Chapter 2 Linear Time-Invariant Systems

2-1 Discrete-Time Linear Time-Invariant (LTI) Systems

1. The representation of discrete-time signals in terms of impulses

$$x[n]\delta[n]=x[0]\delta[n]\underset{\text{generalized}}{\Rightarrow}x[n]\delta[n-k]=x[k]\delta[n-k] \tag{2.1}$$

where $x[k]$ represents a specific value of the signal $x[n]$ at time k. Therefore, $x[n]$ can be expressed as the following weighted sum of time-shifted impulses:

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k] \tag{2.2}$$

Example 2.1:

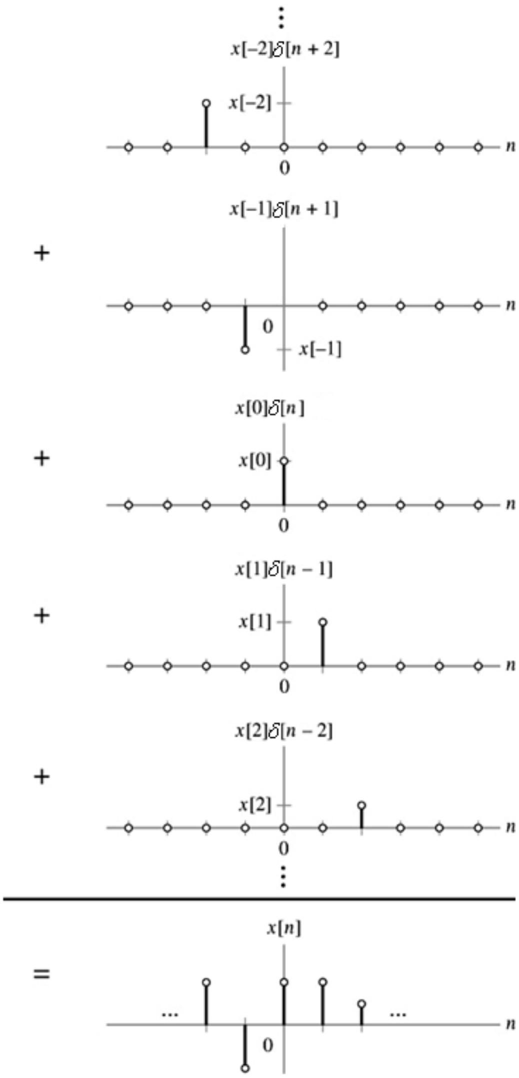

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Figure 2.1 A graphical example illustrating the representation of a signal $x[n]$ as a weighted sum of time-shifted impulses.

Example 2.2: $u[n]=\sum_{k=0}^{\infty}\delta[n-k]$ ■

2. The convolution-sum representation of discrete-time LTI systems

Let $h[n]=H\{\delta[n]\}$ be the impulse response of a discrete-time LTI system. Then $h[n-k]=H\{\delta[n-k]\}$ is the response of the system to the shifted unit sample $\delta[n-k]$. According to the linearity properties of LTI systems, we have

$$\begin{aligned} y[n]=H\{x[n]\}&=H\left\{\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]\right\} \\ &=\sum_{k=-\infty}^{\infty}H\{x[k]\delta[n-k]\} \\ &=\sum_{k=-\infty}^{\infty}x[k]\cdot H\{\delta[n-k]\} \end{aligned} \tag{2.3}$$

With the time-invariance property, (2.3) becomes

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]\cdot h[n-k] \tag{2.4}$$

This result is referred to as the convolution sum or superposition sum and will be represented symbolically as

$$y[n]=x[n]*h[n] \tag{2.5}$$

Interpretation of the convolution of two sequences:

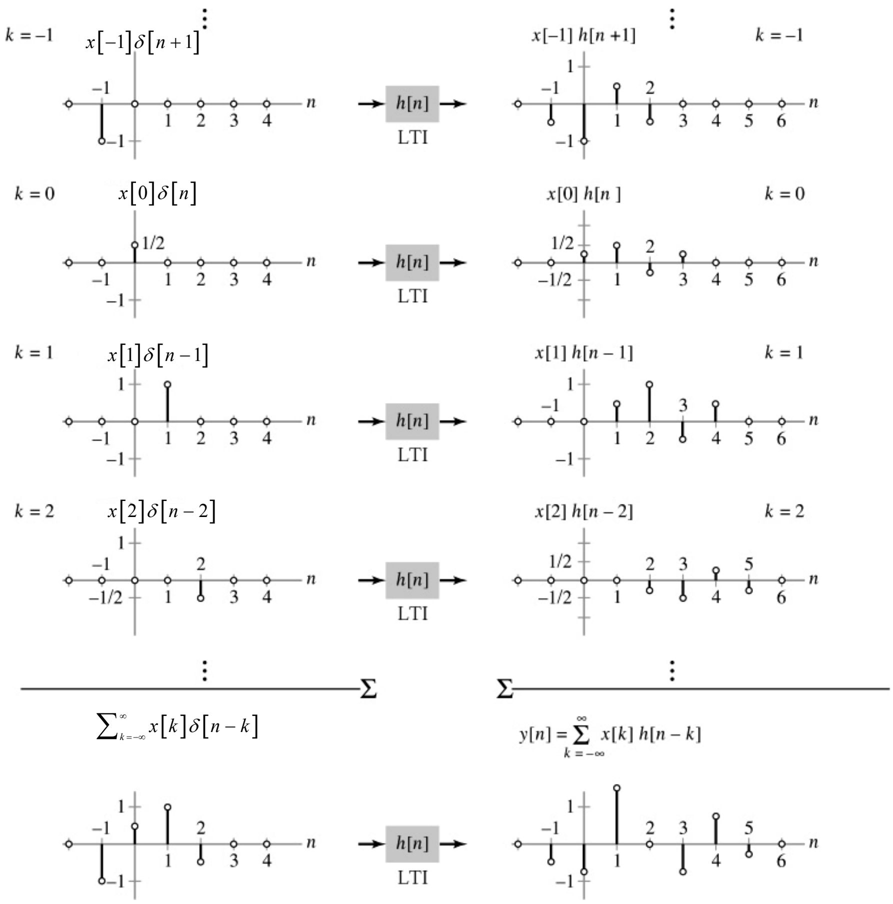

Figure 2.2 Illustration of the convolution sum of two sequences.

Example 2.3:

$$x[n]=\alpha^n u[n];\ h[n]=u[n]$$
$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$
$$x[k]h[n-k]=\begin{cases}\alpha^k\ ,\ 0\le k\le n\\ 0\ ,\ \text{otherwise}\end{cases}\ (\because n-k\ge 0,k\ge 0)$$

(1) For $n\ge 0$, $y[n]=\sum_{k=0}^{n}\alpha^k=\dfrac{1-\alpha^{n+1}}{1-\alpha}$.

