2-4 Systems Described by Differential or Difference Equations

1. The general form of linear constant-coefficient *differential* equations for describing a continuous-time system is

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (2.46)$$

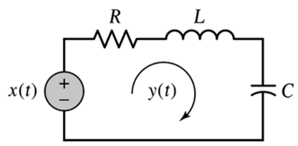
where a_k and b_k are constant coefficients, $x(t)$ is the input signal of the system, and $y(t)$ is the resulting output signal.

2. The general form of linear constant-coefficient *difference* equations for describing a discrete-time system can be similarly expressed by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \qquad (2.47)$$

The order of such a differential or difference equation is (N, M), representing the number of energy storage devices in the system. It is often that $N \geq M$ and the order is described using only N.

Example 2.24: An RLC circuit



$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

$$\overset{\frac{d}{dt}}{\Rightarrow} R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) + \frac{1}{C}y(t) = \frac{d}{dt}x(t)$$

The order is $N = 2$. This implies that the circuit contains two energy storage devices: a capacitor and an inductor. ■

Example 2.25: $y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$

The order is $N = 2$. This implies a maximum memory of 2 in the system output. ■

3. Computing the current output of the system from the input signal and past outputs:

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\} \qquad (2.48)$$

Example 2.26: $y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]+2x[n-1]$

$$y[n]=x[n]+2x[n-1]-y[n-1]-\frac{1}{4}y[n-2]$$

$$y[0]=x[0]+2x[-1]-y[-1]-\frac{1}{4}y[-2]$$

$$y[1]=x[1]+2x[0]-y[0]-\frac{1}{4}y[-1]$$

$$\vdots$$

In order to begin this process at time n = 0, we must know the two most recent past values of the output. These values are known as initial conditions. ■

Note:

- The number of initial conditions required to determine the output is equal to the maximum memory of the system. It is common to choose n = 0 or t = 0 as the starting time for solving a difference or differential equation, respectively. For example, the initial conditions for an Nth-order difference equation and an Nth-order differential equation are the N values given as follows:

$$y[-N], y[-N+1], \ldots, y[-1] \tag{2.49}$$

and

$$y(t)\Big|_{t=0^-}, \frac{dy(t)}{dt}\bigg|_{t=0^-}, \ldots, \frac{d^{N-1}y(t)}{dt^{N-1}}\bigg|_{t=0^-} \tag{2.50}$$

2-5 Solving Differential and Difference Equations

1. The output of a system described by a differential or difference equation may be expressed as the sum of two components.

    (1) Homogeneous solution, $y^{(h)}$: a solution of the homogeneous form (by setting all terms involving the input to zero) of the differential or difference equation.

    (2) Particular solution, $y^{(p)}$: any solution of the original equation for the given input. Thus, the complete solution is

$$y(t)=y^{(h)}(t)+y^{(p)}(t) \quad \text{or} \quad y[n]=y^{(h)}[n]+y^{(p)}[n] \tag{2.51}$$

2. Solutions of linear constant-coefficient differential equations

Consider a continuous-time system described by

$$\frac{d}{dt}y(t)+2y(t)=x(t) \tag{2.52}$$

where $x(t)$ and $y(t)$ are the input and output of the system respectively. The complete solution to the above differential equation can be expressed as

$$y(t)=y^{(p)}(t)+y^{(h)}(t) \tag{2.53}$$

where

$$\begin{cases} y^{(p)}(t): \text{ particular solution} \\ y^{(h)}(t): \text{ homogeneous solution, i.e., solution of } \frac{d}{dt}y(t)+2y(t)=0. \end{cases}$$

(1) Determination of the particular solution

(a) A particular solution is usually obtained by assuming an output of the same general form as the input.

(b) Consider that the input is $x(t)=k\cos(\omega_0 t)u(t)=\text{Re}\{ke^{j\omega_0 t}\}u(t)$. For $t\ge 0$, we can hypothesize a particular solution of the form

$$y^{(p)}(t)=\text{Re}\left\{Ye^{j\omega_0 t}\right\} \tag{2.54}$$

$$\frac{d}{dt}y^{(p)}(t)+2y^{(p)}(t)=\text{Re}\left\{j\omega_0 Ye^{j\omega_0 t}+2Ye^{j\omega_0 t}\right\}=\text{Re}\left\{ke^{j\omega_0 t}\right\} \tag{2.55}$$

$$\Rightarrow j\omega_0 Y+2Y=k \tag{2.56}$$

$$Y=\frac{k}{j\omega_0+2}=\frac{k}{\sqrt{4+\omega_0^2}}e^{-j\theta},\ \theta=\tan^{-1}\left(\frac{\omega_0}{2}\right) \tag{2.57}$$

$$y^{(p)}(t)=\text{Re}\left\{Ye^{j\omega_0 t}\right\}=\frac{k}{\sqrt{4+\omega_0^2}}\cos(\omega_0 t-\theta),\ t>0\,. \tag{2.58}$$

(2) Determination of the homogeneous solution

(a) In order to determine $y^{(h)}(t)$, we hypothesize a solution of the form

$$y^{(h)}(t) = Ae^{st} \tag{2.59}$$

$$\frac{d}{dt}y^{(h)}(t) + 2y^{(h)}(t) = sAe^{st} + 2Ae^{st} = 0 \tag{2.60}$$

$$s + 2 = 0 \Rightarrow s = -2 \tag{2.61}$$

$$y^{(h)}(t) = Ae^{-2t},\ t > 0\,. \tag{2.62}$$

(b) The homogeneous solution of a general linear constant coefficient differential equation can be found in a way given in the Appendix.

(3) Determination of the complete solution

From (1) and (2), we have

$$y(t) = y^{(p)}(t) + y^{(h)}(t) = Ae^{-2t} + \frac{k}{\sqrt{4+\omega_0^2}}\cos(\omega_0 t - \theta),\ t > 0 \tag{2.63}$$

(a) Determination of the constant A by specifying initial (or auxiliary) conditions on the differential equation

If we specify $y(0) = y_0$, then

$$A = y_0 - \frac{k}{\sqrt{4+\omega_0^2}}\cos\theta \tag{2.64}$$

$$y(t) = y_0 e^{-2t} + \frac{k}{\sqrt{4+\omega_0^2}}\left[\cos(\omega_0 t - \theta) - \cos\theta e^{-2t}\right],\ t > 0\,. \tag{2.65}$$

(b) Solution of the differential equation for $t < 0$

For $t < 0$, $x(t) = 0$ and $y(t) = y^{(h)}(t) = Be^{-2t}$. Thus, we have

$$y(t) = y_0 e^{-2t},\ t < 0\ \left(\because y(0) = y_0\right) \tag{2.66}$$

(c) Complete solution

$$y(t) = y_0 e^{-2t} + \frac{k}{\sqrt{4+\omega_0^2}}\left[\cos(\omega_0 t - \theta) - \cos\theta e^{-2t}\right]u(t) \tag{2.67}$$

Note:

- The above system is linear if the initial condition is zero.

