- The Gibbs phenomenon: Consider a periodic square wave $x(t)$ as shown in Example 3.1. Since such a signal meets all the three *Dirichlet* conditions, it has a Fourier series representation in the sense of MSE convergence. More specifically, $x_N(t)$ converges to $x(t)$ as $N \to \infty$ for all t except at the isolated discontinuous time points. There is an interesting effect observed when we use the finite series $x_N(t)$ to approximate $x(t)$ for different values of N, as shown in Fig. 3.2. When N increases, the ripples in the approximating waveform $x_N(t)$ become narrower and narrower, but the overshoot (i.e., the peak amplitude of the ripples) of the waveform beyond the desired amplitude remains constant (about 9% of the height of the discontinuity). This effect is known as the Gibbs phenomenon, observed by Albert Michelson in 1898 and shown/explained by Josiah Gibbs in 1899.

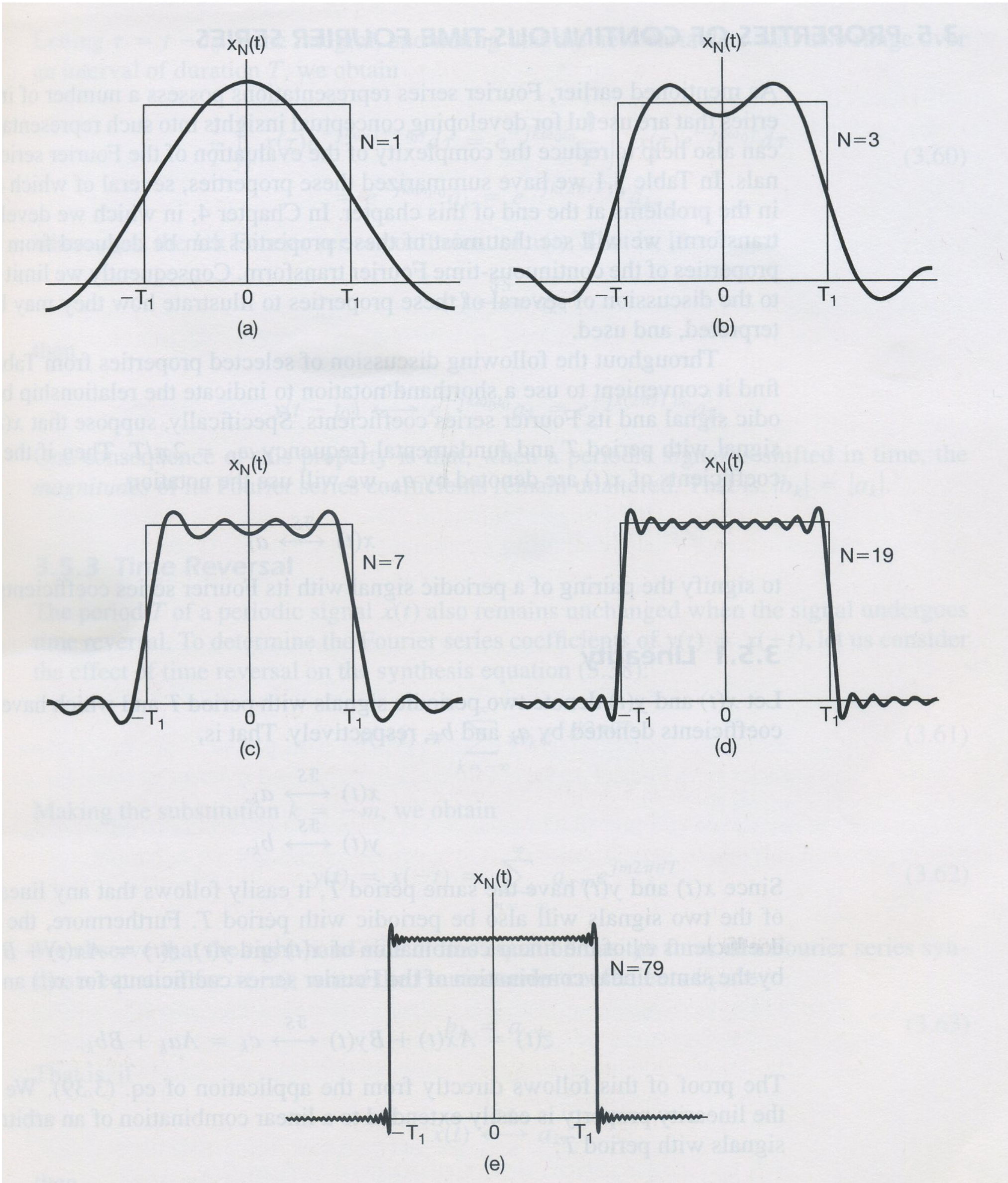


Fig. 3.2 Convergence of the Fourier series of a periodic square wave.