(2) For $n<0$, $y[n]=0$.

$$\Rightarrow y[n]=\frac{1-\alpha^{n+1}}{1-\alpha}u[n]$$ ■

Example 2.4: Multipath communication channels

$$y[n]=x[n]+\frac{1}{2}x[n-1]$$

$$x[n]=\delta[n]\Rightarrow h[n]=\delta[n]+\frac{1}{2}\delta[n-1]$$

Determine the output of this system in response to the input

$$x[n]=2\delta[n]+4\delta[n-1]-2\delta[n-2]$$

$$\Rightarrow y[n]=2h[n]+4h[n-1]-2h[n-2]=2\delta[n]+5\delta[n-1]-\delta[n-3]$$ ■

3. When the sequences are of long duration, the convolution-sum procedure could be cumbersome. So we need to use a systematic approach to for such computation. Rewrite (2.4) as

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}w_n[k] \tag{2.6}$$

where $w_n[k]=x[k]h[n-k]$ is called the intermediate signal. In this equation, k is an independent variable and $h[n-k]=h[-(k-n)]$ is a reflected, time-shifted version of $h[k]$:

$$\begin{cases}n<0,\ h[n-k]=\text{time shift } h[-k] \text{ to the left.}\\ n>0,\ h[n-k]=\text{time shift } h[-k] \text{ to the right.}\end{cases}$$

Example 2.5: Convolution-sum evaluation by using an intermediate signal

$$h[n]=(3/4)^n u[n] \text{ and } x[n] = u[n]$$

Using the intermediate signal $w_n[k]=x[k]h[n-k]$, we can determine the output of the system at time $n = -5$, 5, and 10 as follows:

$$h[n-k]=(3/4)^{n-k} u[n-k]$$

$$n=-5,\ w_{-5}=x[k]h[-5-k]=u[k](3/4)^{-5-k} u[-5-k]=0$$

$$\therefore y[-5]=0$$

$$n=5,\ w_5=x[k]h[5-k]=\begin{cases}(3/4)^{5-k} & ,0\le k\le 5\\ 0 & ,\text{ otherwise}\end{cases}$$

$$\therefore y[5]=\sum_{k=0}^{5}(3/4)^{5-k}=\frac{1-(3/4)^6}{1-3/4}=3.288$$

$$n=10,\ w_{10}=x[k]h[10-k]=\begin{cases}(3/4)^{10-k} & ,0\le k\le 10\\ 0 & ,\text{ otherwise}\end{cases}$$

$$\therefore y[10]=\sum_{k=0}^{10}(3/4)^{10-k}=\frac{1-(3/4)^{11}}{1-3/4}=3.831$$

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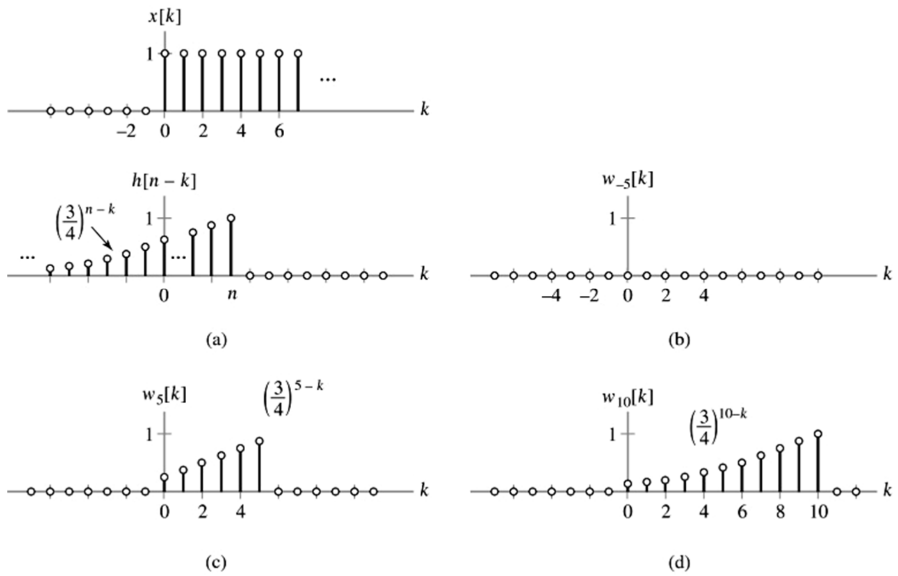


Figure 2.3 Evaluation of the convolution sum for Example 2.5. (a) $x[k]$ and $h[n-k]$, depicted as a function of k. (b) $w_{-5}[k]$. (c) $w_5[k]$. (d) $w_{10}[k]$.

Example 2.6: MA systems

$$y[n]=\frac{1}{4}\sum_{k=0}^{3}x[n-k]\Rightarrow x[n]=\delta[n]\Rightarrow h[n]=\frac{1}{4}\left(u[n]-u[n-4]\right)$$

Determine the output of the system when the input is defined as

$$x[n]=u[n]-u[n-10]$$

$$n<0,\ w_n[k]=0,\ \text{for all } k\Rightarrow y[n]=0$$

$$0\le n\le 3,\ w_n[k]=\begin{cases}1/4 &,0\le k\le n\\ 0 &,\text{otherwise}\end{cases}\Rightarrow y[n]=\sum_{k=0}^{n}1/4=(n+1)/4$$

$$3<n\le 9,\ w_n[k]=\begin{cases}1/4 &,n-3\le k\le n\\ 0 &,\text{otherwise}\end{cases}\Rightarrow y[n]=\sum_{k=n-3}^{n}1/4=1$$ ■

$$9<n\le 12,\ w_n[k]=\begin{cases}1/4 &,n-3\le k\le 9\\ 0 &,\text{otherwise}\end{cases}\Rightarrow y[n]=\sum_{k=n-3}^{9}1/4=(13-n)/4$$

$$12<n,\ w_n[k]=0,\ \text{for all } k\Rightarrow y[n]=0$$

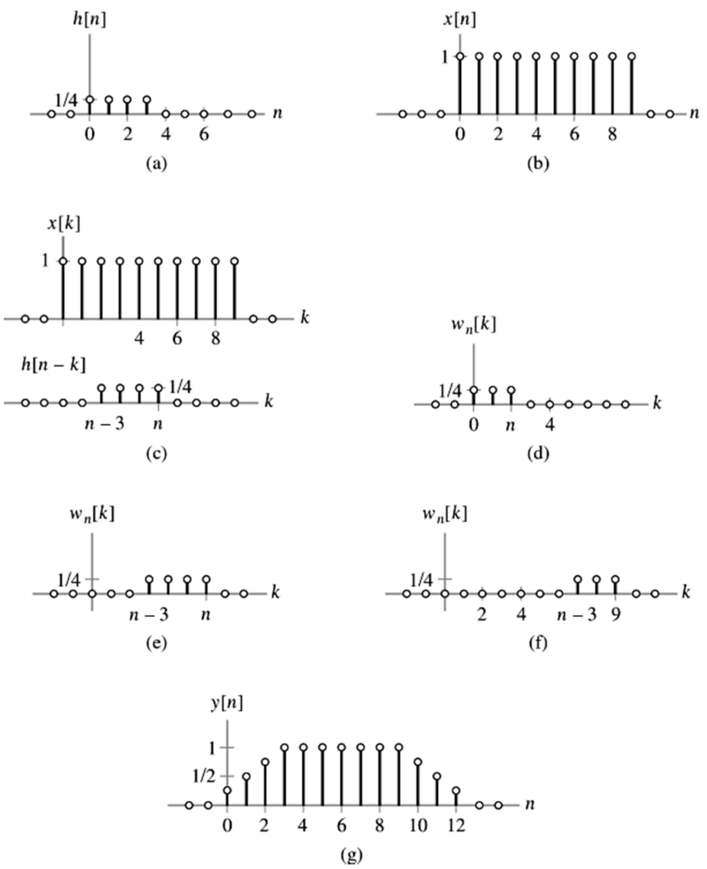


Figure 2.4 Evaluation of the convolution sum for Example 2.6. (a) $h[n]$. (b) $x[n]$. (c) $x[k]$ and $h[n-k]$, depicted as a function of k. (d) $w_n[k]$ for $0 \le n \le 3$. (e) $w_n[k]$ for $3 < n \le 9$. (f) $w_n[k]$ for $9 < n \le 12$. (g) $y[n]$.