  Let $x_1(t)$ and $x_2(t)$ be two input signals, and let $y_1(t)$ and $y_2(t)$ be the corresponding responses with $y_1(0)=y_2(0)=0$, i.e.,

$$\frac{d}{dt}y_1(t)+2y_1(t)=x_1(t),\ y_1(0)=0. \tag{2.68}$$

$$\frac{d}{dt}y_2(t)+2y_2(t)=x_2(t),\ y_2(0)=0. \tag{2.69}$$

  Consider next the input $x_3(t)=\alpha x_1(t)+\beta x_2(t)$, where α and β are any complex numbers. From (2.68) and (2.69), we have

$$\frac{d}{dt}y_3(t)+2y_3(t)=x_3(t),\ y_3(t)=\alpha y_1(t)+\beta y_2(t),\ y_3(0)=0. \tag{2.70}$$

  This means that $y_3(t)=\alpha y_1(t)+\beta y_2(t)$ is the response corresponding to $x_3(t)=\alpha x_1(t)+\beta x_2(t)$. So the superposition principle holds, and the system is linear.

- The above system is incrementally linear if the initial condition is not zero.

$$y(t)=\underbrace{y_0e^{-2t}}_{\text{due to the nonzero auxiliary condition alone}}+\underbrace{\frac{k}{\sqrt{4+\omega_0^2}}\left[\cos(\omega_0 t-\theta)-\cos\theta e^{-2t}\right]u(t)}_{\text{the linear response of the system assuming that the auxiliary condition is zero}} \tag{2.71}$$

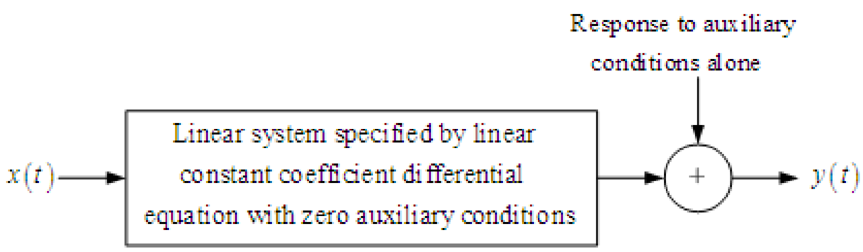


Figure 2.19 A continuous-time incrementally linear system described by a linear constant-coefficient differential equation.

- A general *N*th-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N}a_k\frac{d^ky(t)}{dt^k}=\sum_{k=0}^{M}b_k\frac{d^kx(t)}{dt^k}. \tag{2.72}$$

- The solution $y(t) = y^{(p)}(t) + y^{(h)}(t)$

$$\begin{cases} y^{(p)}(t): \text{ particular solution} \\ y^{(h)}(t): \text{ homogeneous solution} \end{cases}$$

- Initial conditions correspond to the values of

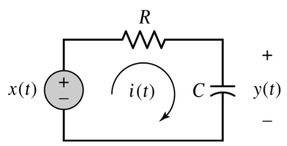
$$y(t)\Big|_{t=0^-},\ \frac{dy(t)}{dt}\Big|_{t=0^-},\ldots,\ \frac{d^{N-1}y(t)}{dt^{N-1}}\Big|_{t=0^-} \qquad (2.73)$$

The system will be linear only if all of these initial conditions are zero.

- A necessary and sufficient condition for the initial conditions at $t = t_0^+$ (e.g., $t_0^+ = 0^+$) to equal the initial conditions at $t = t_0^-$ for a given input is that the right-hand side of the differential equation in (2.72), $\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$, contain no impulses or derivatives of impulses.

***Example** 2.27*: An RC circuit

$$y(t) + RC\frac{d}{dt}y(t) = x(t) = \cos(t)u(t)$$



$R = 1\Omega,\ C = 1F$, and $y(0^-) = 2V$.

The homogeneous solution is

$$y(t) + RC\frac{d}{dt}y(t) = 0$$

The order $N = 1$. $y^{(h)}(t) = ce^{r_1 t}$

where r_1 is the root of the characteristic equation

$$1 + RCr_1 = 0 \Rightarrow r_1 = -\frac{1}{RC}$$

$$\therefore y^{(h)}(t) = ce^{-t/RC} = ce^{-t} \quad (RC = 1)$$

Assume $y^{(p)}(t) = c_1\cos(t) + c_2\sin(t)$. Then

$$c_1\cos(t)+c_2\sin(t)-RCc_1\sin(t)+RCc_2\cos(t)=\cos(t)$$

$$\begin{cases} c_1+RCc_2=1 \\ -RCc_1+c_2=0 \end{cases} \Rightarrow \begin{cases} c_1=\dfrac{1}{1+(RC)^2} \\ c_2=\dfrac{RC}{1+(RC)^2} \end{cases}$$

$$y^{(p)}(t)=\frac{1}{1+(RC)^2}\cos(t)+\frac{RC}{1+(RC)^2}\sin(t)=\frac{1}{2}\cos(t)+\frac{1}{2}\sin(t),\ t>0$$

$\because$ No impulses are introduced. $\therefore y(0^+)=y(0^-)=2$

We have

$$2=y^{(h)}(t)+y^{(p)}(t)=ce^{-0^+}+\frac{1}{2}\cos(0^+)+\frac{1}{2}\sin(0^+)=c+\frac{1}{2}\Rightarrow c=\frac{3}{2}$$

$$y(t)=\frac{3}{2}e^{-t}+\frac{1}{2}\cos(t)+\frac{1}{2}\sin(t),\ t>0$$

■

3. Solutions of linear constant-coefficient difference equations

(1) The *N*th-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{M}b_k x[n-k] \tag{2.74}$$

The solution $y[n]$ can be written as

$$y[n]=y^{(p)}[n]+y^{(h)}[n] \tag{2.75}$$

$$\begin{cases} y^{(p)}[n]:\ \text{particular solution} \\ y^{(h)}[n]:\ \text{homogeneous solution} \rightarrow \sum_{k=0}^{N}a_k y[n-k]=0 \end{cases}$$

Note:

- $y^{(h)}[n]$ is the solution of the homogeneous equation

$$\sum_{k=0}^{N}a_k y^{(h)}[n-k]=0 \tag{2.76}$$

The homogeneous solution for a discrete-time system can be found in a way given in the Appendix.

(2) A system described by the Nth-order linear constant-coefficient difference equation and some initial conditions is incrementally linear.

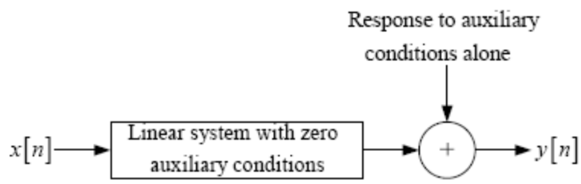


Figure 2.20 A discrete-time incrementally linear system described by a linear constant-coefficient difference equation.

(3) $\because y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$

$\therefore$ A set of initial conditions are needed, such as

$$\underbrace{y[-N], y[-N+1], \ldots, y[-1]}_{N} \tag{2.77}$$

(4) For the order $N > 0$,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$
$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\} \tag{2.78}$$
$\Rightarrow$ a recursive equation

We need initial conditions to determine $y[n]$ from a recursive equation.

