Chapter 1 Fundamentals of Signals and Systems

1-1 Signals

1. Information in a signal is contained in a pattern of variations of some form.
   Example 1.1: The human vocal mechanism produces speech by creating fluctuations in acoustic pressure. ■

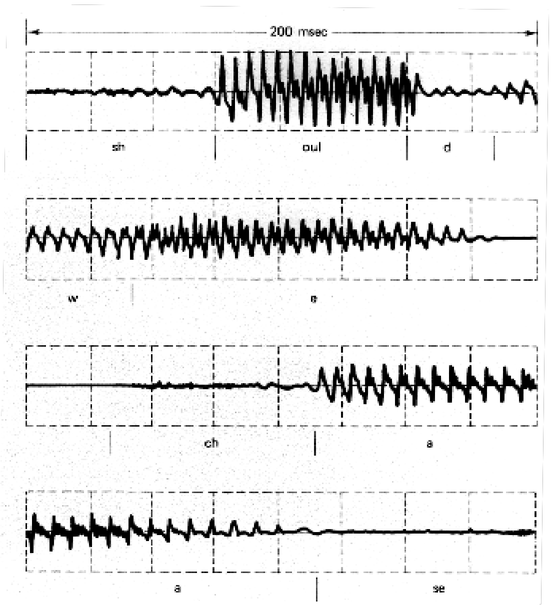

Figure 1.1 Example of a recording of speech [1].

Example 1.2: Monochromatic picture: variation in brightness. ■

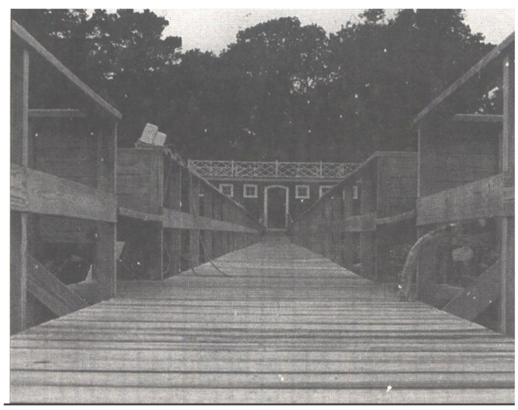

Figure 1.2 A monochromatic picture [1].

2. Signals are represented mathematically as functions of one or more independent variables that convey information on the nature of a physical phenomenon.
   Example 1.3: Speech signal $\rightarrow$ acoustic pressure: function of time (one-dimensional). ■
   Example 1.4: Picture $\rightarrow$ brightness: function of two spatial variables (two-dimensional). ■

Note: For convenience, we will generally refer to the independent variable as time, although it may not in fact represent time in specific applications.

3. Classification of signals

Five methods of classifying signals:

(1) Continuous-time and discrete-time signals

(a) A signal $x(t)$ is said to be a continuous-time signal if it is defined for all time t. The amplitude or value varies continuously with time, e.g., speech signal.

(b) A discrete-time signal is defined only at discrete instants of time. Thus, the independent variable has discrete values only, which are usually uniformly spaced, e.g., stock market index $x[n]$.

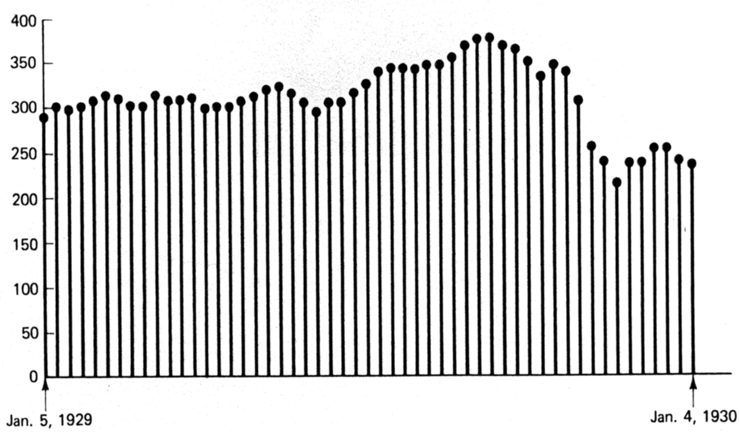

Figure 1.3 An example of a discrete-time signal: the weekly Dow-Jones stock market index from January 5, 1929 to January 4, 1930 [1].