Example 2.7:

$$x[n]=\begin{cases}1\ ,\ 0\le n\le 4\\0,\ \text{otherwise}\end{cases}\quad\text{and}\quad h[n]=\begin{cases}\alpha^{n}\ ,\ 0\le n\le 6\\0\ ,\ \text{otherwise}\end{cases}$$

$$\Rightarrow h[n-k]=\begin{cases}\alpha^{n-k},\ n-6\le k\le n\\0\quad,\ \text{otherwise}\end{cases}$$

Interval 1: $n<0$

$$w_n[k]=x[k]h[n-k]=0\Rightarrow y[n]=0$$

Interval 2: $0\le n\le 4$

$$w_n[k]=\begin{cases}\alpha^{n-k}\ ,0\le k\le n\\0\ ,\ \text{otherwise}\end{cases}\Rightarrow y[n]=\sum_{k=0}^{n}\alpha^{n-k}=\frac{1-\alpha^{n+1}}{1-\alpha}$$

Interval 3: $4<n\le 6$

$$w_n[k]=\begin{cases}\alpha^{n-k}\ ,0\le k\le 4\\0\ ,\ \text{otherwise}\end{cases}\Rightarrow y[n]=\sum_{k=0}^{4}\alpha^{n-k}=\frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$$

Interval 4: $6<n\le 10$

$$w_n[k]=\begin{cases}\alpha^{n-k}\ ,n-6\le k\le 4\\0\ ,\ \text{otherwise}\end{cases}\Rightarrow y[n]=\sum_{k=n-6}^{4}\alpha^{n-k}=\frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$$

Interval 5: $10<n$, $w_n[k]=0\Rightarrow y[n]=0$ ■

4. Basic properties of convolution

(1) Commutative property

$$x[n]*h[n]=h[n]*x[n] \tag{2.7}$$

$$x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{r=-\infty}^{\infty}x[n-r]h[r]=h[n]*x[n] \tag{2.8}$$

(2) Associative property

$$x[n]*(h_1[n]*h_2[n])=(x[n]*h_1[n])*h_2[n] \tag{2.9}$$

Interpretation of the associative property

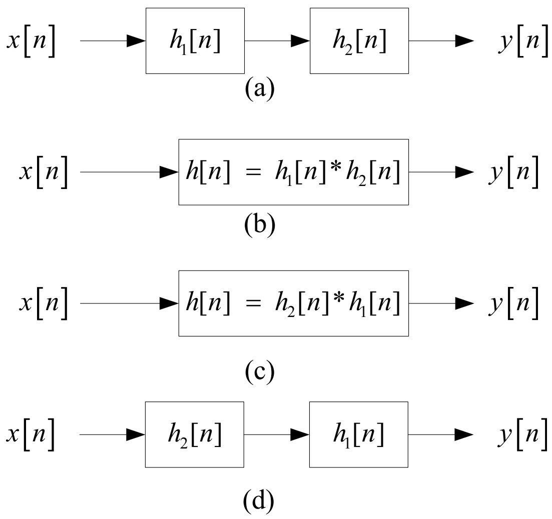


Figure 2.5 The associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

(3) Distributive property

$$x[n]*\left(h_1[n]+h_2[n]\right)=x[n]*h_1[n]+x[n]*h_2[n] \tag{2.10}$$

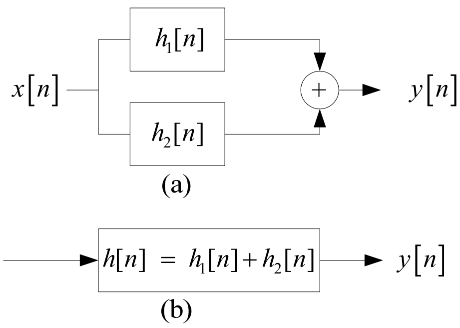


Figure 2.6 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

Note:

- The convolution-sum formula implies that the unit impulse response completely characterizes the behavior of an LTI system.
- The unit impulse response of a nonlinear system does not completely characterize the behavior of the system.

Example 2.8:

$$h[n]=\begin{cases}1 \ , \ n=0,1 \\ 0, \text{ otherwise}\end{cases}$$

(a) LTI systems:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= h[0]x[n] + h[1]x[n-1] \end{aligned}$$

There is exactly one LTI system with $h[n]$ as its impulse response.

(b) Nonlinear systems:

$$\left.\begin{aligned} y[n] &= \left(x[n]+x[n-1]\right)^2 \\ y[n] &= \max\left(x[n],x[n-1]\right) \end{aligned}\right\} \text{ with the same impulse response}$$

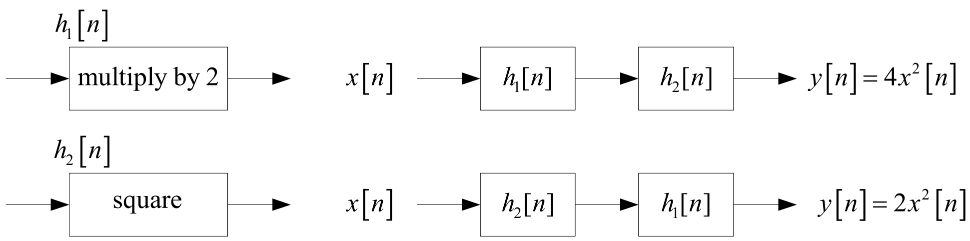
Let $x[n]=\delta[n]$, then $y[n]=h[n]$.

$$\begin{aligned} y[-1] &= \max\left(x[-1],x[-2]\right)=0 \\ y[0] &= \max\left(x[0],x[-1]\right)=1 \\ y[1] &= \max\left(x[1],x[0]\right)=1 \\ y[2] &= \max\left(x[2],x[1]\right)=0 \\ y[3] &= \max\left(x[3],x[2]\right)=0 \\ &\vdots \end{aligned}$$

There would be many nonlinear systems with the same response to the unit impulse input.

(c) It is not true in general that the order in which nonlinear systems are cascaded can be changed without changing the overall response.