For $N = 0$, $y[n] = \sum_{k=0}^{M} (b_k / a_0) x[n-k]$ $\Rightarrow$ a nonrecursive equation

We do not need initial conditions to determine $y[n]$ from a nonrecursive equation.

$$\Rightarrow h[n] = \begin{cases} b_n / a_0, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \quad (\text{let } x[n] = \delta[n], \text{ then } y[n] = h[n]) \tag{2.79}$$

Example 2.28: Example of recursive difference equations

$$y[n] - \frac{1}{2} y[n-1] = x[n],\ y[-1] = a,\ x[n] = k\delta[n]$$

(i) Determine $y[n]$ for $n \geq 0$

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

$$y[0] = x[0] + \frac{1}{2}y[-1] = k + \frac{1}{2}a$$
$$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}\left(k + \frac{1}{2}a\right)$$
$$y[2] = x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2\left(k + \frac{1}{2}a\right)$$
$$\vdots$$
$$y[n] = x[n] + \frac{1}{2}y[n-1]$$
$$= \left(\frac{1}{2}\right)^n\left(k + \frac{1}{2}a\right) = k\left(\frac{1}{2}\right)^n + a\left(\frac{1}{2}\right)^{n+1},\ n \geq 0$$

(ii) Determine $y[n]$ for $n < 0$

$$y[n-1] = 2\{y[n] - x[n]\}$$

$$y[-2] = 2\{y[-1] - x[-1]\} = 2a$$
$$y[-3] = 2\{y[-2] - x[-2]\} = 2^2 a$$
$$y[-4] = 2\{y[-3] - x[-3]\} = 2^3 a$$
$$\vdots$$
$$y[n] = 2\{y[n+1] - x[n+1]\} = 2^{-(n+1)}a = \left(\frac{1}{2}\right)^{n+1}a,\ n < 0$$

Thus, for all values of n,

$$y[n] = \underbrace{\left(\frac{1}{2}\right)^{n+1}a}_{y^{(h)}[n]} + \underbrace{k\left(\frac{1}{2}\right)^n u[n]}_{y^{(p)}[n]}$$

If $a = 0$, the system is linear.

Note:

- Initial conditions are zero. ⇒ The system is linear.
- The recursive difference equation has an impulse response of infinite duration. ⇒ "infinite impulse response" (IIR) system

- The nonrecursive difference equation has an impulse response of finite duration. ⇒ "finite impulse response" (FIR) system ■

Example 2.29: A first-order recursive system

$$y[n]-\rho y[n-1]=x[n]=\left(\frac{1}{2}\right)^n u[n],\ \rho=\frac{1}{4},\ y[-1]=8$$

The homogeneous equation is

$$y[n]-\frac{1}{4}y[n-1]=0 \Rightarrow N=1,\ y^{(h)}[n]=cr_1^n \Rightarrow r_1-\frac{1}{4}=0 \Rightarrow r_1=\frac{1}{4}$$

Assuming $y^{(p)}[n]=c_p\left(\frac{1}{2}\right)^n$, we have

$$c_p\left(\frac{1}{2}\right)^n-\frac{c_p}{4}\left(\frac{1}{2}\right)^{n-1}=\left(\frac{1}{2}\right)^n \Rightarrow c_p\left(1-\frac{2}{4}\right)=1 \Rightarrow c_p=2$$

$$\Rightarrow y^{(p)}[n]=2\left(\frac{1}{2}\right)^n$$

$$y[n]=2\left(\frac{1}{2}\right)^n+c\left(\frac{1}{4}\right)^n,\ n\ge 0$$

$$y[0]=x[0]+\frac{1}{4}y[-1]=3$$

$$3=2\left(\frac{1}{2}\right)^0+c\left(\frac{1}{4}\right)^0 \Rightarrow c=1 \Rightarrow y[n]=2\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n,\ n\ge 0$$

Note: If $\rho=1/2$, there is no coefficient c_p satisfying $c_p(1-2\rho)=1$ and we assume a particular solution of the form $y^{(p)}[n]=c_p n(1/2)^n$. Thus we can obtain a particular solution as follows:

$$c_p n(1-2\rho)+2\rho c_p=1 \Rightarrow c_p n\cdot 0+c_p=1 \Rightarrow c_p=1 \Rightarrow y^{(p)}[n]=n(1/2)^n$$ ■

4. General form of particular solutions for some input signals $x(t)$ and $x[n]$

$x(t)$	Particular Solution
1	c
t^n	$c_1 t^n + c_2 t^{n-1} + \cdots + c_n t + c_{n+1}$
e^{at}	▪ ce^{at} if a is not a characteristic root. ▪ cte^{at} if a is a distinct characteristic root. ▪ $ct^{k-1}e^{at}$ if a is a (k–1)-multiple characteristic root.
$\cos(at)$	$c_1 \cos(at) + c_2 \sin(at)$
$\sin(at)$	$c_1 \cos(at) + c_2 \sin(at)$

$x[n]$	Particular Solution
1	c
n^k	$c_1 n^k + c_2 n^{k-1} + \cdots + c_k n + c_{k+1}$
α^n	▪ $c\alpha^n$ if α is not a characteristic root. ▪ $cn\alpha^n$ if α is a distinct characteristic root. ▪ $cn^{k-1}\alpha^n$ if α is a (k–1)-multiple characteristic root.
$\cos(\Omega n)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

Example 2.30: $\dfrac{d^2 y(t)}{dt^2} + 7\dfrac{dy(t)}{dt} + 6y(t) = 6x(t)$

$$x(t) = \sin(2t),\ y(0) = 0,\ \left.\frac{d}{dt}y(t)\right|_{t=0} = y'(0) = 0$$

Characteristic equation:

$$r^2 + 7r + 6 = 0$$

$$(r+1)(r+6) = 0 \Rightarrow r_1 = -1,\ r_2 = -6$$

$$\Rightarrow y^{(h)}(t) = c_1 e^{-t} + c_2 e^{-6t}$$

$$\because x(t) = \sin 2t$$

$$\therefore y^{(p)}(t) = p_1 \sin 2t + p_2 \cos 2t$$

Substituting $y^{(p)}(t)$ into the differential equation, we obtain

$$-4p_1\sin 2t - 4p_2\cos 2t + 14p_1\cos 2t - 14p_2\sin 2t + 6p_1\sin 2t + 6p_2\cos 2t = 6\sin 2t$$

$$\begin{aligned}&\left(-4p_1 - 14p_2 + 6p_1 - 6\right)\sin 2t + \left(-4p_2 + 14p_1 + 6p_2\right)\cos 2t\\ &= \left(2p_1 - 14p_2 - 6\right)\sin 2t + \left(14p_1 + 2p_2\right)\cos 2t = 0\\ &\Rightarrow \begin{cases}2p_1 - 14p_2 - 6 = 0\\ 14p_1 + 2p_2 = 0\end{cases} \Rightarrow \begin{cases}p_1 = 3/50\\ p_2 = -21/50\end{cases}\\ &y(t) = y^{(h)}(t) + y^{(p)}(t) = c_1e^{-t} + c_2e^{-6t} + 3/50\sin 2t - 21/50\cos 2t\\ &\because y(0) = 0 \text{ and } y'(0) = 0\\ &\therefore \begin{cases}c_1 + c_2 - 21/50 = 0\\ -c_1 - 6c_2 + 6/50 = 0\end{cases} \Rightarrow c_1 = 12/25,\ c_2 = -3/50\end{aligned}$$

$$\Rightarrow y(t) = 12/25e^{-t} - 3/50e^{-6t} + 3/50\sin 2t - 21/50\cos 2t$$ ■

Example 2.31: $y[n] + 2y[n-1] = x[n] - x[n-1],\ x[n] = n^2,\ y[0] = 1$

$$\begin{aligned}&r + 2 = 0 \Rightarrow r = -2\\ &\therefore y^{(h)}[n] = c(-2)^n\\ &\because x[n] = n^2 \ \therefore x[n] - x[n-1] = 2n - 1\\ &\Rightarrow y^{(p)}[n] = p_1n + p_2\end{aligned}$$