(c) Digital signal: discrete-time and discrete-state signal

(d) Analog signal: continuous-time and continuous-state signal

Note:

- If the signal amplitude is continuous, the signal is called "continuous-state" signal; otherwise, it is called "discrete-state" signal.
- A discrete-time signal is often referred to as a discrete-time sequence.
- continuous-time signal $\xrightarrow{\text{sampling}}$ discrete-time signal

(2) Even and odd signals

(a) Even: $x(t) = x(-t)$ for all $t \rightarrow$ symmetric about vertical axis

(b) Odd: $x(-t) = -x(t)$ for all $t \rightarrow$ anti-symmetric about vertical axis

Example 1.5:

$$x(t)=\begin{cases}\sin\left(\dfrac{\pi t}{T}\right), & -T\le t\le T\\ 0, & \text{otherwise}\end{cases}$$

$$x(-t)=\sin\left(-\frac{\pi t}{T}\right)=-\sin\left(\frac{\pi t}{T}\right)=-x(t)\Rightarrow \text{odd signal}$$

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(c) Any signal can be broken into a sum of an odd signal and an even signal

$$x(t)=x_e(t)+x_o(t) \tag{1.1}$$

where $x_e(t)$ and $x_o(t)$ mean even and odd signals, respectively.

$$\because\ x_e(t)=x_e(-t)\ \text{and}\ x_o(t)=-x_o(-t) \tag{1.2a}$$

$$\therefore\ x(-t)=x_e(-t)+x_o(-t)=x_e(t)-x_o(t) \tag{1.2b}$$

$$\Rightarrow x_e(t)=\frac{1}{2}\left[x(t)+x(-t)\right]\ \text{and}\ x_o(t)=\frac{1}{2}\left[x(t)-x(-t)\right] \tag{1.2c}$$

Example 1.6:

$$x(t)=e^{-2t}\cos(t)$$
$$x(-t)=e^{2t}\cos(-t)=e^{2t}\cos(t)$$

$$\therefore\begin{cases}x_e(t)=\dfrac{1}{2}\left[e^{-2t}\cos(t)+e^{2t}\cos(t)\right]=\cosh(2t)\cos(t)\\ x_o(t)=-\sinh(2t)\cos(t)\end{cases}$$

■

(d) A complex-valued signal $x(t)$ is said to be conjugate symmetric if

$$x(-t)=x^*(t) \tag{1.3}$$

Let $x(t)=a(t)+jb(t)$. Then $x^*(t)=a(t)-jb(t)$.

$$\begin{aligned}\Rightarrow x(-t)&=a(-t)+jb(-t)\\ &=a(t)-jb(t)=x^*(t)\\ \Rightarrow \underbrace{a(-t)=a(t)}_{even}&,\ \underbrace{b(-t)=-b(t)}_{odd}\end{aligned} \tag{1.4}$$

(3) Periodic signals and aperiodic signals

(a) Periodic signal: $x(t+T) = x(t)$ for all t, where T is a positive constant. $T = T_0,\ 2T_0,\ 3T_0, \ldots$

- $T=T_0$: fundamental period
- $1/T$: fundamental frequency, $f = 1/T$ Hz or cycles/sec
- $\omega = 2\pi f = 2\pi/T$: angular frequency (radians/sec)

Note: $x(t)$ is a constant

- ➢ The fundamental period is undefined.
- ➢ The fundamental frequency is defined to be zero.

(b) $x[n] = x[n+N]$ for integer n, where N is a positive integer.

- fundamental period: smallest N (samples)
- fundamental angular frequency: $\Omega=2\pi/N$ (radians or radians/sample)

(4) Deterministic signals and random signals

(a) A deterministic signal is a signal about which there is no uncertainty with respect to its value at any time.

(b) A random signal is a signal about which there is uncertainty before it occurs.

(5) Energy signals and power signals

(a) The instantaneous power dissipated in the resistor R is defined by

$$\begin{aligned} p(t) &= v^2(t)/R = i^2(t)\cdot R \\ &= v^2(t) = i^2(t),\ R = 1 \text{ ohm} \end{aligned} \tag{1.5}$$

We may express the instantaneous power of the signal as

$$p(t) = x^2(t) \tag{1.6}$$

the total energy of the non-periodic continuous-time signal $x(t)$ as

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\,dt = \int_{-\infty}^{\infty} x^2(t)\,dt \tag{1.7}$$

and its time-averaged, or average, power as

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt \\ &\left(= \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt, \text{ for periodic signal}\right) \end{aligned} \tag{1.8}$$

$\sqrt{P}$ means root mean-squared (rms) value of the periodic signal $x(t)$.