Example 2.9:



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2-2 Continuous-Time LTI Systems

1. The representation of continuous-time signals in terms of impulses

   Staircase approximation:

$$\delta_\Delta(t)=\begin{cases}1/\Delta\ , & 0<t<\Delta\\ 0\ , & \text{otherwise}\end{cases},\ \Delta\delta_\Delta(t)=1\ \ (\text{area}=1) \tag{2.11}$$

$$\hat{x}(t)=\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \tag{2.12}$$

$$\begin{aligned}x(t)&=\lim_{\Delta\to 0}\hat{x}(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_\Delta(t-k\Delta)\Delta\\ &=\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau\end{aligned} \tag{2.13}$$

- $x(t)$ equals the limit as $\Delta\to 0$ of the area under $x(\tau)\delta_\Delta(t-\tau)$

$$\delta_\Delta(t-\tau)\xrightarrow{\Delta\to 0}\delta(t-\tau) \tag{2.14}$$

- $\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau=\int_{-\infty}^{\infty}x(t)\delta(t-\tau)d\tau=x(t)\int_{-\infty}^{\infty}\delta(t-\tau)d\tau=x(t)$

$$(\because x(\tau)\delta(t-\tau)=x(t)\delta(t-\tau))$$

Example 2.10:

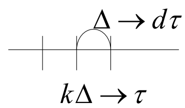
$$u(t)=\int_{-\infty}^{\infty}u(\tau)\delta(t-\tau)d\tau=\int_{0}^{\infty}\delta(t-\tau)d\tau$$ ■

2. The convolution-integral representation of continuous-time LTI systems

   From (2.13), the output $y(t)$ of an LTI system corresponding to the input $x(t)$ can be expressed as

$$\begin{aligned}y(t)&=\lim_{\Delta\to 0}H\left\{\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_\Delta(t-k\Delta)\Delta\right\}\\ &=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)H\{\delta_\Delta(t-k\Delta)\}\Delta\ \ (\because \text{linear property})\end{aligned} \tag{2-15}$$

   where $H\{\delta_\Delta(t-k\Delta)\}$ is defined as the response of the LTI system to the input $\delta_\Delta(t-k\Delta)$. As $\Delta\to 0$, $H\{\delta_\Delta(t-k\Delta)\}\to H\{\delta(t-k\Delta)\}$. Thus we have



$$y(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)H\{\delta(t-k\Delta)\}\Delta \tag{2-16}$$

$$\Rightarrow y(t)=\int_{-\infty}^{\infty}x(\tau)H\{\delta(t-\tau)\}d\tau \tag{2-17}$$

Let $H\{\delta(t)\} = h(t)$ be the impulse response of the LTI system. Then

$$H\{\delta(t-\tau)\} = h(t-\tau) \text{ } (\because \text{time-invariance property}) \tag{2-18}$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} w_t(\tau)d\tau \tag{2-19}$$

where $w_t(\tau) = x(\tau)h(t-\tau)$ is called the intermediate signal. This result is referred to as the convolution integral or convolution of *x*(*t*) and *h*(*t*). The convolution operation will be represented symbolically as

$$y(t) = x(t) * h(t) \tag{2-20}$$

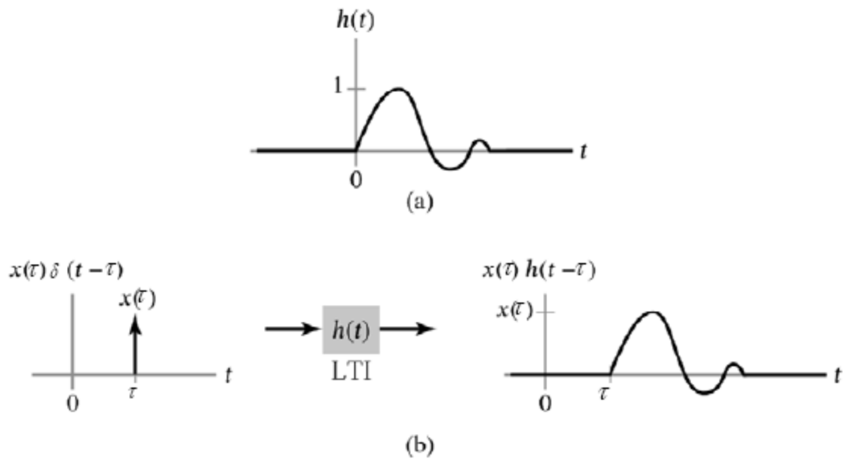


Figure 2.7 (a) Impulse response of an LTI system *H*. (b) The output of an LTI system to a time-shifted and amplitude-scaled impulse is a time-shifted and amplitude-scaled impulse response.

3. Properties of the continuous-time convolution

(1) Commutativity

$$x(t) * h(t) = h(t) * x(t) \tag{2-21}$$

The roles of input signal and impulse response are interchangeable.

(2) Associativity

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t) \tag{2-22}$$

A cascade combination of LTI systems can be condensed into a single system whose impulse response is the convolution of the individual impulse responses.

(3) Distributivity

$$x(t)*\left[h_1(t)+h_2(t)\right]=\left[x(t)*h_1(t)\right]+\left[x(t)*h_2(t)\right] \tag{2-23}$$

A parallel combination of LTI systems is equivalent to a single system whose impulse response is the sum of the individual impulse response in the parallel configuration.

Note:

- The overall impulse response of a cascade of two nonlinear systems (or even linear but time-varying system) does depend upon the order in which the systems are cascaded.
- A nonlinear continuous-time system is not completely described by its unit impulse response.

Example 2.11:

Let $x(t)=e^{-at}u(t)$ and $h(t)=u(t)$, where $a>0$.

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau \Rightarrow w_t(\tau)=x(\tau)h(t-\tau)=\begin{cases} e^{-a\tau} & , 0<\tau<t \\ 0 & , \text{otherwise}\end{cases}$$

For $t>0$, $y(t)=\int_0^t e^{-a\tau}d\tau=\frac{1}{a}\left(1-e^{-at}\right)$

For $t<0$, $y(t)=0$

$$y(t)=\frac{1}{a}\left(1-e^{-at}\right)u(t)$$ ■

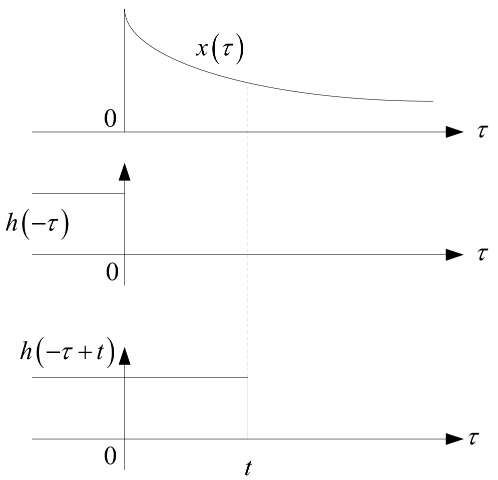


Figure 2.8 Evaluation of the convolution integral for Example 2.11.

