Chapter 3 Fourier Representations of Periodic Signals

3-1 Eigenfunctions and Eigenvalues of Continuous-Time LTI Systems

1. A signal for which the system output $y(t)$ is just a (possibly complex) constant times the input $x(t)$ is referred to as an eigenfunction of the system, and the amplitude factor is referred to as the eigenvalue. That is,

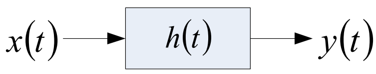
$$x(t) \rightarrow y(t) = \underbrace{H}_{eigenvalue} \cdot \underbrace{x(t)}_{eigenfunction}$$

Note:

- $x(t)$ is called an eigenfunction of the system if $T\{x(t)\} = H \cdot x(t)$, where H is called the eigenvalue corresponding to the eigenfunction.

2. Complex exponentials e^{st} are eigenfunctions of continuous time LTI systems:

$x(t) = e^{st}$ and



$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\cdot e^{s(t-\tau)}d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\ &= e^{st}\cdot H(s) = H(s)\cdot x(t) \end{aligned} \tag{3.1}$$

(1) $H(s)$ is a complex constant whose value depends on "s".

(2) e^{st} ………………………………… eigenfunction

(3) $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$ ……………… eigenvalue

3. $x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$

Note:

- $T\{x(t)\} = \sum_k a_k T\{e^{s_k t}\} = \sum_k a_k H(s_k) e^{s_k t}$
- $s \rightarrow s_k$: Fourier series
- $s = j\omega$: Fourier transform
- $s = \sigma + j\omega$: Laplace transform

3-2 Fourier Series Representation of Periodic Continuous-Time Signals: The Continuous-Time Fourier Series

1. Fourier series representation

Harmonically related complex exponentials:

$$\phi_k(t) = e^{jk\omega_0 t} \quad , k = 0, \ \pm 1, \ \pm 2, \ ....$$

Note:

- Each of these exponentials is periodic with period $T_0 = 2\pi/\omega_0$.
- Any linear combination of these exponentials is also periodic with period T_0.

Let $x(t)$ be a periodic continuous-time signal with fundamental period T_0. Then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{3.2}$$

is referred to as the Fourier series representation of $x(t)$.

Note:

- The terms for $k = \pm 1$: the fundamental components or the lst harmonic components.
  The terms for $k = \pm 2$: the 2nd harmonic components.
  ⋮
  The terms for $k = \pm N$: the *N*th harmonic components.
- Alternative forms for the Fourier series of real periodic signals:

$$\begin{cases} x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \\ x(t) = a_0 + 2\sum_{k=1}^{\infty} \left[b_k \cos(k\omega_0 t) - c_k \sin(k\omega_0 t) \right] \end{cases} \tag{3.3}$$

$\because x(t)$ is real $\quad \therefore x^*(t) = x(t)$.

Let $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$. Then

$$x^*(t) = x(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} \underset{k \text{ replaced with } -k}{=} \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} \tag{3.4}$$

$$\Rightarrow a_k = a_{-k}^* \text{ or } a_k^* = a_{-k}$$

$$\begin{aligned} \Rightarrow x(t) &= a_0 + \sum_{k=-\infty}^{-1} a_k e^{jk\omega_0 t} + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} \\ &= a_0 + \sum_{k=1}^{\infty} \left[a_{-k} e^{-jk\omega_0 t} + a_k e^{jk\omega_0 t} \right] \\ &= a_0 + \sum_{k=1}^{\infty} \left[\left(a_k e^{jk\omega_0 t} \right)^* + a_k e^{jk\omega_0 t} \right] \\ &= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{ a_k e^{jk\omega_0 t} \right\} \\ &= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{ A_k e^{j(k\omega_0 t + \theta_k)} \right\} \quad \left(a_k = A_k e^{j\theta_k} \right) \\ &= a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \end{aligned} \tag{3.5}$$