(b) For a non-periodic discrete-time signal $x[n]$, the total energy is defined by

$$E = \sum_{n=-\infty}^{\infty} x^2[n] \tag{1.9}$$

and its average power is defined by

$$P = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N+1}^{N} x^2[n]$$
$$(= \frac{1}{N} \sum_{n=0}^{N-1} x^2[n], \text{ for periodic signals}) \tag{1.10}$$

(c) A signal is referred to as an energy signal if and only if the total energy of the signal satisfies the condition

$$0 < E < \infty \tag{1.11}$$

A signal is referred to as a power signal if and only if the average power of the signal satisfies the condition

$$0 < P < \infty \tag{1.12}$$

(d) An energy signal has zero time-average power and a power signal has infinite energy. They are mutually exclusive.

Note:

- Periodic signals and random signals are usually viewed as power signals, whereas signals that are both deterministic and non-periodic are usually viewed as energy signals.
- Signals that satisfy neither property are referred to as neither energy signals nor power signals [3].

1-2 Basic Operations on Signals

1. Operations performed on dependent variables
   (1) Amplitude scaling

$$y(t) = cx(t) \tag{1.13}$$

where c is a scaling factor.

$$y[n] = cx[n] \tag{1.14}$$

   (2) Addition, e.g., mixer

$$y(t) = x_1(t) + x_2(t) \tag{1.15}$$

$$y[n] = x_1[n] + x_2[n] \tag{1.16}$$

(3) Multiplication, e.g., amplitude modulation (AM) radio signal

$$y(t) = x_1(t)x_2(t) \tag{1.17}$$

$$y[n] = x_1[n]x_2[n] \tag{1.18}$$

(4) Differentiation: $\frac{d}{dt}x(t)$, e.g., inductor, $v(t) = L\frac{d}{dt}i(t)$.

(5) Integration: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$, e.g., capacitor, $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$.

2. Operations performed on the independent variable

(1) Time scaling:

$$y(t) = x(at) \tag{1.19}$$

$$\begin{cases} a > 1 \Rightarrow y(t) \text{ is a compressed version of } x(t). \\ 0 < a < 1 \Rightarrow y(t) \text{ is an expanded (stretched) version of } x(t). \end{cases}$$

$$y[n] = x[kn],\ k > 0 \tag{1.20}$$

$k > 1 \Rightarrow$ some values of $x[n]$ are lost.

(2) Reflection:

$$y(t) = x(-t) \tag{1.21}$$

$y(t)$ represents a reflected version of $x(t)$ about $t = 0$. An even signal is the same as its reflected version. An odd signal is the negative of its reflected version.

(3) Time shifting:

$$y(t) = x(t - t_0) \tag{1.22}$$

$$\begin{cases} t_0 > 0 \Rightarrow y(t) \text{ is obtained by shifting } x(t) \text{ toward the right.} \\ t_0 < 0 \Rightarrow y(t) \text{ is obtained by shifting } x(t) \text{ toward the left.} \end{cases}$$

$$y[n] = x[n - m] \tag{1.23}$$

where the shift m must be a positive or negative integer.

3. Precedence rule for time shifting and time scaling

   Let $y(t)$ is derived from another signal $x(t)$ through a combination of time shifting and time scaling; that is,

$$y(t) = x(at - b) \tag{1.24}$$

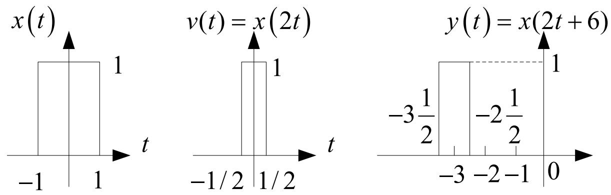
   To obtain $y(t)$ from $x(t)$, the time-shifting and time-scaling operations must be performed in the correct order: time-shifting $\rightarrow$ time-scaling

   (1) The time-shifting operation always replaces t by $t–b$.
   (2) The scaling operation always replaces t by at.

