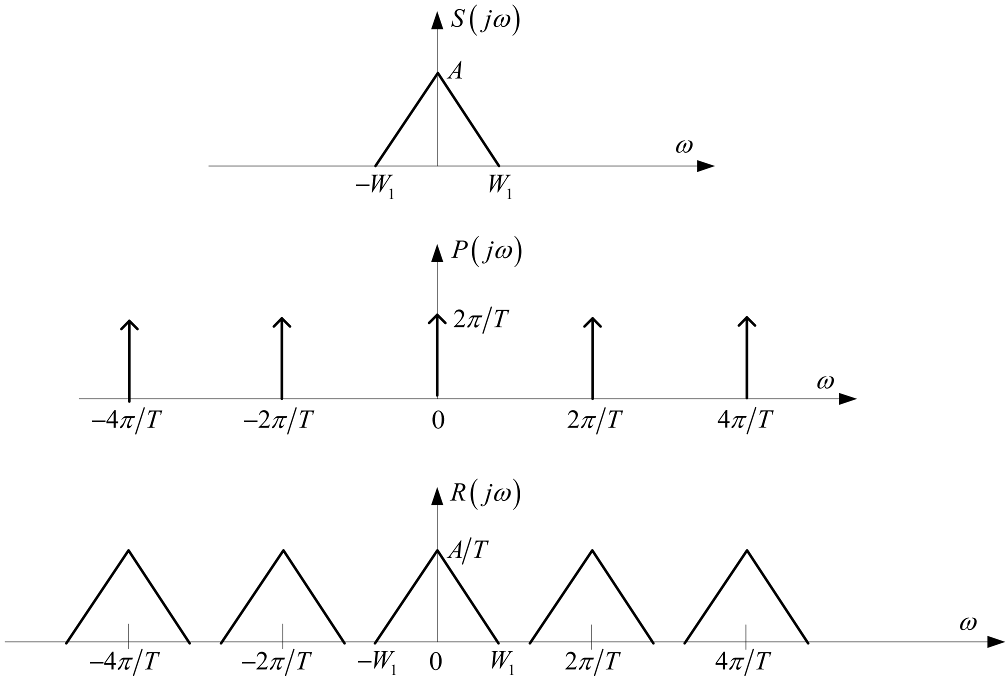


$$R(j\omega)=\frac{1}{2\pi}\left[S(j\omega)*P(j\omega)\right]=\frac{1}{T}\sum_{k=-\infty}^{\infty}S(j\omega)*\delta\left(\omega-\frac{2\pi k}{T}\right)$$
$$=\frac{1}{T}\sum_{k=-\infty}^{\infty}S\left(j\left(\omega-\frac{2\pi k}{T}\right)\right)$$

■

Example 4.14: (Sampling Theorem)

$$r(t) = s(t)\,p(t)$$



■

- $\frac{2\pi}{T} \geq 2W_1$ i.e., $\omega_0 \geq 2W_1$

  Sampling Frequency≥2×(Signal Bandwidth)

  ⇒ No aliasing in $R(j\omega)$.

  ⇒ $s(t)$ can be reconstructed from $r(t)$.

4-4 The Frequency Response of Systems Characterized by Linear Constant-Coefficient Differential Equations

1. Calculation of the Frequency Response and the Impulse Response

$$\sum_{k=0}^{N} a_k \cdot \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \cdot \frac{d^k x(t)}{dt^k}$$

$$\updownarrow \mathcal{F} \qquad\qquad \updownarrow \mathcal{F}$$

$$\sum_{k=0}^{N} a_k \cdot (j\omega)^k \cdot Y(j\omega) = \sum_{k=0}^{M} b_k \cdot (j\omega)^k \cdot X(j\omega)$$

$$\Rightarrow Y(j\omega)\left[\sum_{k=0}^{N} a_k \cdot (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k \cdot (j\omega)^k\right]$$

$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{\sum_{k=0}^{M} b_k \cdot (j\omega)^k}{\sum_{k=0}^{N} a_k \cdot (j\omega)^k} \quad \cdots\cdots \text{ Frequency Response}$$

Example 4.15: Determine the impulse response $h(t)$ of the LTI system described by the following differential equation that is initially at rest.

$$\frac{dy(t)}{dt} + ay(t) = x(t) \qquad (a > 0)$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega) \Rightarrow Y(j\omega)[a + j\omega] = X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a + j\omega} \Rightarrow h(t) = e^{-at}u(t)$$