Substituting $y^{(p)}[n]$ into the difference equation, we obtain

$$\begin{aligned}&p_1n + p_2 + 2\left[p_1(n-1) + p_2\right] = 2n - 1\\ &\Rightarrow 3p_1n + 3p_2 - 2p_1 = 2n - 1\\ &\Rightarrow (3p_1 - 2)n + (3p_2 - 2p_1 + 1) = 0\\ &\Rightarrow \begin{cases}3p_1 - 2 = 0\\ 3p_2 - 2p_1 + 1 = 0\end{cases} \Rightarrow \begin{cases}p_1 = 2/3\\ p_2 = 1/9\end{cases}\\ &\Rightarrow y[n] = c(-2)^n + 2n/3 + 1/9\\ &\because y[0] = 1 \therefore c = 8/9\end{aligned}$$

$$\Rightarrow y[n] = 8/9(-2)^n + 2n/3 + 1/9$$ ■

2-6 Characteristics of Systems Described by Differential or Difference Equations

1. It is informative to express the output of a system described by a differential or difference equation as the sum of two components:
   (1) One associated only with the initial conditions.
   $\Rightarrow$ This component is called the *natural response* (or zero-input response), $y^{(n)}$.
   (2) The other associated only with the input signal.
   $\Rightarrow$ This component is called the *forced response* (or zero-state response), $y^{(f)}$.
2. The *natural response* is the system output for zero input and thus describes the behavior the system dissipates any stored energy or memory of the past represented by non-zero initial conditions.

   Zero input $\Rightarrow$ $y^{(h)}(t)$ or $y^{(h)}[n]$
   $\Rightarrow$ Determine the coefficients c_i of the homogeneous solutions such that the initial conditions are satisfied.

Example 2.32: An RC circuit (same as Example 2.27)

$$y(t)+RC\frac{d}{dt}y(t)=x(t),\ \ R=1\Omega,\ C=1F,\ \text{and}\ y(0^-)=2V$$

$$y^{(h)}(t)=ce^{-t}\Rightarrow y^{(n)}(0)=2\Rightarrow c=2\Rightarrow y^{(n)}(t)=2e^{-t},\ t\geq 0.$$ ■

Example 2.33: A First-order recursive system (same as Example 2.29)

$$y[n]-\rho y[n-1]=x[n],\ \ x[n]=\left(\frac{1}{2}\right)^n u[n],\ \rho=\frac{1}{4},\ y[-1]=8$$

$$y^{(h)}[n]=c\left(\frac{1}{4}\right)^n\Rightarrow 8=c\left(\frac{1}{4}\right)^{-1}\Rightarrow c=2\Rightarrow y^{(n)}[n]=2\left(\frac{1}{4}\right)^n,\ n\geq -1.$$ ■

3. The *forced response* is the system output due to the input signal with zero initial conditions. Thus, the forced response is of the same form as the complete solution of the differential or difference equation.

   Zero initial conditions
   $\Rightarrow$ The system is initially "at rest", and there is no stored energy or memory in the system before the input signal is applied.
   $\Rightarrow$ The system behavior is "forced" by the input signal.
   The forced response depends on the particular solution, $y^{(p)}$, which is valid only for time $t > 0$ or $n \geq 0$.

Note: As before, the forced response for continuous-time systems is considered only when the inputs do not result in impulses on the right-hand side of the differential equation, i.e., $y(0^-) = y(0^+)$.

Example 2.34: An RC circuit (same as Example 2.27)

$$y(t) + RC\frac{d}{dt}y(t) = x(t),\ \ x(t) = \cos(t)u(t),\ \ R = 1\Omega,\ C = 1F$$

$$y(t) = ce^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t),\ t > 0$$

Assume that the system is initially at rest, i.e., $y(0) = 0 \Rightarrow c = -1/2$.

$$\Rightarrow\ y^{(f)}(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t),\ t > 0.$$ ■

Example 2.35: An first-order recursive system (same as Example 2.29)

$$y[n] - \rho y[n-1] = x[n],\ \ x[n] = \left(\frac{1}{2}\right)^n u[n],\ \rho = \frac{1}{4}$$

$$y[0] = x[0] + \frac{1}{4}y[-1] = 1 + 0 = 1$$

$$y[n] = 2\left(\frac{1}{2}\right)^n + c\left(\frac{1}{4}\right)^n \Rightarrow c = -1$$

$$\Rightarrow\ y^{(f)}[n] = 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n$$ ■

4. The basic difference between impulse-response descriptions and differential- or difference-equation system descriptions:
   (1) Impulse response $\Rightarrow$ no provision for initial conditions
   It applies only to systems that are initially at rest or when the input is known for all time.
   (2) Differential- and difference-equation system descriptions are more flexible in this respect, since they apply to systems either initially at rest or with nonzero initial conditions.

5. Linearity and time-invariance
   (1) The forced response of a system described by a constant-coefficient differential or difference equation is linear with respect to the input.

$$\left.\begin{array}{l} x_1(t) \to y_1^{(f)}(t) \\ x_2(t) \to y_2^{(f)}(t) \end{array}\right\} \Rightarrow \alpha x_1(t) + \beta x_2(t) \to \alpha y_1^{(f)}(t) + \beta y_2^{(f)}(t) \qquad (2.80)$$

(2) The natural response is linear with respect to the initial conditions:

$$\left.\begin{array}{l}\text{initial conditions } I_1 \to y_1^{(n)}(t) \\ \text{initial conditions } I_2 \to y_2^{(n)}(t)\end{array}\right\} \Rightarrow \alpha I_1 + \beta I_2(t) \to \alpha y_1^{(n)}(t) + \beta y_2^{(n)}(t) \quad (2.81)$$

(3) Time invariance

(a) The forced response is also *time-invariant* since the system is initially at rest.

(b) The complete response of an LTI system described by a differential or difference equation is not time-invariant, since the initial conditions will result in an output term that does not shift with a time shift of the input.

(4) The forced response is also *causal* since the system is initially at rest, i.e., the output does not begin prior to the time at which the input is applied to the system.

6. Roots of the characteristic equation

The roots of the characteristic equation afford considerable information about the LTI system behavior.

(1) The forced response depends on both the input and the roots of the characteristic equation, since it involves both the homogeneous and particular solutions.

(2) The basic form of the natural response is dependent entirely on the roots of the characteristic equation.

(3) The impulse response of an LTI system also depends on the roots of the characteristic equation, since it contains the same terms as the natural response.

(4) Stability

For a BIBO stable LTI system, the output must be bounded for any set of initial conditions.

⇒ The natural response of the system must be bounded.

⇒ Each term in the natural response must be bounded.

(a) In discrete-time LTI systems,

$$\left|r_i^n\right| \text{ is bounded or } \left|r_i\right| < 1 \text{ for all } i \quad (2.82)$$

(b) In continuous-time LTI systems,

$$\left|e^{r_i t}\right| \text{ is bounded or } \operatorname{Re}\{r_i\} < 0 \quad (2.83)$$

$\operatorname{Re}\{r_i\} = 0$ means that the system is on the verge of instability.

In a stable LTI system with zero input, the stored energy eventually dissipates and the output approaches zero.