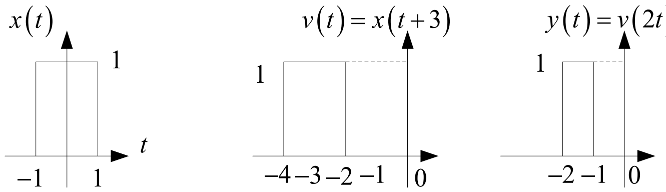
$$\begin{aligned} v(t) &= x(t - b) \\ y(t) &= v(at) = x(at - b) \end{aligned} \tag{1.25}$$

Example 1.7: $y(t) = x(2t + 3)$

Time scaling $\rightarrow$ time shifting: $y(t) = v(t + 3) = x(2(t + 3)) \neq x(2t + 3)$



Time shifting $\rightarrow$ time scaling: $y(t) = v(2t) = x(2t + 3)$



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1-3 Basic Continuous-Time Signals

1. Complex exponential signal

$$x(t) = Be^{at} \tag{1.26}$$

   Sinusoidal signal

$$x(t) = A\cos(\omega t + \phi) \tag{1.27}$$

(1) B and a are real $\Rightarrow$ real exponential signal

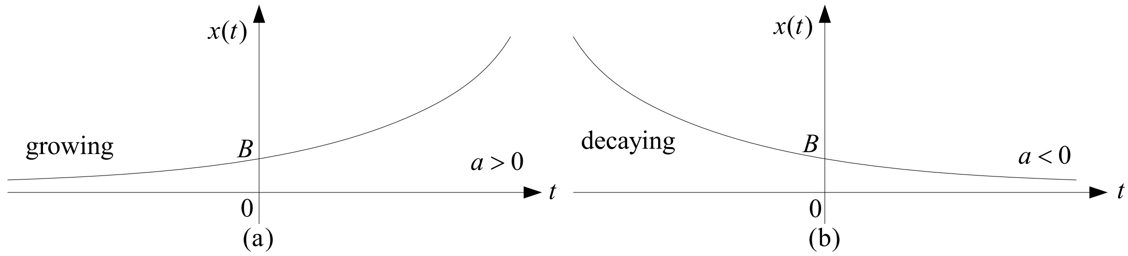
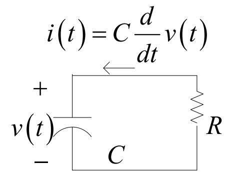


Figure 1.4 Continuous-time real exponential $x(t) = Be^{at}$: (a) $a > 0$; (b) $a < 0$ [1].

Example 1.8: The operation of the capacitor for $t \geq 0$

$$RC\frac{d}{dt}v(t) + v(t) = 0$$

$$\Rightarrow v(t) = V_0 e^{-t/RC}$$



where V_0 denotes the initial value of the voltage developed across the capacitor.

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(2) B is real and a is pure imaginary $\Rightarrow$ periodic complex exponential

$$x(t) = Be^{j\omega_0 t},\ a = j\omega_0\text{: periodic} \tag{1.28}$$

$$e^{j\omega_0 t} = e^{j\omega_0(t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T} \tag{1.29}$$

$\Rightarrow e^{j\omega_0 T} = 1 \Rightarrow$ the fundamental period is $T_0 = \dfrac{2\pi}{|\omega_0|}$.

Euler's relation:

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t \tag{1.30}$$

$$e^{j(\omega_0 t+\phi)} = \cos(\omega_0 t+\phi) + j\sin(\omega_0 t+\phi) \tag{1.31}$$

$$e^{-j(\omega_0 t+\phi)} = \cos(\omega_0 t+\phi) - j\sin(\omega_0 t+\phi) \tag{1.32}$$

$$\cos(\omega_0 t+\phi) = \frac{1}{2}\left[e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right] \text{ or } \cos(\omega_0 t+\phi) = \text{Re}\left\{e^{j(\omega_0 t+\phi)}\right\} \tag{1.33}$$

where Re{·} denotes the real part of the complex quantity enclosed inside the braces.

$$\therefore A\cos(\omega_0 t+\phi) = \text{Re}\left\{Be^{j\omega_0 t}\right\},\ B = Ae^{j\phi} \tag{1.34}$$

Note:

- Fundamental period = T_0
- Fundamental frequency = $\omega_0 = 2\pi/T_0$
- The fundamental frequency of a constant signal is zero.
- Harmonically related complex exponentials:

$$\phi_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{1.35}$$

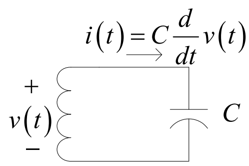
$k = 0 \Rightarrow \phi_k$ is a constant

$k \neq 0 \Rightarrow \phi_k$ is periodic with fundamental period $2\pi/(|k|\omega_0)$ or fundamental frequency $|k|\omega_0$. $\phi_k(t)$ has a common period of $2\pi/\omega_0$.