■

Example 4.16: Determine the impulse response $h(t)$ of the LTI system described by the following differential equation that is initially at rest.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$(j\omega)^2 Y(j\omega) + 4j\omega Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega)$$

$$\Rightarrow Y(j\omega)\left[(j\omega)^2 + 4j\omega + 3\right] = X(j\omega)[j\omega + 2]$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}$$

$$= \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}$$

$$= \frac{A}{j\omega + 1} + \frac{B}{j\omega + 3}$$

$$j\omega + 2 = A(j\omega + 3) + B(j\omega + 1)$$

$$= (A + B)j\omega + (3A + B)$$

$$\left.\begin{array}{l} A + B = 1 \\ 3A + B = 2 \end{array}\right\} \Rightarrow A = B = \frac{1}{2}$$

$$\Rightarrow H(j\omega) = \frac{1/2}{j\omega + 1} + \frac{1/2}{j\omega + 3}$$

$$\Rightarrow h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

■

Example 4.17:

$$x(t) = e^{-t}u(t)$$

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

$$y(t) = x(t) * h(t) = ?$$

$$
\begin{aligned}
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= \frac{1}{j\omega+1}\cdot\frac{j\omega+2}{(j\omega+1)(j\omega+3)} \\
&= \frac{A}{j\omega+1}+\frac{B}{(j\omega+1)^2}+\frac{C}{j\omega+3}
\end{aligned}
$$

$j\omega+2 = A(j\omega+1)(j\omega+3)+B(j\omega+3)+C(j\omega+1)^2$

Let $s = j\omega$

$\Rightarrow s+2 = A(s+1)(s+3)+B(s+3)+C(s+1)^2$

Set $s=-1$

$\Rightarrow 1 = B\cdot 2 \Rightarrow B = 1/2$

Set $s=-3$

$\Rightarrow -1 = 4C \Rightarrow C = -1/4$

$A+C=0 \qquad (\text{the } s^2 \text{ term})$

$\Rightarrow A = -C = 1/4 \Rightarrow y(t) = \left[\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{1}{4}e^{-3t}\right]u(t)$

■

2. Cascade and Parallel-Form Structures for Implementation of LTI Systems
   (i) Cascade-form structure

$$H(j\omega) = \frac{b_M\prod_{k=1}^{M}(\lambda_k+j\omega)}{a_N\prod_{k=1}^{N}(v_k+j\omega)} \tag{4.40}$$

where λ_k and v_k may be complex.

By multiplying together the two first-order terms involving complex conjugate λ_k's or v_k's, we obtain second-order terms with real coefficients. For example,

$$(\lambda+j\omega)(\lambda^*+j\omega) = |\lambda|^2+2\,\mathrm{Re}\{\lambda\}j\omega+(j\omega)^2$$

$$\Rightarrow H(j\omega) = \frac{b_M}{a_N}\frac{\prod_{k=1}^{P}\left[\beta_{0k}+\beta_{1k}(j\omega)+(j\omega)^2\right]\prod_{k=1}^{M-2P}(\lambda_k+j\omega)}{\prod_{k=1}^{Q}\left[\alpha_{0k}+\alpha_{1k}(j\omega)+(j\omega)^2\right]\prod_{k=1}^{N-2Q}(v_k+j\omega)} \tag{4.41}$$

where the coefficients are all real.

$\Rightarrow$ The system can be implemented using a cascade (let *P*=*Q*) of *P* second-order systems and (*N-2P*) first-order systems.

- Realization of a second-order system

$$H_k(j\omega)=\frac{\beta_{0k}+j\omega\beta_{1k}+(j\omega)^2}{\alpha_{0k}+j\omega\alpha_{1k}+(j\omega)^2}=\frac{Y_k(j\omega)}{X_k(j\omega)}$$

$$\Rightarrow \alpha_{0k}Y_k(j\omega)+j\omega\alpha_{1k}Y_k(j\omega)+(j\omega)^2 Y_k(j\omega)$$
$$=\beta_{0k}X_k(j\omega)+j\omega\beta_{1k}X_k(j\omega)+(j\omega)^2 X_k(j\omega)$$
$$\Rightarrow \alpha_{0k}y_k(t)+\alpha_{1k}\frac{dy_k(t)}{dt}+\frac{d^2y_k(t)}{dt^2}$$
$$=\beta_{0k}x_k(t)+\beta_{1k}\frac{dx_k(t)}{dt}+\frac{d^2x_k(t)}{dt^2}$$