7. Response time
   (1) The natural response has decayed to zero.
      (a) The system behavior is governed only by the particular solution.
      (b) The transition of the system from its initial condition to an equilibrium condition (determined by the input) has been completed.
   (2) The response time of an LTI system to a transient is therefore proportional to

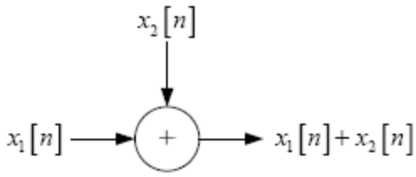
$$\begin{cases} \max|r_i| & \text{for the discrete-time case} \\ \max\left(\text{Re}\{r_i\}\right) & \text{for the continuous-time case} \end{cases}. \tag{2.84}$$

2-7 Block-Diagram Representations of LTI Systems Described by Differential or Difference Equations

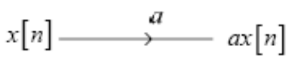
1. Realization of difference equations

   Notations of basic elements:

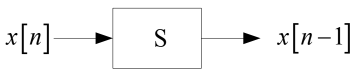
   (1) Adder



   (2) Multiplication by a coefficient

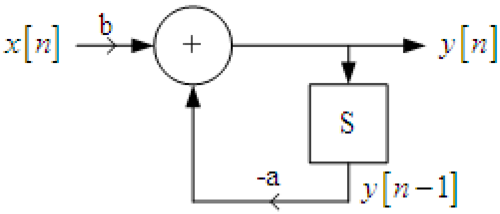


   (3) Unit delay



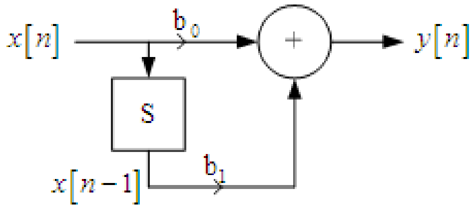
Example 2.36: $y[n]+ay[n-1]=bx[n]$ (initial rest)

$$\Rightarrow y[n]=-ay[n-1]+bx[n]$$



■

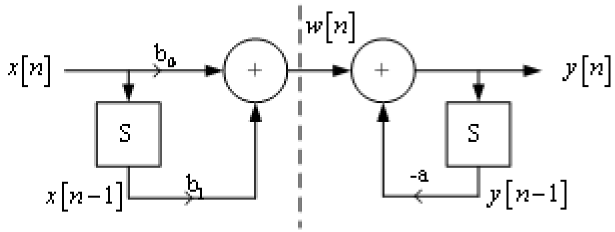
Example 2.37: $y[n]=b_0x[n]+b_1x[n-1]$



■

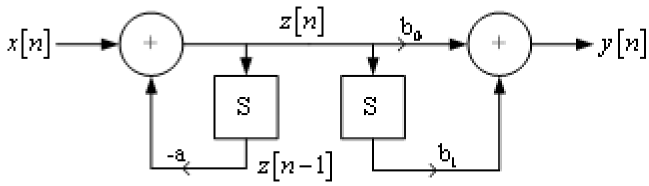
Example 2.38: $y[n]+ay[n-1]=b_0x[n]+b_1x[n-1]$ (initial rest)

$$\Rightarrow y[n]=-ay[n-1]+\underbrace{b_0x[n]+b_1x[n-1]}_{w[n]=b_0x[n]+b_1x[n-1]}=-ay[n-1]+w[n]$$



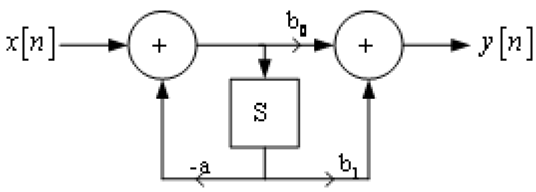
This realization requires two delay elements.

⇓ Interchange the order of cascade interconnection.



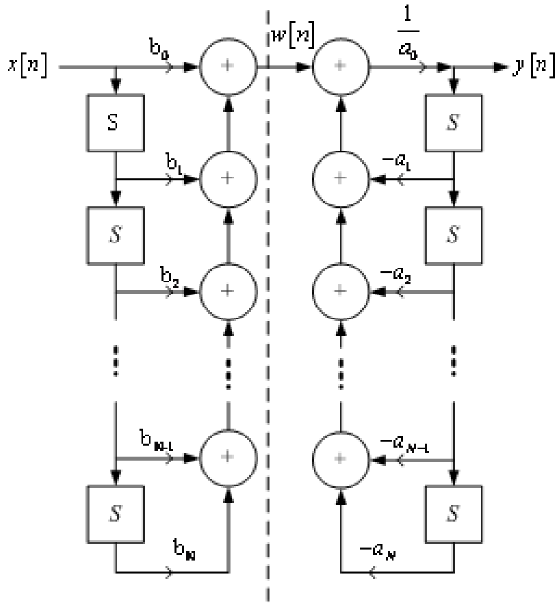
$$\begin{cases} z[n]=-az[n-1]+x[n] \\ y[n]=b_0z[n]+b_1z[n-1] \end{cases}$$

⇓

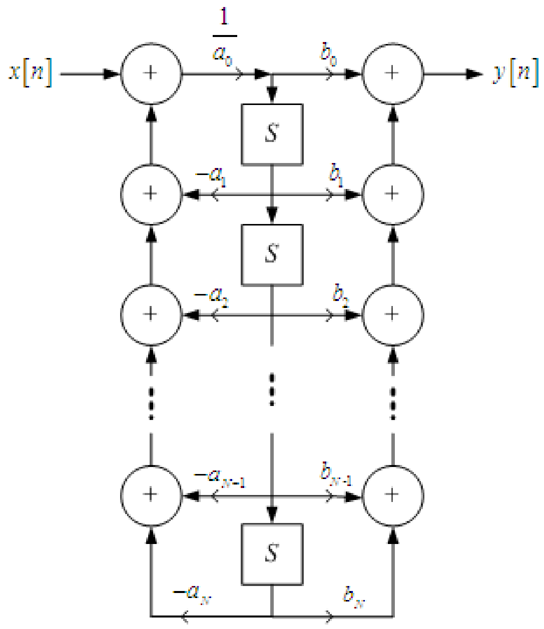


This realization requires only one delay element. ■

Example 2.39: $y[n]=\frac{1}{a_0}\left\{\underbrace{\sum_{k=0}^{N} b_k x[n-k]}_{w[n]}-\sum_{k=1}^{N} a_k y[n-k]\right\}$ (initial rest)



"Direct form I realization" (with $2N$ delay elements)



"Direct form II realization" or "canonic realization" (with N delay elements) ■

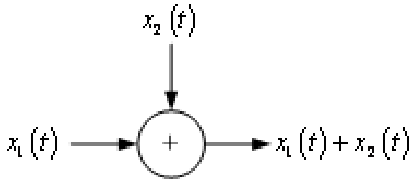
Note:

- The direct form II realization requires fewer delay elements than the direct form I realization.
- In fact, the direct form II realization requires the minimum number of delay elements.