Example 1.9:

$$LC\frac{d^2}{dt^2}v(t) + v(t) = 0$$

$$\Rightarrow v(t) = V_0\cos(\omega_0 t),\ t \geq 0$$



where $\omega_0 = 1/\sqrt{LC}$ is the natural angular frequency of oscillation of the circuit.

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(3) B is complex and a is complex: general complex exponential function

$$B = |B|e^{j\theta} \text{ and } a = r + j\omega_0 \tag{1.36}$$

$$\begin{aligned} Be^{at} &= |B|e^{j\theta}e^{(r+j\omega_0)t} = |B|e^{rt} \cdot e^{j(\omega_0 t + \theta)} \\ &= |B|e^{rt}\cos(\omega_0 t + \theta) + j|B|e^{rt}\sin(\omega_0 t + \theta) \\ &= |B|e^{rt}\cos(\omega_0 t + \theta) + j|B|e^{rt}\cos(\omega_0 t + \theta - \pi/2) \end{aligned} \tag{1.37}$$

$r = 0 \Rightarrow$ the real and imaginary parts are sinusoidal.

$r > 0 \Rightarrow$ the real and imaginary parts are sinusoidal signals multiplied by a growing exponential.

$r < 0 \Rightarrow$ the real and imaginary parts are sinusoidal signals multiplied by a decaying exponential.

2. The continuous-time unit-step function

$$u(t)=\begin{cases}0, t<0\\ 1, t>0\end{cases} \quad (1.38)$$
$$(t=0,\ \text{undefined})$$

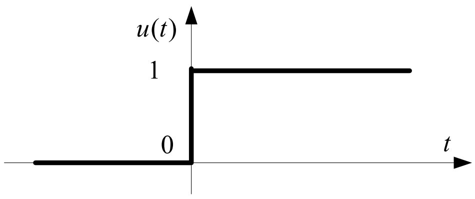

Figure 1.5 Continuous-time version of the unit-step function of unit amplitude.

Example 1.10: Rectangular pulse

$$x(t)=\begin{cases}A, 0\le |t|<0.5\\ 0\quad, |t|>0.5\end{cases}$$
$$x(t)=Au(t+0.5)-Au(t-0.5)$$

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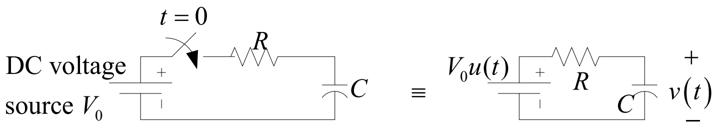
Example 1.11: RC circuit



$$\left.\begin{aligned}v(0)&=0\\ v(\infty)&=V_0\end{aligned}\right\}\Rightarrow v(t)=V_0\left(1-e^{-t/RC}\right)u(t)$$

■

3. The continuous-time unit impulse function

   (1) The continuous-time version of the unit impulse is defined by the following pair of relations:

$$\delta(t)=0 \text{ for } t\neq 0 \text{ and } \int_{-\infty}^{\infty}\delta(t)\,dt=1 \quad (1.39)$$

   The impulse $\delta(t)$ is also referred to as the *Dirac delta function*.

(2) The impulse and the unit-step function are related to each other in that if we are given either one, we can uniquely determine the other.

$$\delta(t) \Rightarrow u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \tag{1.40}$$

$$\delta(t) = \frac{du(t)}{dt} \quad \text{(in a restricted sense)} \tag{1.41}$$

$u(t)$ is discontinuous at $t = 0$. $\Rightarrow$ not differentiable at $t = 0$

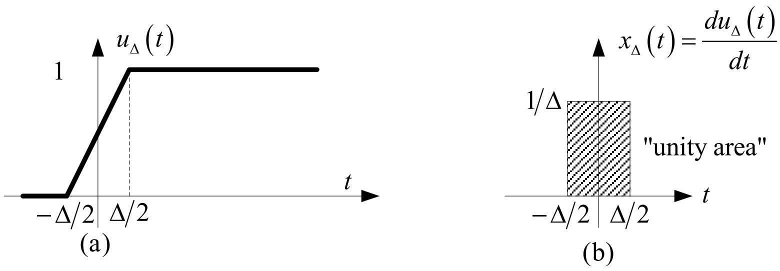


Figure 1.6 (a) Continuous approximation to the unit step; (b) derivative of $u_\Delta(t)$ [1].