Let $a_k = b_k + jc_k$. Then

$$\begin{aligned} x(t) &= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\} \\ &= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{(b_k + jc_k)\left(\cos(k\omega_0 t) + j\sin(k\omega_0 t)\right)\right\} \\ &= a_0 + 2\sum_{k=1}^{\infty}\left[b_k \cos(k\omega_0 t) - c_k \sin(k\omega_0 t)\right] \end{aligned} \tag{3.6}$$

- $x(t) = \sum_k a_k e^{jk\omega_0 t} \rightarrow y(t) = \sum_k a_k \cdot H(jk\omega_0) e^{jk\omega_0 t}$ with

$$H(jk\omega_0) = \int_{-\infty}^{\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau \tag{3.7}$$

2. Determination of the Fourier series coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{3.8}$$

$$x(t)\cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \tag{3.9}$$

$$\begin{aligned} \int_0^{T_0} x(t)\cdot e^{-jn\omega_0 t} dt &= \int_0^{T_0} \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \int_0^{T_0} e^{j(k-n)\omega_0 t} dt \end{aligned} \tag{3.10}$$

$$\int_0^{T_0} e^{j(k-n)\omega_0 t} dt = \int_0^{T_0} \cos(k-n)\omega_0 t \; dt + j\int_0^{T_0} \sin(k-n)\omega_0 t \; dt$$

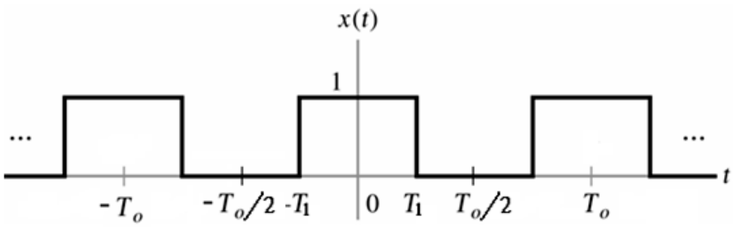
$$= \begin{cases} 0 \;,\; \text{for } k \neq n \rightarrow \begin{pmatrix} \sin(k-n)\omega_0 t \text{ and } \cos(k-n)\omega_0 t \\ \text{are periodic with fundamental period } T_0/|k-n| \end{pmatrix} \\ T_0 \;,\; \text{for } k = n \end{cases}$$

$$\Rightarrow a_n = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jn\omega_0 t} dt = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jn\omega_0 t} dt \tag{3.11}$$

where $\int_{T_0}$ denotes an integral over any interval of length T_0.

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k &= \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jk\omega_0 t} dt \end{aligned}} \tag{3.12}$$

Example 3.1: Derive the Fourier series of the following signal:



$$x(t)=\begin{cases}1 \ , & |t|<T_1 \\ 0 \ , & T_1<|t|<T_0/2\end{cases}$$ for one period. Fundamental period $=T_0$.

The Fourier series representation of $x(t)$ is as follows:

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ , \quad \omega_0 = 2\pi/T_0$$

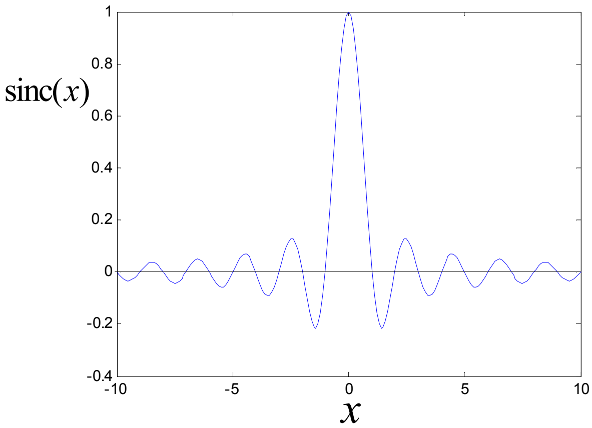
$$a_k=\frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jk\omega_0 t}dt=\frac{1}{T_0}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt$$

$$=\begin{cases}2T_1/T_0 & , \text{ for } k=0 \\ \dfrac{\sin(k\omega_0 T_1)}{k\pi} & , \text{ for } k\neq 0\end{cases} \ . \tag{3.13}$$

Note:

- $\because \operatorname{sinc}(x)=\dfrac{\sin(\pi x)}{\pi x}$

$$\therefore \frac{\sin(k\omega_0 T_1)}{k\pi}=\frac{\omega_0 T_1}{\pi}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}=\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$



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3-3 Approximation of Periodic Signals Using Finite Fourier Series and the Convergence of Fourier Series

1. Finite Fourier series

$$\begin{cases}x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} & \cdots\cdots \text{ infinite series} \\ x_N(t)=\sum_{k=-N}^{N} a'_k e^{jk\omega_0 t} & \cdots\cdots \text{ finite series}\end{cases} \tag{3.14}$$

Let $e_N(t)$ denote the approximation error given by

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a'_k e^{jk\omega_0 t}. \tag{3.15}$$

To determine how good a particular approximation is, a quantitative measure of the size of the approximation error is needed. One commonly used measure for this is the mean-squared error (MSE) criterion defined as

$$E_N = \frac{1}{T_0}\int_{T_0} |e_N(t)|^2 dt = \frac{1}{T_0}\int_{T_0} e_N(t) e_N^*(t) dt \tag{3.16}$$

where T_0 is the fundamental period of the periodic signal $x(t)$ and E_N is the average power of the error signal over one period. To minimize E_N, the coefficients of $x_N(t)$ in (3.14) can be determined as follows:

$$E_N = \frac{1}{T_0}\int_{T_0} \left| x(t) - \sum_{k=-N}^{N} a'_k e^{jk\omega_0 t} \right|^2 dt \tag{3.17}$$

$$\frac{\partial E_N}{\partial a'_k} = 0 \Rightarrow \boxed{a'_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt = a_k}. \tag{3.18}$$

This means that, if $x(t)$ has a Fourier series representation, the best approximation using $x_N(t)$ (formed by a finite number of harmonically related complex exponentials) can be obtained by truncating the Fourier series of $x(t)$ to the desired number of terms. As N increases, E_N decreases.

(1) If $x(t)$ has a Fourier series representation, then $\lim_{N\to\infty} E_N = 0$.

(2) If $\lim_{N\to\infty} E_N = 0$, then the Fourier series $\lim_{N\to\infty} x_N(t)$ is said to converge to $x(t)$.

The latter is called the MSE convergence of the Fourier series for $x(t)$.

2. Any one of the following conditions is sufficient to ensure the MSE convergence of the Fourier series for $x(t)$:

   Condition 1: If the periodic signal $x(t)$ is a continuous function of t, the Fourier series converges. Actually, in the case, the convergence is uniform, which is a stronger criterion than MSE convergence.

   Condition 2: If $x(t)$ is *square-integrable* over a period *T*, that is, if

$$\int_T |x(t)|^2 dt < \infty$$

   the Fourier series converges in the MSE sense. Since this condition is equivalent to the requirement that the average power in $x(t)$ be finite, it clearly applies to all periodic signals encountered in the laboratory, as well as to most theoretical signals of interest. In particular, note that any bounded signal satisfies the condition because, if $|x(t)| < B$ for finite B and all t, then

$$\int_T |x(t)|^2 \, dt < TB^2.$$

Condition 3 (*Dirichlet Conditions*):

(1) Over any period, $x(t)$ must be absolutely integrable, i.e.,

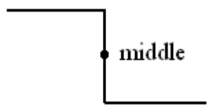
$$\int_{T_0} |x(t)| \, dt < \infty . \tag{3.19}$$

(2) In any finite interval of time, $x(t)$ is of bounded variation, i.e., there are no more than a finite number of maxima and minima during any single period of the signal.

(3) In a finite number of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities must be finite.

Note:

- The three *Dirichlet* conditions guarantee that:
  - (i) $x(t)$ has a Fourier series representation in the sense of MSE convergence.
  - (ii) $x(t)$ is equal to its Fourier series representation for all t except at isolated values of t for which $x(t)$ is discontinuous. At the isolated points of discontinuity, the series converges to the average value of the discontinuity.



  $\Rightarrow$ The integrals of both signals, $x(t)$ and its Fourier series, over any interval are identical.