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $w_0 = 2\pi / T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jkw_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jMw_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a_k^*
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (period with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau) y(t-\tau) d\tau$	$Ta_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jkw_0 a_k = jk \frac{2\pi}{T} a_k$
Integration	$\int_{-\infty}^{t} x(t) dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jkw_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a_{-k}^* \\ Re\{a_k\} = Re\{a_{-k}\} \\ Im\{a_k\} = -Im\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = \sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = Ev\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = Od\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\begin{cases} Re\{a_k\} \\ jIm\{a_k\} \end{cases}$

Parserval's Relation for Periodic Signals

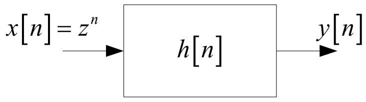
$$\frac{1}{T} \int_T \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{+\infty} \lvert a_k \rvert^2$$

3-4 Eigenfunctions and Eigenvalues of Discrete-Time LTI Systems

1. Similar to the continuous-time case, an eigenfunction and an eigenvalue of a discrete-time LTI system with input $x[n]$ and output $y[n]$ are defined as follows:

$$x[n] \to y[n] = \underbrace{H}_{eigenvalue} \cdot \underbrace{x[n]}_{eigenfunction}$$

2. Complex exponentials z^n are eigenfunctions of discrete-time LTI systems:



$$\begin{aligned} y[n] &= x[n]*h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = z^n H(z) \end{aligned} \tag{3.20}$$

$$\Rightarrow y[n] = H(z)x[n] \tag{3.21}$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ is the eigenvalue associated with the eigenfunction z^n.

Note:

- $x[n] = \sum_k a_k z_k^n \to y[n] = \sum_k a_k H(z_k) z_k^n$ (superposition principle)
- In this chapter, we restrict ourself to the case of z^n with $|z| = 1$, i.e., the complex exponentials of the form $e^{j\Omega n}$.

3-5 Fourier Series Representation of Periodic Discrete-Time Signals: The Discrete-Time Fourier Series (DTFS)

1. Discrete-time Fourier series representation

(1) Harmonically related discrete-time complex exponentials:

$$\phi_k[n] = e^{jk\frac{2\pi}{N}n}, \ k = 0, \ \pm 1, \ \pm 2, \ldots \tag{3.22}$$

where the fundamental period is N and the fundamental frequency is $\Omega_0 = 2\pi/N$.

(2) There are only N different signals in the set of $\phi_k[n]$, $k = 0, \ \pm 1, \ \pm 2, \ldots$.

$\because e^{j(k+N)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n}$ $\quad \therefore \phi_k[n] = \phi_{k+Nr}[n]$, r is an integer.

(3) If $x[n]$ is periodic with period N, then the discrete-time Fourier series

representation of $x[n]$ is

$$x[n]=\sum_{k=<N>} a_k\phi_k[n]=\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{3.23}$$

where $k=<N>$ means k varies over a range of N successive integers, e.g.,

$$\begin{cases} 0,\ 1,\ 2,\ 3,\ldots,\ N-1 \\ 1,\ 2,\ 3,\ 4,\ldots,\ N \\ 2,\ 3,\ 4,\ 5,\ldots,\ N+1 \\ \qquad\vdots \end{cases}$$

Note: Discrete-time Fourier series coefficients are the sampled values of the discrete-time Fourier transform.

2. Determination of the discrete-time Fourier series coefficients

If $x[n]$ is periodic with period N and its Fourier series representation is

$$x[n]=\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{3.24}$$

then $a_k=\dfrac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n}$. ■

Proof:

$$x[n]=\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{3.25}$$

$$\begin{aligned}\sum_{n=<N>} x[n]e^{-jr\frac{2\pi}{N}n} &= \sum_{n=<N>}\left(\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}\right)e^{-jr\frac{2\pi}{N}n} \\ &= \sum_{n=<N>}\sum_{k=<N>} a_k e^{j(k-r)\frac{2\pi}{N}n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)\frac{2\pi}{N}n}\end{aligned} \tag{3.26}$$

$$\because \sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n} = \begin{cases} N &, \ k=mN \\ \dfrac{1-\left(e^{j\frac{2\pi}{N}k}\right)^N}{1-e^{j\frac{2\pi}{N}k}}=0 &, \ \text{otherwise} \end{cases} \tag{3.27}$$

$$\therefore \sum_{n=<N>} e^{j(k-r)\frac{2\pi}{N}n} = \begin{cases} N,\ k=r+mN \\ 0\ \ ,\ \text{otherwise} \end{cases} \tag{3.28}$$