Example 2.12:

$$x(t)=\begin{cases}1\ ,\ 0<t<T\\0,\ \text{otherwise}\end{cases}\quad\text{and}\quad h(t)=\begin{cases}t\ ,\ 0<t<2T\\0,\ \text{otherwise}\end{cases}$$

$t<0,\ y(t)=0$

$0<t<T,\ 0<\tau<t$

$$w_t(\tau)=x(\tau)h(t-\tau)=t-\tau\Rightarrow y(t)=\int_0^t(t-\tau)d\tau=t^2-\frac{1}{2}t^2=\frac{1}{2}t^2$$

$T<t<2T,\ 0<\tau<T$

$$y(t)=\int_0^T(t-\tau)d\tau=Tt-\frac{1}{2}T^2$$

$2T<t<3T,\ t-2T<\tau<T$

$$y(t)=\int_{t-2T}^T(t-\tau)d\tau=-\frac{1}{2}t^2+Tt+\frac{3}{2}T^2$$

$3T<t,\ \ y(t)=0$

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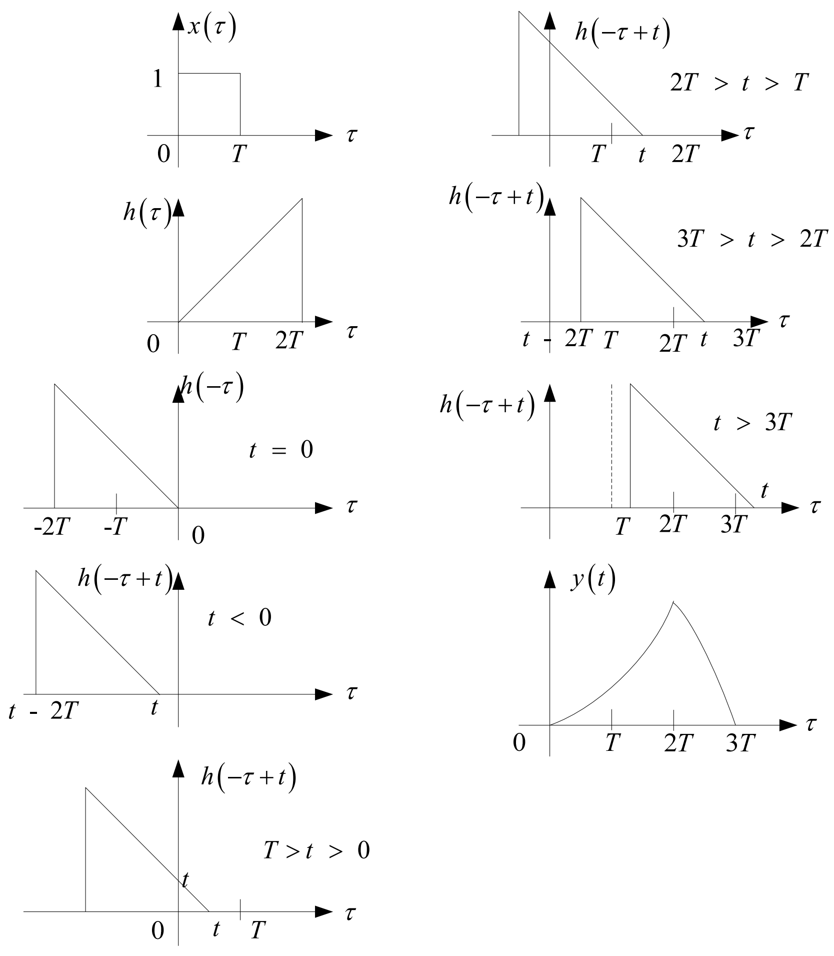

Figure 2.9 Evaluation of the convolution integral for Example 2.12.

Example 2.13:

$$x(t)=u(t-1)-u(t-3) \text{ and } h(t)=u(t)-u(t-2)$$

Evaluate the convolution integral for a system with $x(t)$ and $h(t)$.

$$t<1,\ w_t(\tau)=0 \Rightarrow y(t)=0$$

$$1\le t<3,\ w_t(\tau)=\begin{cases}1\ ,\ 1<\tau<t\\ 0,\ \text{otherwise}\end{cases} \Rightarrow y(t)=t-1$$

$$3\le t<5,\ w_t(\tau)=\begin{cases}1,\ t-2<\tau<3\\ 0\ ,\ \text{otherwise}\end{cases} \Rightarrow y(t)=5-t$$

$$5\le t,\ w_t(\tau)=0 \Rightarrow y(t)=0$$

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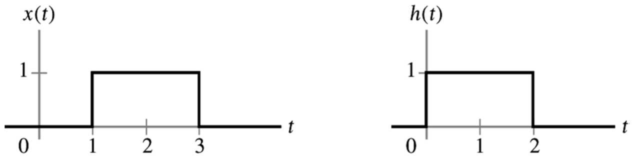

Figure 2.10 The input signal $x(t)$ and the impulse response $h(t)$ for Example 2.13.

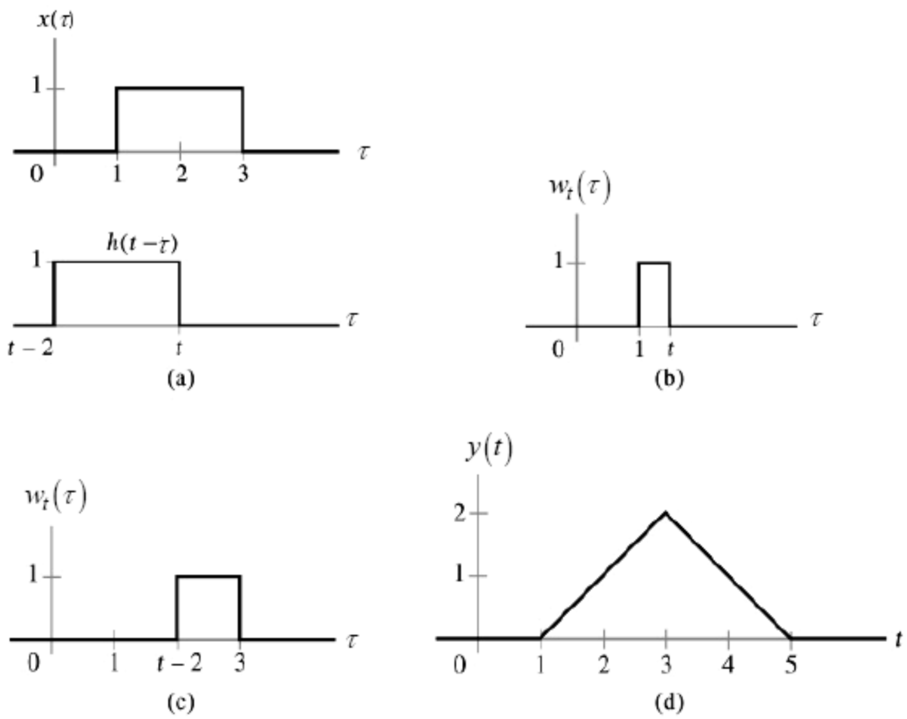

Figure 2.11 Evaluation of the convolution integral for Example 2.13.

Example 2.14: An RC circuit's output

Assume RC = 1 sec. $h(t)=e^{-t}u(t)$ and $x(t)=u(t)-u(t-2)$.