$$\Rightarrow \delta(t) = \lim_{\Delta \to 0} x_\Delta(t) \tag{1.42}$$

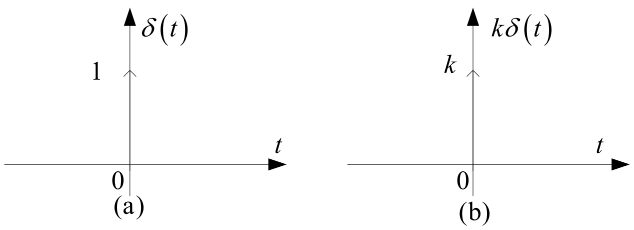
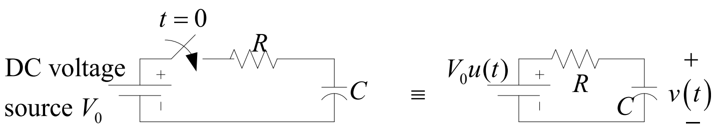


Figure 1.7 (a) Unit impulse; (b) scaled impulse [1].

“The height of the arrow used to depict the scaled impulse will be chosen to be representative of its area.”

Example 1.12: RC circuit (continued)



$$\because v(t) = V_0 u(t)$$

$$\therefore i(t) = C\frac{d}{dt}v(t) = CV_0\delta(t)$$

■

(3) Graphical interpretation of

$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau \tag{1.43}$$

Alternative interpretation:

$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau \underset{(\tau=t-\sigma\Rightarrow d\tau=-d\sigma)}{=} \int_{\infty}^{0}\delta(t-\sigma)(-d\sigma)=\int_{0}^{\infty}\delta(t-\sigma)d\sigma \tag{1.44}$$

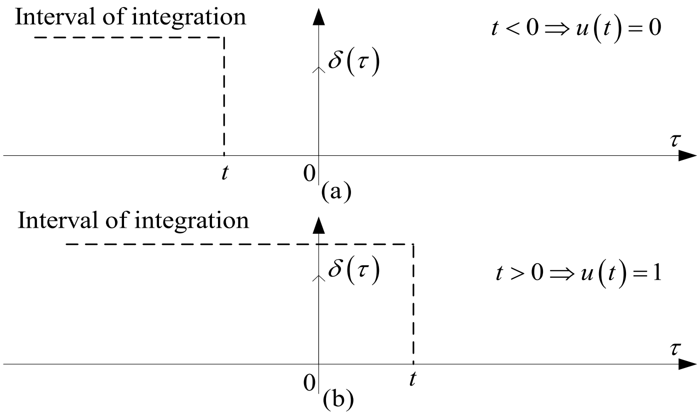


Figure 1.8 Running integral given in (1.43): (a) $t<0$; (b) $t>0$.

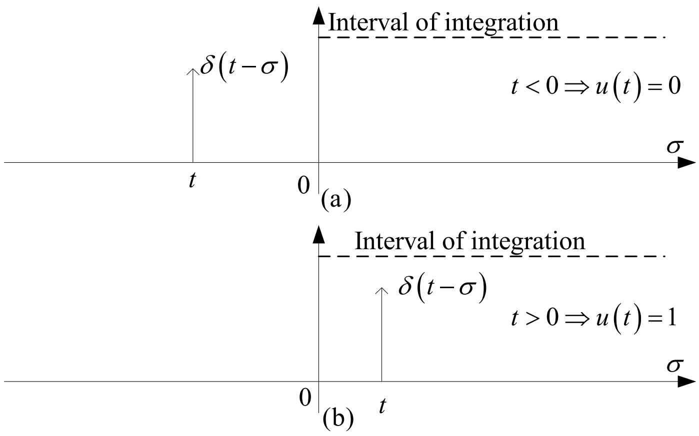


Figure 1.9 Relationship given in (1.44): (a) $t<0$; (b) $t>0$.

(4) Equivalence property: Product of $x(t)$ and $\delta(t)$

$$x_1(t)=x(t)x_\Delta(t)\approx x(0)x_\Delta(t) \text{ and } \lim_{\Delta\to 0}x_\Delta(t)=\delta(t)$$
$$\Rightarrow x(t)\delta(t)=x(0)\delta(t) \tag{1.45}$$
$$x(t)\delta(t-t_0)=x(t_0)\delta(t-t_0)$$

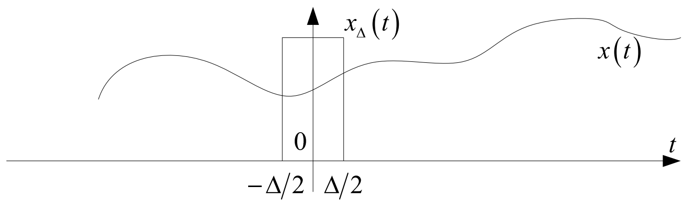


Figure 1.10 The product $x(t)x_\Delta(t)$ [1].