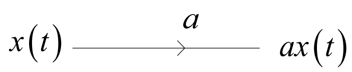
2. Realization of differential equations:

   Notation of basic elements:

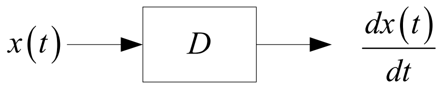
   (1) Adder



   (2) Multiplication by a coefficient



   (3) Differentiator



Consider the following linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k} \qquad (M = N \text{ here}) \tag{2.85}$$

$$\Rightarrow \; y(t) = \frac{1}{a_0}\left\{\sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k} - \sum_{k=1}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} \tag{2.86}$$

The direct form I and direct form II realizations of this differential equation could be the same as those of the difference equation described above except that the delay elements used in the realizations are replaced by differentiators. However, a differentiation element is often difficult to realize, and some other realization methods are necessary. Let us see how to realize the differential equation using integrators, rather than differentiators. Assume the following notations:

$$\begin{aligned}
&y^{(0)}(t) = y(t)\\
&y^{(1)}(t) = y(t)*u(t) = \int_{-\infty}^{t} y(\tau)\,d\tau\\
&y^{(2)}(t) = y(t)*u(t)*u(t) = y^{(1)}(t)*u(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} y(\sigma)\,d\sigma\right]d\tau\\
&\qquad\vdots\\
&y^{(k)}(t) = y^{(k-1)}(t)*u(t) = \int_{-\infty}^{t} y^{(k-1)}(\tau)\,d\tau
\end{aligned} \tag{2.87}$$

$$\begin{aligned}
&x^{(0)}(t) = x(t)\\
&x^{(1)}(t) = x(t)*u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau\\
&x^{(2)}(t) = x(t)*u(t)*u(t) = x^{(1)}(t)*u(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} x(\sigma)\,d\sigma\right]d\tau\\
&\qquad\vdots\\
&x^{(k)}(t) = x^{(k-1)}(t)*u(t) = \int_{-\infty}^{t} x^{(k-1)}(\tau)\,d\tau
\end{aligned} \tag{2.88}$$

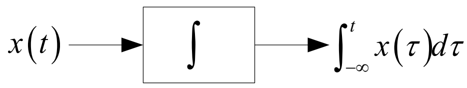
Then the *N*th integral of $d^k y(t)/dt^k$ and the *N*th integral of $d^k x(t)/dt^k$ are precisely $y^{(N-k)}(t)$ and $x^{(N-k)}(t)$, respectively, if the system is initially at rest (i.e., the initial conditions for all the integration terms are zero). Applying these results to (2.85) yields

$$\sum_{k=0}^{N} a_k y^{(N-k)}(t) = \sum_{k=0}^{N} b_k x^{(N-k)}(t) \tag{2.89}$$

$$\Rightarrow \; y^{(0)}(t) = y(t) = \frac{1}{a_N}\left\{\sum_{k=0}^{N} b_k x^{(N-k)}(t) - \sum_{k=0}^{N-1} a_k y^{(N-k)}(t)\right\} \tag{2.90}$$

This implies that the integrator is a basic element for realization of (2.90).

(4) Integrator



According to (2.90), the corresponding direct form I and direct form II realizations can be illustrated as follows:

(a) Direct form I realization:

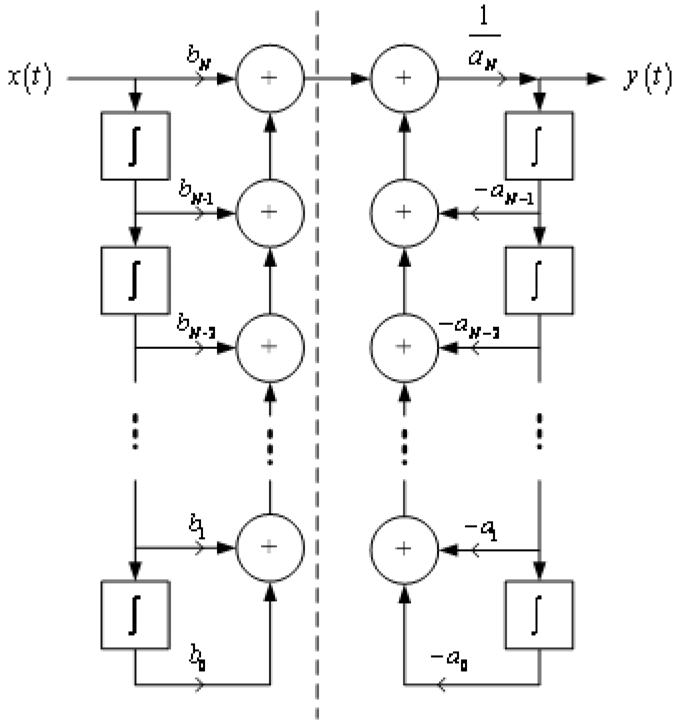


Figure 2.21 Direct form I realization of the LTI system described by (2.90). This approach requires 2*N* integrators.

(b) Direct form II realization:

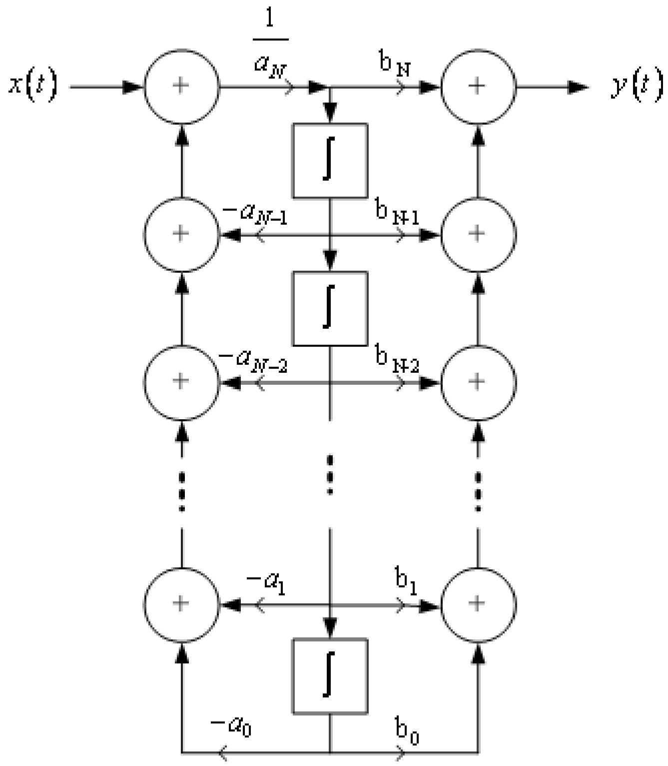


Figure 2.22 Direct form II realization of the LTI system described by (2.90). This approach requires only *N* integrators.

2-9 State-Variable Descriptions of LTI Systems

1. The *state* of a system may be defined as a minimal set of signals that represent the system's entire memory of the past. That is, given only the value of the state at an initial point in time n_i (or t_i) and the input for times $n \geq n_i$ (or $t \geq t_i$), we can determine the output for all times $n \geq n_i$ (or $t \geq t_i$).
2. A state-variable description for direct form II implementation of a second-order LTI system is depicted in Fig. 2.23. Given the input for $n \geq n_i$ and the outputs of the time-shift operation labeled $q_1[n]$ and $q_2[n]$ at $n = n_i$, we can determine the output of the system for $n \geq n_i$. This suggests that we may choose $q_1[n]$ and $q_2[n]$ to form the state of the system at $n = n_i$. It can be seen from Fig. 2.23 that the state at $n = n_i+1$, formed by $q_1[n+1]$ and $q_2[n+1]$, can be obtained from the state at $n = n_i$ as follows:

$$\begin{cases} q_1[n+1] = -a_1 q_1[n] - a_2 q_2[n] + x[n] \\ q_2[n+1] = q_1[n] \end{cases} \tag{2.91}$$

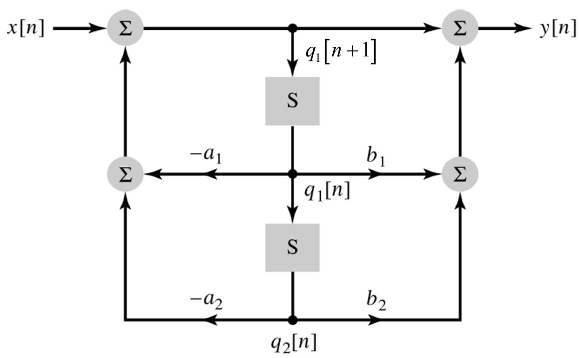


Figure 2.23 Direct form II representation of a second-order discrete-time LTI system depicting state variables $q_1[n]$ and $q_2[n]$.