Thus

$$\sum_{n=<N>} x[n]e^{-jr\frac{2\pi}{N}n} = Na_r \Rightarrow a_r=\frac{1}{N}\sum_{n=<N>} x[n]e^{-jr\frac{2\pi}{N}n}. \tag{3.29}$$

Note:

- The discrete-time Fourier series coefficients are often referred to as the spectral coefficients of $x[n]$.
- $a_k = a_{k+N}$

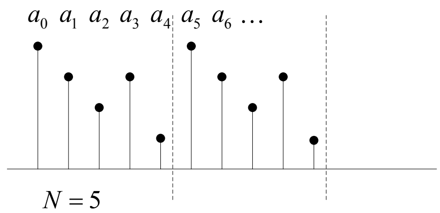


Figure 3.3 The discrete-time Fourier series coefficients a_k repeating periodically with period N.

- The discrete-time Fourier series representation is a finite series with N terms. Only N successive elements of the a_k sequence are used in the Fourier series representation.

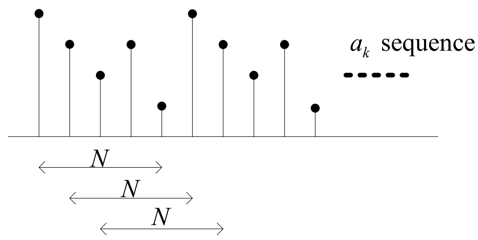


Figure 3.4 The discrete-time Fourier series with N successive elements of the a_k sequence.

Example 4.1: $x[n] = \sin(\Omega_0 n)$, period $= 2\pi/\Omega_0$.

Three situations:

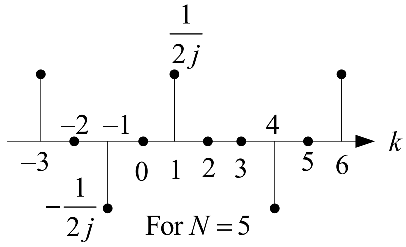
$$\begin{cases} \left.\begin{array}{l} 2\pi/\Omega_0 \text{ is an integer.} \\ 2\pi/\Omega_0 \text{ is a ratio of integers} \end{array}\right\} \Rightarrow \text{ periodic} \\ 2\pi/\Omega_0 \text{ is an irrational number} \Rightarrow \text{ aperiodic} \end{cases}$$

(1) $2\pi/\Omega_0 = N \Rightarrow x[n] = \sin\left(\frac{2\pi}{N}n\right) = \frac{1}{2j}\left(e^{j\frac{2\pi}{N}n} - e^{-j\frac{2\pi}{N}n}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$$

$$\Rightarrow a_1 = \frac{1}{2j} \text{ and } a_{-1} = -\frac{1}{2j}$$

and the remaining coefficients are zero.



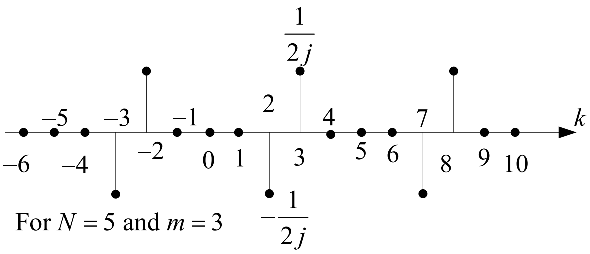
(2) $2\pi/\Omega_0 = m/N$; m and N have no common factors.

$$\Rightarrow \Omega_0 = 2\pi m/N$$

$$\Rightarrow x[n] = \sin\left(\frac{2\pi m}{N} n\right) = \frac{1}{2j}\left(e^{jm\frac{2\pi}{N}n} - e^{-jm\frac{2\pi}{N}n}\right)$$

$$\Rightarrow a_m = \frac{1}{2j} \text{ and } a_{-m} = -\frac{1}{2j}$$

and the remaining coefficients are zero.