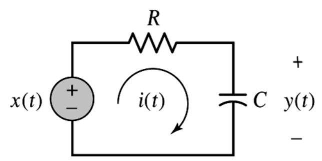

Figure 2.12 An RC circuit with the voltage source *x*(*t*) as input and the voltage measured across the capacitor, *y*(*t*), as output.

$$h(t-\tau)=e^{-(t-\tau)}u(t-\tau)$$
$$t<0,\ w_t(\tau)=0 \Rightarrow y(t)=0$$
$$0\le t<2,\ w_t(\tau)=\begin{cases} e^{-(t-\tau)} & , 0<\tau<t \\ 0 & , \text{otherwise} \end{cases} \Rightarrow y(t)=\int_0^t e^{-(t-\tau)}d\tau = 1-e^{-t}$$
$$2\le t,\ w_t(\tau)=\begin{cases} e^{-(t-\tau)} & , 0<\tau<2 \\ 0 & , \text{otherwise} \end{cases} \Rightarrow y(t)=\int_0^2 e^{-(t-\tau)}d\tau = (e^2-1)e^{-t}$$

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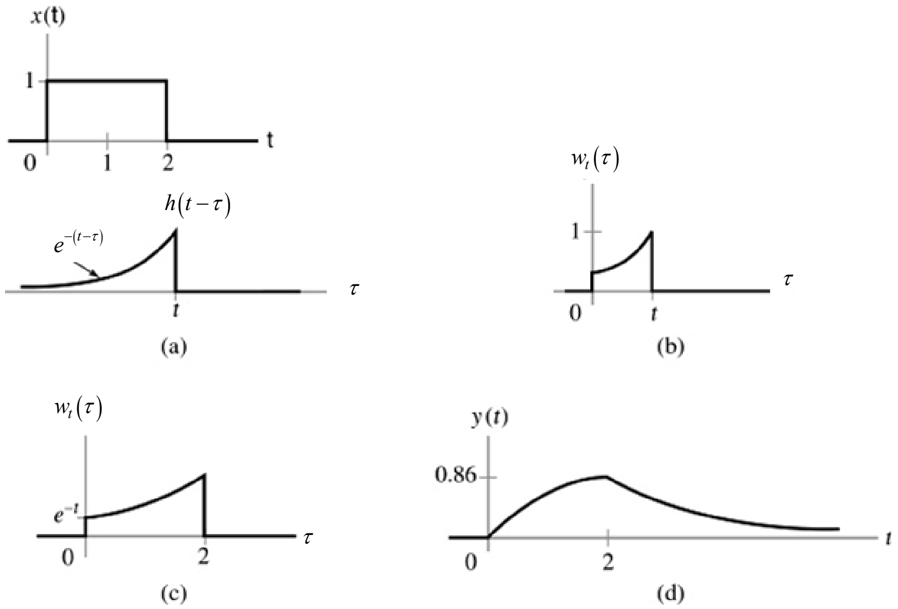


Figure 2.13 Evaluation of the convolution integral for Example 2.14.

Example 2.15: Radar range measurement: propagation model

$$x(t)=\begin{cases} \sin(\omega_c t) & , 0\le t\le T_0 \\ 0 & , \text{otherwise} \end{cases} \quad \text{and} \quad h(t)=a\delta(t-\beta)$$

where *a* represents the attenuation factor and β the round-trip time delay.

$$h(\tau) = a\delta(\tau - \beta) \Rightarrow h(-\tau) = a\delta(-\tau - \beta) = a\delta(-(\tau + \beta)) = a\delta(\tau + \beta) \quad (\because \text{ even symmetry})$$

$$h(t - \tau) = a\delta(\tau - (t - \beta)) \qquad \blacksquare$$

$$r(t) = \int_{-\infty}^{\infty} x(\tau) a\delta(\tau - (t - \beta)) d\tau = ax(t - \beta)$$

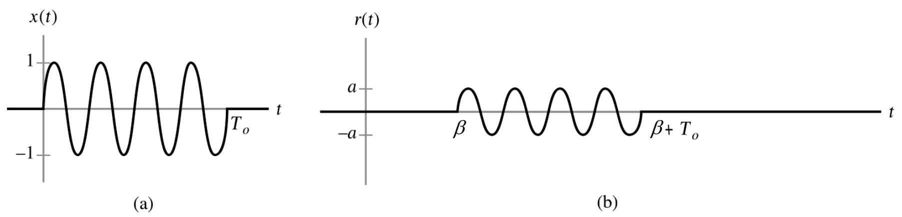
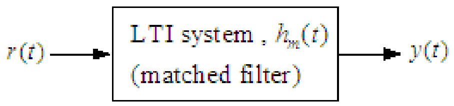


Figure 2.14 Radar range measurement. (a) Transmitted RF pulse. (b) The received echo is an attenuated and delayed version of the transmitted pulse.

Example 2.16: Radar range measurement: the matched filter



$$h_m(t) = x(-t) = \begin{cases} -\sin(\omega_c t), & -T_0 \le t \le 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$w_t(\tau) = r(\tau) h_m(t - \tau) = r(\tau) x(\tau - t)$$

$$t + T_0 < \beta \Rightarrow t < \beta - T_0, \; w_t(\tau) = 0 \Rightarrow y(t) = 0$$

$$\beta \le t + T_0 < \beta + T_0 \Rightarrow \beta - T_0 < t \le \beta,$$

$$w_t(\tau) = \begin{cases} a\sin(\omega_c(\tau - \beta))\sin(\omega_c(\tau - t)), & \beta < \tau < t + T_0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$\begin{aligned}
\Rightarrow \; y(t) &= \int_{\beta}^{t+T_0} a\sin(\omega_c(\tau - \beta))\sin(\omega_c(\tau - t)) d\tau \\
&= \frac{a}{2}\int_{\beta}^{t+T_0} \left[\cos(\omega_c(t - \beta)) - \cos(\omega_c(2\tau - \beta - t))\right] d\tau \\
&= \frac{a}{2}\cos(\omega_c(t - \beta))[t + T_0 - \beta] + \frac{a}{4\omega_c}\sin(\omega_c(2\tau - \beta - t))\Big|_{\beta}^{t+T_0} \\
&= \frac{a}{2}\cos(\omega_c(t - \beta))[t + T_0 - \beta] \\
&\quad + \frac{a}{4\omega_c}\left[\sin(\omega_c(t + 2T_0 - \beta)) - \sin(\omega_c(\beta - t))\right] \\
&\approx \frac{a}{2}\cos(\omega_c(t - \beta))[t + T_0 - \beta] \quad (\because \omega_c > 10^6 \text{ rad/s})
\end{aligned}$$

$\beta < t < \beta + T_0$,

$$w_t(\tau) = \begin{cases} a\sin\left(\omega_c(\tau-\beta)\right)\sin\left(\omega_c(\tau-t)\right), & t < \tau < \beta + T_0 \\ 0 & \text{, otherwise} \end{cases}$$

$$\Rightarrow \begin{aligned} y(t) &= \int_t^{\beta+T_0} a\sin\left(\omega_c(\tau-\beta)\right)\sin\left(\omega_c(\tau-t)\right)d\tau \\ &\approx \frac{a}{2}\cos\left(\omega_c(t-\beta)\right)\left[\beta+T_0-t\right] \end{aligned}$$