Thus, the output of the system can be expressed by

$$\begin{aligned} y[n] &= x[n] - a_1 q_1[n] - a_2 q_2[n] + b_1 q_1[n] + b_2 q_2[n] \\ &= (b_1 - a_1) q_1[n] + (b_2 - a_2) q_2[n] + x[n] \end{aligned} \tag{2.92}$$

We can rewrite (2.91) and (2.92) in matrix-vector form as

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n] \tag{2.93}$$

$$\Rightarrow \mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \tag{2.94}$$

$$y[n] = \begin{bmatrix} b_1 - a_1 & b_2 - a_2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + x[n] \tag{2.95}$$

$$\Rightarrow y[n] = \mathbf{c}\mathbf{q}[n] + Dx[n] \tag{2.96}$$

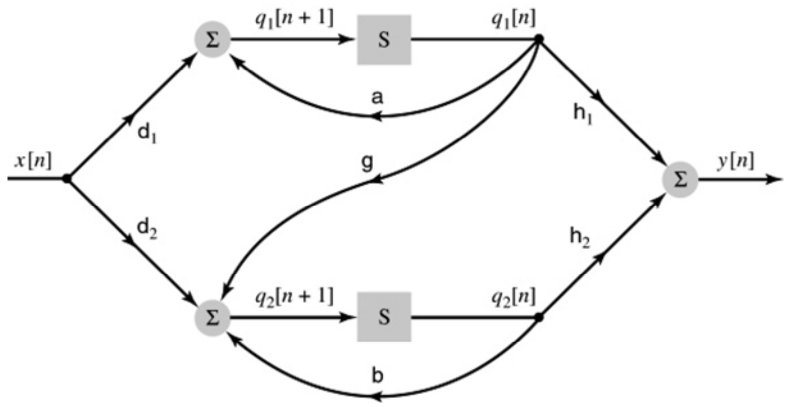
where

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} b_1 - a_1 & b_2 - a_2 \end{bmatrix}, \text{ and } D = 1. \tag{2.97}$$

Equations (2.94) and (2.96) are the general form of a state-variable description corresponding to a discrete-time system with $q_1[n]$ and $q_2[n]$ as state variables. Discrete-time systems with different internal structures will have different **A**'s, **B**'s, **c**'s, and D's in their state-variable descriptions. Such a system representation is useful for applications where the internal system structure needs to be considered.

3. If a discrete-time system is described by an Nth-order difference equation, then the corresponding state vector **q**[n] is $N \times 1$, **b** is $N \times 1$, **A** is $N \times N$, and **c** is $1 \times N$.

Example 2.40: The state-variable description of a second-order system



$$\begin{cases} q_1[n+1] = \alpha q_1[n] + \delta_1 x[n] \\ q_2[n+1] = \gamma q_1[n] + \beta q_2[n] + \delta_2 x[n] \end{cases}$$

$$y[n] = \eta_1 q_1[n] + \eta_2 q_2[n]$$ ■

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix}, \text{ and } D = 0.$$

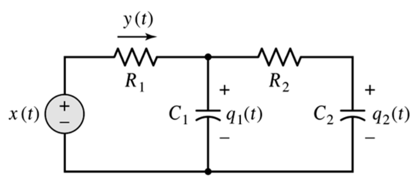
4. The state-variable description of continuous-time systems is analogous to that of discrete-time systems. Consider a continuous-time system with input $x(t)$, output $y(t)$, and state vector $\mathbf{q}(t)$ of dimension $N \times 1$. Then the corresponding state description (state-variable equations) can be expressed as follows:

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t). \tag{2.98}$$

$$y(t) = \mathbf{c}\mathbf{q}(t) + Dx(t). \tag{2.99}$$

where **A** is an $N \times N$ matrix, **b** is an $N \times 1$ vector , **c** is a $1 \times N$ vector, and D is a scalar. These parameters describe the internal structure of the continuous-time system.

Example 2.41: The state-variable description of an electrical circuit



Considering $x(t)$ as the input, $y(t)$ as the output, and $q_1(t)$ and $q_2(t)$ as the state variables, we can derive the following equations:

$$x(t) = y(t)R_1 + q_1(t) \Rightarrow y(t) = -\frac{1}{R_1}q_1(t) + \frac{1}{R_1}x(t)$$

The latter equation expresses the output $y(t)$ as a linear combination of the state variable $q_1(t)$ and the input $x(t)$. Let $i_2(t)$ be the current through R_2. Then we have

$$q_1(t) = R_2 i_2(t) + q_2(t) \Rightarrow i_2(t) = \frac{1}{R_2}q_1(t) - \frac{1}{R_2}q_2(t)$$

$$\because i_2(t) = C_2\frac{d}{dt}q_2(t)$$

$$\therefore \frac{d}{dt}q_2(t) = \frac{1}{R_2C_2}q_1(t) - \frac{1}{R_2C_2}q_2(t)$$

This is a state equation for $q_2(t)$. We need another state equation for $q_1(t)$. With $i_1(t)$ denoting the current through C_1, the following equations can be obtained:

$$i_1(t) = C_1\frac{d}{dt}q_1(t)$$

$$y(t) = i_1(t) + i_2(t)$$

$$-\frac{1}{R_1}q_1(t) + \frac{1}{R_1}x(t) = C_1\frac{d}{dt}q_1(t) + C_2\frac{d}{dt}q_2(t)$$

$$\Rightarrow \frac{d}{dt}q_1(t) = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)q_1(t) + \frac{1}{R_2C_1}q_2(t) + \frac{1}{R_1C_1}x(t)$$

This is state equation for $q_1(t)$. Accordingly, the corresponding state-variable description is as follows:

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$

$$y(t) = -\frac{1}{R_1}q_1(t) + \frac{1}{R_1}x(t)$$

with

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

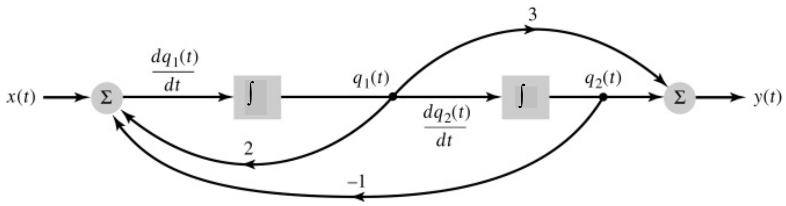
$$\mathbf{A} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} -\frac{1}{R_1} & 0 \end{bmatrix}, \text{ and } D = \frac{1}{R_1}$$

■

Example 2.42: The state-variable description from a block diagram.