  $\Rightarrow$ The two signals behave identically under convolution and consequently are identical from the standpoint of the analysis of LTI systems.

- For a periodic signal that varies continuously, the Fourier series representation converges and equals the original signal at any *t*.

Example 3.2:

(a) $x(t) = 1/t$, $0 < t \le 1$, period $= 1$, as shown in Fig. 3.1(a).

This signal violates Conditions 1-3, and does not have a Fourier series representation.

(b) $x(t) = \sin(2\pi/t)$, $0 < t \leq 1$, period $= 1$, as shown in Fig. 3.1(b).

This signal meets the 1st Dirichlet condition (and Condition 2), but not the 2nd one, since

$$\int_0^1 |\sin(2\pi/t)| dt < 1$$

and there are an infinite number of maxima and minima in each period. The corresponding Fourier series is of MSE convergence.

(c) The periodic signal (with period 8) shown in Fig. 3.1(c) has an infinite number of discontinuities in each period. It meets Condition 2 but violates the 3rd Dirichlet condition. The corresponding Fourier series is of MSE convergence.

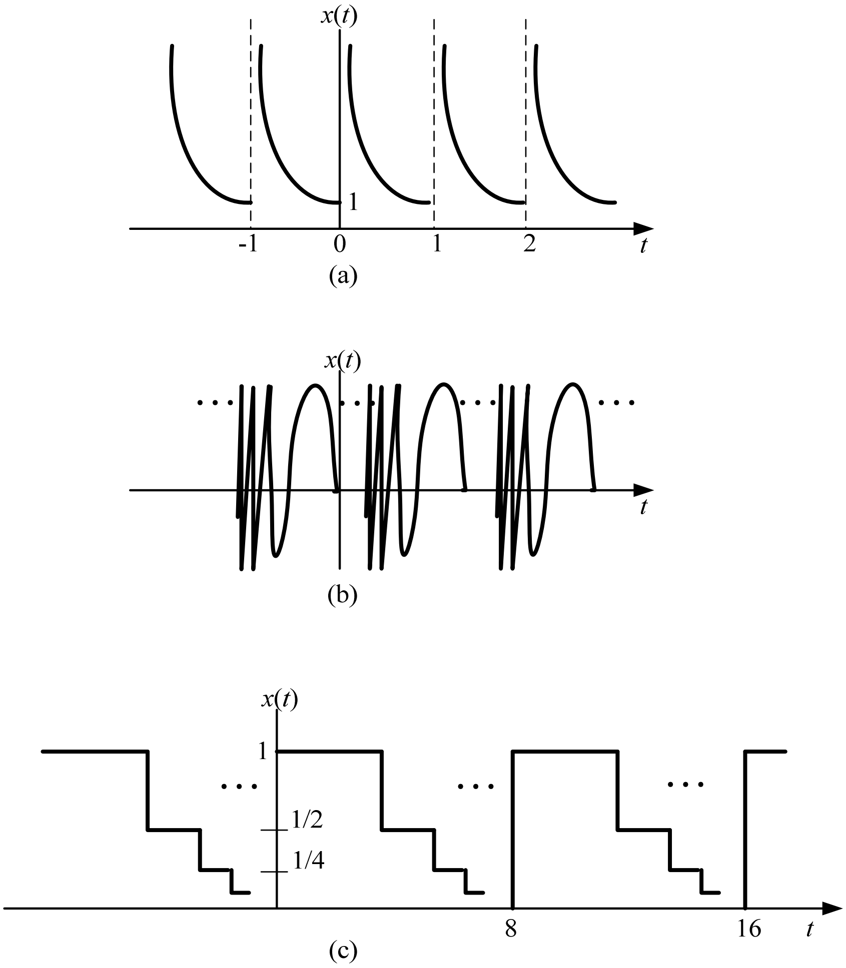


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Figure 3.1 Some signals violating the Dirichlet conditions.

Note:

- The signals that do not satisfy the Dirichlet conditions are generally pathological in nature and thus are not particularly important in the study of signals and systems.