■

$\beta + T_0 < t$, $w_t(\tau) = 0 \Rightarrow y(t) = 0$

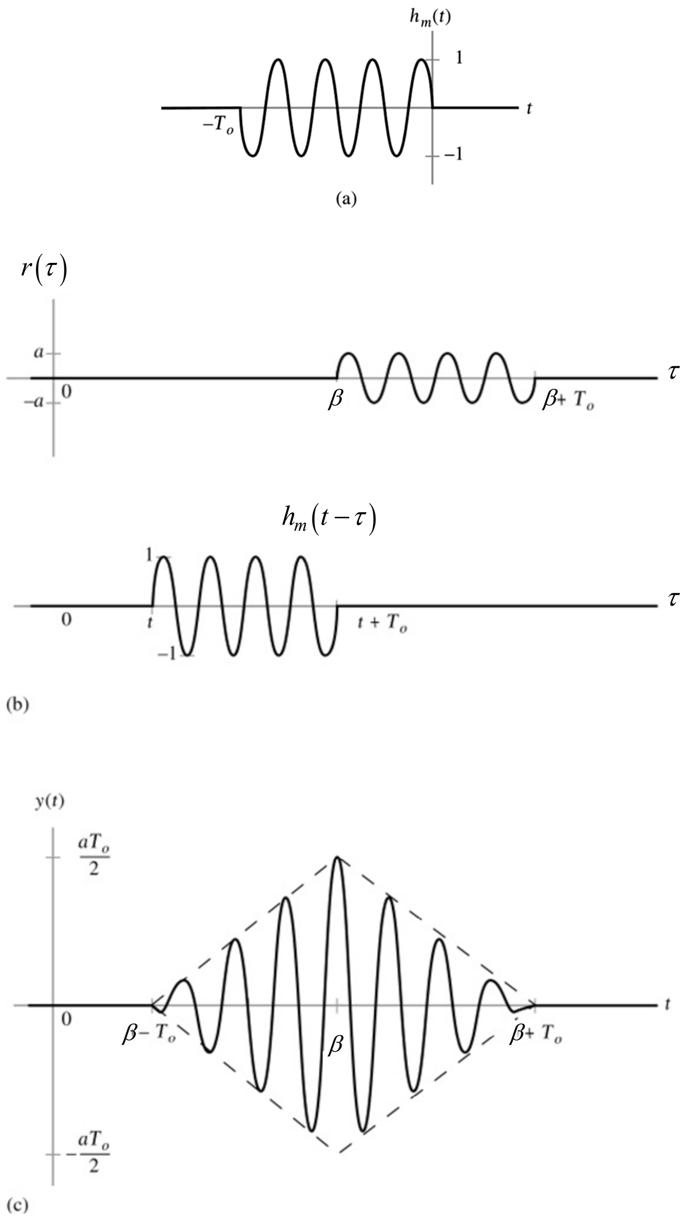


Figure 2.15 (a) The impulse response of the matched filter. (b) The received signal $r(\tau)$ superimposed on the reflected and time-shifted matched filter impulse response $h_m(t-\tau)$, depicted as function of τ. (c) The matched filter output $y(t)$.

2-3 Properties of Linear Time-Invariant Systems

1. LTI systems with or without memory

(1) Discrete-time memoryless systems: $y[n]$ depends only on $x[n]$.

$$\Rightarrow h[n] = 0 \text{ for } n \neq 0 \tag{2-24}$$

$$\Rightarrow h[n] = c\delta[n],\ c = h[0] \tag{2-25}$$

$$\Rightarrow y[n] = cx[n] \tag{2-26}$$

If a discrete-time LTI system has an impulse response $h[n]$ which is not identically zero for $n \neq 0$, then the system has memory.

Example 2.17: $y[n] = x[n] + x[n-1]$

$$h[n] = \begin{cases} 1 & , n = 0,1 \\ 0, & \text{otherwise} \end{cases}$$ ■

(2) Continuous-time memoryless systems:

$$\begin{cases} h(t) = 0 \text{ for } t \neq 0 \\ y(t) = cx(t) \Rightarrow h(t) = c\delta(t) \end{cases} \tag{2-27}$$

$h(t) \neq 0$ for some nonzero value of $t \Rightarrow$ a "memory" system.

If c = 1, then the convolution sum and convolution integral formulas of memoryless LTI systems imply that

$$\begin{aligned} x[n] &= x[n] * \delta[n] \\ x(t) &= x(t) * \delta(t) \end{aligned} \tag{2-28}$$

2. Invertibility of LTI systems

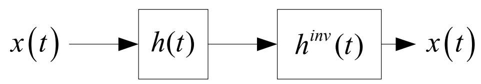


Figure 2.16 Concept of an inverse system for continuous-time LTI systems.

$$\begin{cases} h(t) * h^{inv}(t) = \delta(t) \\ h[n] * h^{inv}[n] = \delta[n] \end{cases} \tag{2-29}$$

The process of recovering $x(t)$ from $h(t) * x(t)$ is termed *deconvolution*, since it corresponds to recovering or undoing the convolution operation.

Example 2.18:

$$y(t) = x(t - t_0)$$

$$\begin{cases} h(t) = \delta(t - t_0) \\ x(t - t_0) = x(t) * \delta(t - t_0) \ \ (\text{delay by } t_0) \end{cases}$$

$$\begin{cases} h^{inv}(t) = \delta(t + t_0) \\ x(t + t_0) = x(t) * \delta(t + t_0) \ \ (\text{advance by } t_0) \end{cases}$$

$$\begin{aligned} h(t) * h^{inv}(t) &= \delta(t - t_0) * \delta(t + t_0) \\ &= \int_{-\infty}^{\infty} \delta(\tau - t_0)\delta(t - \tau + t_0)\, d\tau \\ &= \delta(t) \int_{-\infty}^{\infty} \delta(\tau - t_0)\, d\tau = \delta(t) \end{aligned}$$

■

Example 2.19: Multipath communication channels: compensation by means of an inverse system

$$\begin{aligned} & y[n] = x[n] + ax[n-1] \\ \Rightarrow\ & h[n] = \delta[n] + a\delta[n-1] \end{aligned}$$

Find a causal and stable inverse system that recovers $x[n]$ from $y[n]$.

$$h[n] * h^{inv}[n] = \delta[n]$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} h[k] h^{inv}[n-k] = h^{inv}[n] + ah^{inv}[n-1] = \delta[n]$$

$$\text{For } n < 0,\ h^{inv}[n] = 0 \quad (\because \text{causal})$$

$$\text{For } n = 0,\ h^{inv}[0] + ah^{inv}[-1] = 1 \Rightarrow h^{inv}[0] = 1$$

$$\text{For } n > 0,\ h^{inv}[n] + ah^{inv}[n-1] = 0 \Rightarrow h^{inv}[n] = -ah^{inv}[n-1]$$

$$\Rightarrow h^{inv}[1] = -a,\ h^{inv}[2] = a^2,\ h^{inv}[3] = -a^3, \ldots$$

$$\therefore h^{inv}[n] = (-a)^n u[n]$$

$$\sum_{k=-\infty}^{\infty} \left| h^{inv}[k] \right| = \sum_{k=-\infty}^{\infty} |a|^k < \infty \text{ when } |a| < 1$$

■

3. Causality for LTI systems

   The output of a causal system depends only on the present and past values of the input.