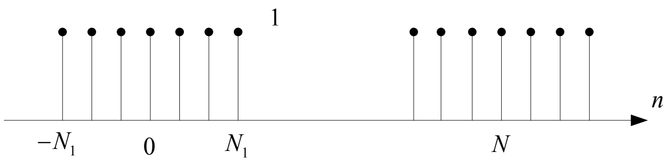


$a_2 = a_{-3+5}$

$a_{-2} = a_{3-5}$

■

***Example* 4.2**: Discrete-time periodic square wave



For $k \neq 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\frac{2\pi}{N}(m-N_1)} \quad (m = n + N_1)$$

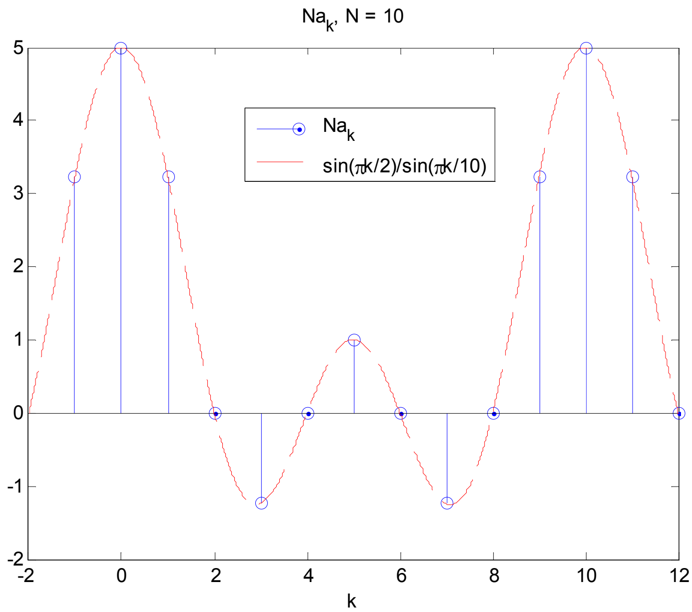
$$= \frac{1}{N} e^{jk\frac{2\pi}{N}N_1} \sum_{m=0}^{2N_1} e^{-jk\frac{2\pi}{N}m}$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}N_1} \left(\frac{1-\left(e^{-jk\frac{2\pi}{N}}\right)^{2N_1+1}}{1-e^{-jk\frac{2\pi}{N}}}\right) = \frac{1}{N}\cdot\frac{\sin\left(2\pi k\left(N_1+\frac{1}{2}\right)\Big/N\right)}{\sin\left(2\pi k/(2N)\right)}.$$

For $k = 0, \pm N, \pm 2N, \ldots, \ a_k = (2N_1+1)/N$.

Note:

The discrete-time counterpart of the sinc function is of the form $\sin(\beta x)/\sin(x)$. The coefficients a_k for $2N_1 + 1 = 5$ and $N = 10$ are sketched as follows:



$$\Rightarrow Na_k = \left.\frac{\sin\left((2N_1+1)\Omega/2\right)}{\sin(\Omega/2)}\right|_{\Omega=2\pi k/N} \qquad \blacksquare$$

3. Approximation of a discrete-time periodic signal using a truncated Fourier series

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} = \begin{cases} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} a_k e^{jk\frac{2\pi}{N}n}, & N \text{ is even} \\ \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} a_k e^{jk\frac{2\pi}{N}n}, & N \text{ is odd} \end{cases} \tag{3.30}$$

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk\frac{2\pi}{N}n}, \quad M < N/2 \text{ or } M < (N-1)/2 \tag{3.31}$$

(1) When $M \to N/2$ or $(N-1)/2$, the approximation of $x[n]$ by $\hat{x}[n]$ is shown in Fig. 3.4, where $x[n]$ is a square wave. There are no convergence issues and no **Gibbs phenomenon**. **The Gibbs phenomenon** exists in the continuous-time case, where the ripples in the partial sum become compressed toward the discontinuity with the peak amplitude of the ripples remaining constant independent of the number of terms in the partial sum.

(2) In general, there are no convergence issues with the discrete-time Fourier series. Any discrete-time periodic sequence $x[n]$ is completely specified by a finite number of parameters, namely the values of the a_k sequence over one period.