For convenience, we only consider the following second-order terms for realization of a cascade system (as shown in Fig. 4.4):

$$H(j\omega)=\frac{b_M}{a_N}\frac{\prod_{k=1}^{P}\left[\beta_{0k}+j\omega\beta_{1k}+(j\omega)^2\right]}{\prod_{k=1}^{Q}\left[\alpha_{0k}+j\omega\alpha_{1k}+(j\omega)^2\right]} \quad (4.42)$$

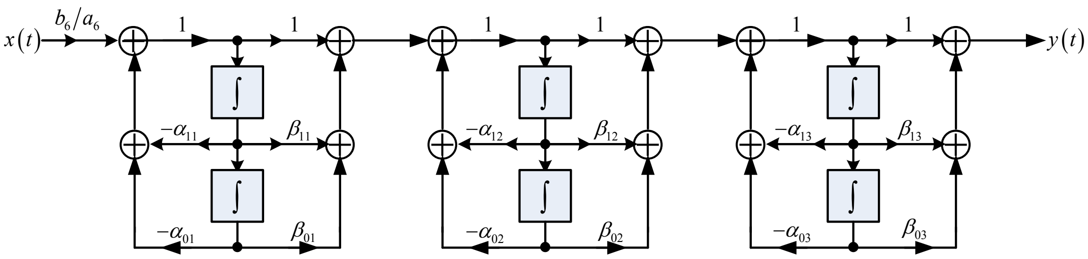


Figure 4.4 A cascade-form structure of second-order subsystems with N=M=6 and P=Q=3.

(ii) Parallel-form structure

$$H(j\omega)=\frac{b_N\prod_{k=1}^{N}(\lambda_k+j\omega)}{a_N\prod_{k=1}^{N}(v_k+j\omega)} \quad (4.43)$$

If all the v_k's are distinct, then $H(j\omega)$ can be expressed as

$$H(j\omega)=\left(\frac{b_N}{a_N}\right)+\sum_{k=1}^{N}\frac{A_k}{v_k+j\omega}. \quad (4.44)$$

Adding together the pairs involving complex conjugate v_k's, we obtain

$$H(j\omega)=\left(\frac{b_N}{a_N}\right)+\sum_{k=1}^{Q}\frac{\gamma_{0k}+j\omega\gamma_{1k}}{\alpha_{0k}+j\omega\alpha_{1k}+(j\omega)^2} + \sum_{l=1}^{N-2Q}\frac{A_l}{v_l+j\omega}. \tag{4.45}$$

(All the coefficients are real.)

⇒ We can implement the system by using a parallel interconnection of Q second-order systems and (N-$2Q$) first-order systems.
For convenience, we only consider the following second-order terms with an additional constant for realization of a parallel system (as shown in Fig. 4.5):

$$H(j\omega)=\left(\frac{b_N}{a_N}\right)+\sum_{k=1}^{Q}\frac{\gamma_{0k}+j\omega\gamma_{1k}}{\alpha_{0k}+j\omega\alpha_{1k}+(j\omega)^2}. \tag{4.46}$$