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \Rightarrow y[n] = \sum_{k=-\infty}^{n} x[k] h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \Rightarrow y[n] = \sum_{k=0}^{\infty} h[k] x[n-k] \end{aligned} \qquad (2\text{-}30)$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \Rightarrow y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \Rightarrow y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned} \tag{2-31}$$

$$\Rightarrow \begin{cases} h[n] = 0 \text{ for } n < 0 \\ h(t) = 0 \text{ for } t < 0 \end{cases} \tag{2-32}$$

Example 2.20:

$$\left.\begin{aligned} h[n] &= u[n] \\ h[n] &= \delta[n] - \delta[n-1] \end{aligned}\right\} \Rightarrow \text{causal}$$

$h(t) = \delta(t - t_0)$ is causal for $t_0 \geq 0$ and noncausal for $t_0 < 0$. ■

4. Stability for LTI systems

BIBO stability: bounded input $\rightarrow$ bounded output

(1) The impulse response $h[n]$ is absolutely summable, i.e., $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.
$\Leftrightarrow$ The discrete-time system is BIBO stable.

(a) $\sum_{k=-\infty}^{\infty} |h[k]| < \infty \Rightarrow$ BIBO stable

Consider $|x[n]| \leq M_x$ for all n

$$\begin{aligned} |y[n]| &= \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \quad (\because |a+b| \leq |a|+|b|) \\ &\leq M_x \sum_{k=-\infty}^{\infty} |h[k]| \quad \text{for all } n \end{aligned} \tag{2-33}$$

Thus, if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, then the system is BIBO stable.

(b) BIBO stable $\Rightarrow$ $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

$\equiv$ $\sum_{k=-\infty}^{\infty} |h[k]| \rightarrow \infty \Rightarrow$ not BIBO stable (i.e., there exists a bounded input that can generate an unbounded output.)

The system output at index $n = n_0$ for the input $x[n]$ is

$$y[n_0] = \sum_{k=-\infty}^{\infty} h[k]x[n_0 - k] \tag{2-34}$$

Consider a bounded input of the form

$$x[n] = \pm B_1,\ \text{for all } n \tag{2-35}$$

and let $x[n_0 - k] = \text{sign}(h[k])B_1$, then

$$y[n_0] = B_1 \sum_{k=-\infty}^{\infty} |h[k]| \ \left(\because \text{sign}(h[k])h[k] = |h[k]|\right) \tag{2-36}$$

$$\because \sum_{k=-\infty}^{\infty} |h[k]| \to \infty \quad \therefore y[n_0] \to \infty$$

Therefore, "$h[n]$ is absolutely summable" is a sufficient and necessary condition to guarantee the BIBO stability of a discrete-time LTI system.

(2) The impulse response $h(t)$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$.
$\Leftrightarrow$ The continuous-time system is BIBO stable.

(a) $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \ \Rightarrow$ BIBO stable

Consider $|x(t)| \le M_x$ for all t

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \\ &\le \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)| d\tau \\ &\le M_x \int_{-\infty}^{\infty} |h(\tau)| d\tau \end{aligned} \tag{2-37}$$

Thus, if $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$, then the system is BIBO stable.

(b) BIBO stable $\Rightarrow \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

$\equiv \int_{-\infty}^{\infty} |h(\tau)| d\tau \to \infty \ \Rightarrow$ not BIBO stable (i.e., there exists a bounded input that can generate an unbounded output.)

The system output at time $t = t_0$ for the input $x(t)$ is

$$y(t_0) = \int_{-\infty}^{\infty} h(\tau)x(t_0 - \tau)d\tau \tag{2-38}$$

Consider a bounded input of the form

$$x(t) = \pm B_2,\ \text{for all } t \tag{2-39}$$

and let $x(t_0-\tau)=\operatorname{sign}(h(\tau))B_2$, then

$$y(t_0)=B_2\int_{-\infty}^{\infty}|h(\tau)|d\tau \ \left(\because \operatorname{sign}(h(\tau))h(\tau)=|h(\tau)|\right) \tag{2-40}$$

$$\because \int_{-\infty}^{\infty}|h(\tau)|d\tau\to\infty \quad \therefore\ y(t_0)\to\infty$$

The system is BIBO stable if and only if the impulse response is absolutely integrable given as

$$\int_{-\infty}^{\infty}|h(\tau)|\,d\tau<\infty \tag{2-41}$$

Example 2.21: $h[n]=\delta[n-n_0]$

$$\sum_{n=-\infty}^{\infty}|h[n]|=\sum_{n=-\infty}^{\infty}|\delta[n-n_0]|=1\Rightarrow \text{ stable}$$
$$h[n]=u[n]\Rightarrow y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{n}x[k]\Rightarrow \text{accumulator}$$
$$\sum_{k=-\infty}^{\infty}|u[n]|=\sum_{k=0}^{\infty}|u[n]|=\infty\Rightarrow \text{ unstable}$$

■

Example 2.22: Properties of the first-order recursive system

$$y[n]=\rho y[n-1]+x[n],\ |\rho|<1$$

The impulse response of the system is $h[n]=\rho^n u[n]$.

$$\begin{cases} h[n]=0 \text{ for } n<0\Rightarrow \text{ The system is causal.} \\ h[n]\neq 0 \text{ for } n>0\Rightarrow \text{ The system is with memory} \\ \sum_{k=-\infty}^{\infty}|h[k]|=\sum_{k=-\infty}^{\infty}|\rho^k|=\sum_{k=-\infty}^{\infty}|\rho|^k<\infty\Rightarrow \text{ The system is not BIBO stable.} \end{cases}$$

■

5. The unit step response of an LTI system

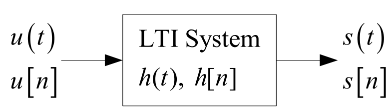


Figure 2.17 The step response of an LTI system *H*.

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k] \tag{2-42}$$

$$h[n] = s[n] - s[n-1] \tag{2-43}$$

$$s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \tag{2-44}$$

$$h(t) = \frac{d}{dt} s(t) = s'(t) \tag{2-45}$$

In both continuous-time and discrete-time cases, the unit step response also completely characterizes the behavior of an LTI system.

Example 2.23: An RC circuit

From chapter 1, the impulse response of an RC circuit is $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$. Accordingly, the corresponding step response can be determined as follows:

$$s(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-\tau/RC} u(\tau) d\tau = \begin{cases} \frac{1}{RC} \int_{0}^{t} e^{-\tau/RC} u(\tau) d\tau, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$= \begin{cases} 1 - e^{-t/RC}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

■

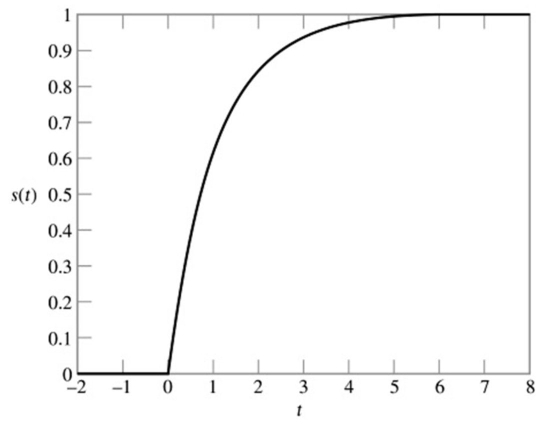


Figure 2.18 The step response of an RC circuit with $RC = 1$.