(5) Shifting property: $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = \left[\int_{-\infty}^{\infty} \delta(t-t_0)dt\right]x(t_0) = x(t_0) \tag{1.46}$$

It is assumed that $x(t)$ is continuous at time $t = t_0$.

(6) Time-scaling property:

$$\delta(at) = \frac{1}{a}\delta(t) \tag{1.47}$$

$$\lim_{\Delta \to 0} x_\Delta(at) = \frac{1}{a}\delta(t) \tag{1.48}$$

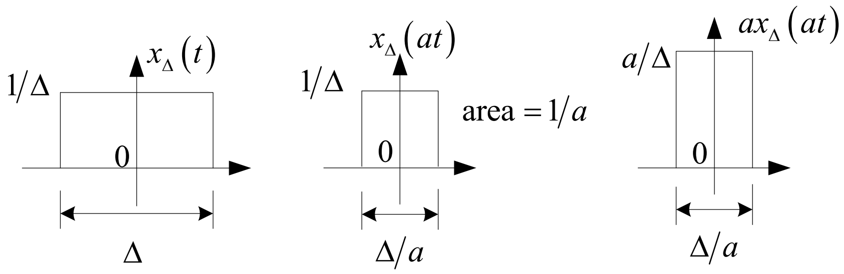


Figure 1.11 Steps involved in proving the time-scaling property of the unit impulse.

4. Ramp function

The integral of the step function $u(t)$ is a ramp function of unit slope. The ramp function is defined as

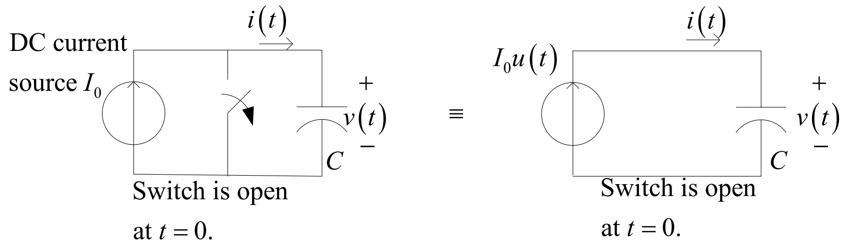
$$r(t) = \begin{cases} t, & t \ge 0 \\ 0, & t < 0 \end{cases} = tu(t) \tag{1.49}$$

Example 1.13: Parallel circuit

$$i(t) = I_0 u(t)$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau = \frac{1}{C}\int_{-\infty}^{t} I_0 u(\tau)\,d\tau = \begin{cases} 0 &, t<0 \\ I_0 t/C &, t \ge 0 \end{cases}$$

$$= \frac{I_0}{C} t u(t) = \frac{I_0}{C} r(t)$$



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1-4 Basic Discrete-Time Signals

1. Discrete-time unit step sequence

$$u[n] = \begin{cases} 0, n<0 \\ 1, n \ge 0 \end{cases} \tag{1.50}$$

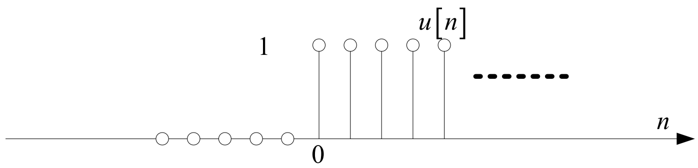

Figure 1.12 Discrete-time version of step function of unit amplitude.