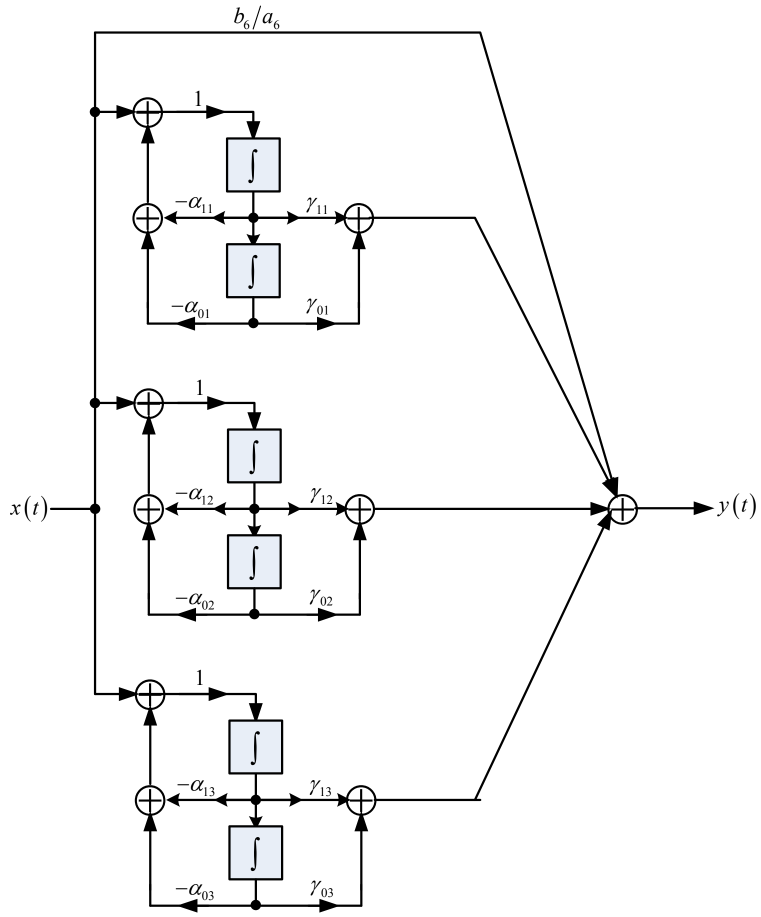


Figure 4.5 A parallel-form structure of second-order subsystems with N=6 and Q=3.

Appendix
Partial Fraction Expansion

- $G(v) = \dfrac{b_2 v^2 + b_1 v + b_0}{(v-\rho_1)^2 (v-\rho_2)}$

$$G(v) = \frac{A_{11}}{v-\rho_1} + \frac{A_{12}}{(v-\rho_1)^2} + \frac{A_{21}}{v-\rho_2}$$

(i) $(v-\rho_1)^2 G(v) = A_{11}(v-\rho_1) + A_{12} + \dfrac{A_{21}(v-\rho_1)^2}{v-\rho_2}$

$$\Rightarrow \left[(v-\rho_1)^2 G(v)\right]\Big|_{v=\rho_1} = A_{12}$$

(ii) $\dfrac{d\left[(v-\rho_1)^2 G(v)\right]}{dv} = A_{11} + A_{21}\left[\dfrac{2(v-\rho_1)}{v-\rho_2} - \dfrac{(v-\rho_1)^2}{(v-\rho_2)^2}\right]$

$$\Rightarrow \left.\frac{d\left[(v-\rho_1)^2 G(v)\right]}{dv}\right|_{v=\rho_1} = A_{11}$$

(iii) $(v-\rho_2)G(v) = \dfrac{A_{11}}{(v-\rho_1)}(v-\rho_2) + \dfrac{A_{12}}{(v-\rho_1)^2}(v-\rho_2) + A_{21}$

$$\Rightarrow \left[(v-\rho_2)G(v)\right]\Big|_{v=\rho_2} = A_{21}$$

- $G(v) = \dfrac{b_{n-1}v^{n-1} + \cdots + b_1 v + b_0}{(v-\rho_1)^{\sigma_1}(v-\rho_2)^{\sigma_2}\cdots(v-\rho_r)^{\sigma_r}}$

$$= \frac{A_{11}}{v-\rho_1} + \frac{A_{12}}{(v-\rho_1)^2} + \cdots + \frac{A_{1\sigma_1}}{(v-\rho_1)^{\sigma_1}}$$

$$+ \frac{A_{21}}{v-\rho_2} + \frac{A_{22}}{(v-\rho_2)^2} + \cdots + \frac{A_{2\sigma_2}}{(v-\rho_2)^{\sigma_2}}$$

$$\vdots$$

$$+ \frac{A_{r1}}{v-\rho_r} + \frac{A_{r2}}{(v-\rho_r)^2} + \cdots + \frac{A_{r\sigma_r}}{(v-\rho_r)^{\sigma_r}}$$

$$= \sum_{i=1}^{r}\sum_{k=1}^{\sigma_i} \frac{A_{ik}}{(v-\rho_i)^k}$$

$$\boxed{A_{ik} = \frac{1}{(\sigma_i - k)!}\left\{\frac{d^{\sigma_i - k}}{dv^{\sigma_i - k}}\left[(v-\rho_i)^{\sigma_i} G(v)\right]\right\}\Big|_{v=\rho_i}}$$

References

[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, Signals and Systems, 2nd Ed., Pearson Education Limited, 2014 (or Prentice-Hall, 1997).

[2] Leland B. Jackson, Signals, Systems, and Transforms, Addison-Wesley, 1991.