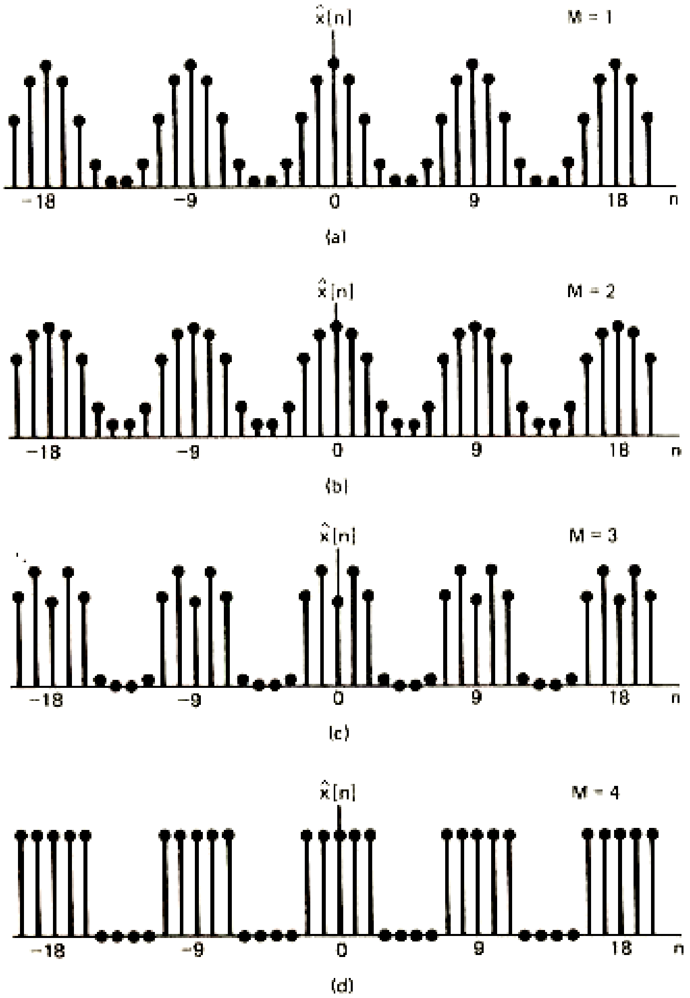


Figure 3.5 Partial sums of Eq. (3.31) for the periodic square wave with N = 9 and $2N_1+1$ = 5: (a) M = 1; (b) M = 2; (c) M = 3; (d) M = 4.

4. The input-output relationship of an LTI system with impulse response $h[n]$:

$$x[n]=\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \rightarrow y[n]=\sum_{k=<N>} a_k H\left(j\frac{2\pi k}{N}\right)e^{jk\frac{2\pi}{N}n}$$

with $H\left(j\frac{2\pi k}{N}\right)=\sum_{n=-\infty}^{\infty} h[n]e^{-jk\frac{2\pi}{N}n}$.

The result comes from (3.20) and (3.21), where $e^{jk\frac{2\pi}{N}n}$ is an eigenfunction and the superposition principle can be applied.

***Example* 4.3**:

$$\begin{cases} h[n]=\alpha^n u[n],\ |\alpha|<1 \\ x[n]=\cos(2\pi n/N) \end{cases}$$

$$x[n]=\frac{1}{2}\left(e^{j\frac{2\pi}{N}n}+e^{-j\frac{2\pi}{N}n}\right)$$

$$H\left(j\frac{2\pi k}{N}\right)=\sum_{n=0}^{\infty}\alpha^{n}e^{-jk\frac{2\pi}{N}n}=\sum_{n=0}^{\infty}\left(\alpha e^{-j\frac{2\pi k}{N}}\right)^{n}$$

$$=\frac{1}{1-\alpha e^{-j\frac{2\pi k}{N}}}\quad\left(\because\left|\alpha e^{-j\frac{2\pi k}{N}}\right|<1\right)$$

$$\Rightarrow y[n]=\frac{1}{2}H\left(j\frac{2\pi}{N}\right)e^{j\frac{2\pi}{N}n}+\frac{1}{2}H\left(-j\frac{2\pi}{N}\right)e^{-j\frac{2\pi}{N}n}$$

$$=\frac{1}{2}\left(\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}\right)e^{j\frac{2\pi}{N}n}+\frac{1}{2}\left(\frac{1}{1-\alpha e^{j\frac{2\pi}{N}}}\right)e^{-j\frac{2\pi}{N}n}$$

■

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi / N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n]+By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n-n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=<N>} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
First Difference	$x[n]-x[n-1]$	$\left(1-e^{-jk(2\pi/N)}\right)a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{1-e^{-jk(2\pi/N)}}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ Re\{a_k\} = Re\{a_{-k}\} \\ Im\{a_k\} = -Im\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = Ev\{x[n]\} & [\text{x[n] real}] \\ x_o[n] = Od\{x[n]\} & [\text{x[n] real}] \end{cases}$	$Re\{a_k\}$ $jIm\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=<N>} \lvert x[n] \rvert^2 = \sum_{k=<N>} \lvert a_k \rvert^2$$

References

[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, Signals and Systems, 2nd Ed., Pearson Education Limited, 2014 (or Prentice-Hall, 1997).

[2] Leland B. Jackson, Signals, Systems, and Transforms, Addison-Wesley, 1991.