In a block diagram representation of a continuous-time system, the state variables correspond to the outputs of the integrators. So the input to an integrator is derivative of the corresponding state variable.



$$\frac{d}{dt}q_1(t) = 2q_1(t) - q_2(t) + x(t)$$

$$\frac{d}{dt}q_2(t) = q_1(t)$$

$$y(t) = 3q_1(t) + q_2(t)$$

$$\therefore \mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} 3 & 1 \end{bmatrix},\ \text{and } D = 0$$

■

5. Transformations of the state

(1) There is no unique state-variable description of a system with a given input-output characteristic. Different state-variable descriptions may be obtained by transforming the state-variables.

Example 2.43: Consider Example 2.42 again. Let us define a set of new state variables by

$$q_1'(t) = q_2(t) \text{ and } q_2'(t) = q_1(t).$$

Then a state description different from that in Example 2.42 can be obtained, where corresponding parameters are as follows:

$$\mathbf{A}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},\ \mathbf{b}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathbf{c}' = \begin{bmatrix} 1 & 3 \end{bmatrix},\ \text{and } D' = 0.$$

■

It can be checked that the state vectors of Examples 2.42 and 2.43 have the following relationship:

$$\mathbf{q}' = \mathbf{T}\mathbf{q} \tag{2.100}$$

where **T** is referred to as a state-transformation matrix. In order for the new state to represent the entire system's memory, the relationship between **q** and **q′** must be one to one. This means that **T** must be a nonsingular matrix, i.e., $\mathbf{T}^{-1}$ exists and $\mathbf{q} = \mathbf{T}^{-1}\mathbf{q}'$.

(2) The original state-variable description is given by

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{A}\mathbf{q} + \mathbf{b}x \\ y &= \mathbf{c}\mathbf{q} + Dx \end{aligned} \tag{2.101}$$

where the dot over **q** denotes differentiation in continuous time or time advance ([n+1]) in discrete time. From (2.100), we have

$$\begin{aligned} \dot{\mathbf{q}}' &= \mathbf{T}\dot{\mathbf{q}} = \mathbf{T}\mathbf{A}\mathbf{q} + \mathbf{T}\mathbf{b}x = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{q}' + \mathbf{T}\mathbf{b}x = \mathbf{A}'\mathbf{q}' + \mathbf{b}'x \\ y &= \mathbf{c}\mathbf{q} + Dx = \mathbf{c}\mathbf{T}^{-1}\mathbf{q}' + Dx = \mathbf{c}'\mathbf{q}' + D'x \end{aligned} \tag{2.102}$$

where $\mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$, $\mathbf{b}' = \mathbf{T}\mathbf{b}$, $\mathbf{c}' = \mathbf{c}\mathbf{T}^{-1}$, and $D' = D$.

Example 2.44: Transforming the state

Consider that a discrete-time system has a state-variable description with the following parameters:

$$\mathbf{A} = \frac{1}{10}\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix},\ \mathbf{c} = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix},\ \text{and } D = 2.$$

Find the parameters $\mathbf{A}'$, $\mathbf{b}'$, $\mathbf{c}'$, and D' of the new state-variable description under the following state transformation:

$$\begin{cases} q_1'[n] = -\frac{1}{2}q_1[n] + \frac{1}{2}q_2[n] \\ q_2'[n] = \frac{1}{2}q_1[n] + \frac{1}{2}q_2[n] \end{cases}$$

It is clear that $\mathbf{q}' = \mathbf{T}\mathbf{q}$ with

$$\mathbf{T} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

$$\Rightarrow \det(\mathbf{T}) \neq 0 \Rightarrow \text{ nonsingular} \Rightarrow \mathbf{T}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix} -1/2 & 0 \\ 0 & 3/10 \end{bmatrix},\ \mathbf{b}' = \mathbf{T}\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\mathbf{c}' = \mathbf{c}\mathbf{T}^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix},\ \text{and } D = 2.$$

■

Note:

(a) $\mathbf{A}'$ is a diagonal matrix and thus separates the state update into two decoupled first-order difference equations given by

$$\begin{cases} q_1[n+1] = -\dfrac{1}{2}q_1[n] + x[n] \\ q_2[n+1] = \dfrac{3}{10}q_2[n] + 3x[n] \end{cases}.$$

(b) Different state-variable descriptions for a given system's input-output characteristic correspond to different ways of determining the LTI system output from the input.

(c) Powerful tools from linear algebra may be used to systematically study and design the internal structure of the system.

(d) The ability to transform the internal structure without changing the input-output characteristic of the system is advantageous to identify an implementation structure that optimizes a performance criterion not directly related to input-output behavior.

Appendix
Characteristic Equations of Differential and Difference Equations

1. Differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{2.A1}$$

The characteristic equation:

$$a_N r^N + a_{N-1} r^{N-1} + \cdots + a_1 r + a_0 = 0 \tag{2.A2}$$

The roots of this equation ($r_1, r_2, \ldots, r_N$) are called the characteristic roots.

Note:

- When the characteristic roots are all distinct, the homogeneous solution $y^{(h)}(t)$ will be

$$y^{(h)}(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_N e^{r_N t} \tag{2.A3}$$

- Suppose that r_1 is a *k*-multiple root of the characteristic equation. Then, corresponding to r_1, there will be *k* terms in the homogeneous solution:

$$c_1 t^{k-1} e^{r_1 t} + c_2 t^{k-2} e^{r_1 t} + \cdots + c_{k-1} t e^{r_1 t} + c_k e^{r_1 t} \tag{2.A4}$$

***Example* A1**: $\frac{d^3 y(t)}{dt^3} + 7\frac{d^2 y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 12 y(t) = x(t)$

Characteristic equation: $r^3 + 7r^2 + 16r + 12 = 0$

$\Rightarrow (r+2)^2 (r+3) = 0 \Rightarrow y_h(t) = c_1 t e^{-2t} + c_2 e^{-2t} + c_3 e^{-3t}$ ■

2. Difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{2.A5}$$

The characteristic equation:

$$a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-1} r + a_N = 0 \tag{2.A6}$$

The roots of this equation ($r_1, r_2, \ldots, r_N$) are also called the characteristic roots.

Note:

- When $r_1, r_2, \ldots, r_N$ are all distinct , the homogeneous solution $y^{(h)}[n]$ will be

$$y^{(h)}[n] = c_1 r_1^n + c_2 r_2^n + \cdots + c_N r_N^n \quad (2.A7)$$

- When the characteristic equation contains multiple roots, the homogeneous solution of a difference equation will be of slightly different form. Specifically, let r_1 be a k-multiple characteristic root; then its corresponding terms in the homogeneous solution are

$$c_1 n^{k-1} r_1^n + c_2 n^{k-2} r_1^n + \cdots + c_{k-1} n r_1^n + c_k r_1^n \quad (2.A8)$$

***Example* A2**: $y[n] + 6y[n-1] + 12y[n-1] + 8y[n-3] = x[n]$

Characteristic equation: $r^3 + 6r^2 + 12r + 8 = 0$

$$\Rightarrow (r+2)^3 = 0 \Rightarrow r = -2, -2, -2$$

$$\Rightarrow y^{(h)}[n] = \left(c_1 n^2 + c_2 n + c_3\right)(-2)^n$$

■

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