2. Discrete-time unit impulse (or unit sample)

$$\delta[n] = \begin{cases} 0, n \ne 0 \\ 1, n = 0 \end{cases} \tag{1.51}$$

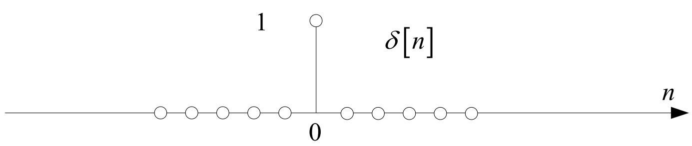

Figure 1.13 Discrete-time form of the unit impulse.

$$x[n]\delta[n]=x[0]\delta[n] \tag{1.52}$$

$$\delta[n]=u[n]-u[n-1] \tag{1.53}$$

$$u[n]=\sum_{m=-\infty}^{n}\delta[m] \tag{1.54}$$

$$\text{or}\quad u[n]=\sum_{k=0}^{\infty}\delta[n-k] \tag{1.55}$$

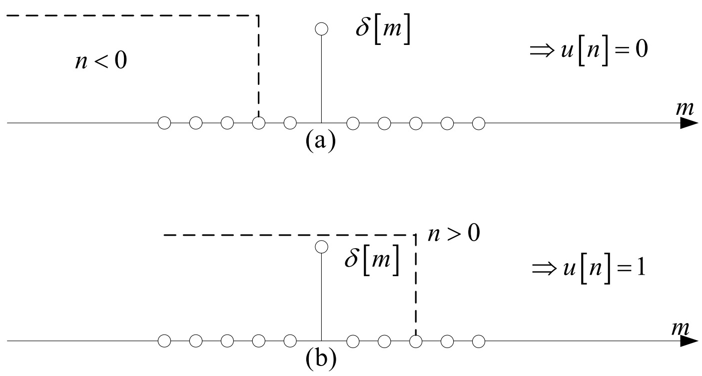

Figure 1.14 Running sum of (54): (a) $n < 0$; (b) $n > 0$ [1].

3. Discrete-time ramp function

$$r[n]=\begin{cases} n, & n\geq 0 \\ 0, & n<0 \end{cases}=nu[n] \tag{1.56}$$

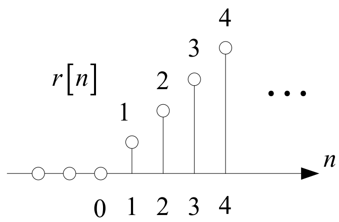

Figure 1.15 Discrete-time version of the ramp function.

4. Discrete-time complex exponential signals

$$x[n]=Br^{n}=Be^{\alpha n}\quad \left(r=e^{\alpha},\ \alpha \text{ may be any complex number.}\right) \tag{1.57}$$

sinusoidal signals

$$x[n]=A\cos(\Omega n+\phi) \tag{1.58}$$

(1) B and r are real

$|r| > 1 \Rightarrow$ the signal grows exponentially with n.

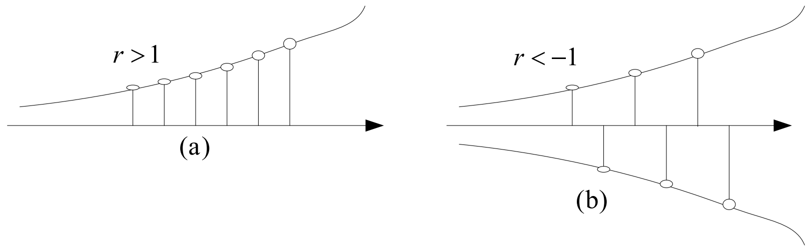

Figure 1.16 $x[n] = Br^n$: (a) $r > 1$; (b) $r < -1$ [1].

$|r| < 1 \Rightarrow$ the signal decays exponentially with n.

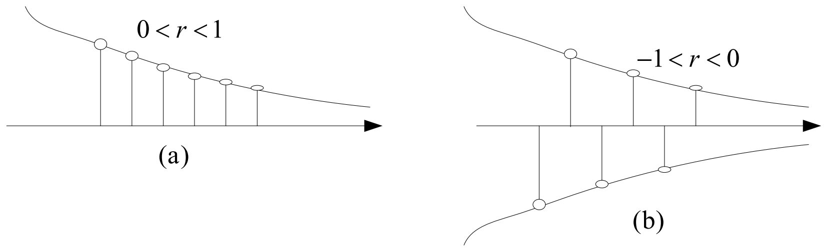

Figure 1.17 $x[n] = Br^n$: (a) $0 < r < 1$; (b) $-1 < r < 0$ [1].

(2) α is pure imaginary

$$x[n] = e^{j\Omega_0 n} = \cos\Omega_0 n + j\sin\Omega_0 n \tag{1.59}$$

$$A\cos(\Omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\Omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\Omega_0 n} \tag{1.60}$$

Both Ω_0 and ϕ have units of radians.

Example 1.14: sinusoidal sequences: ■

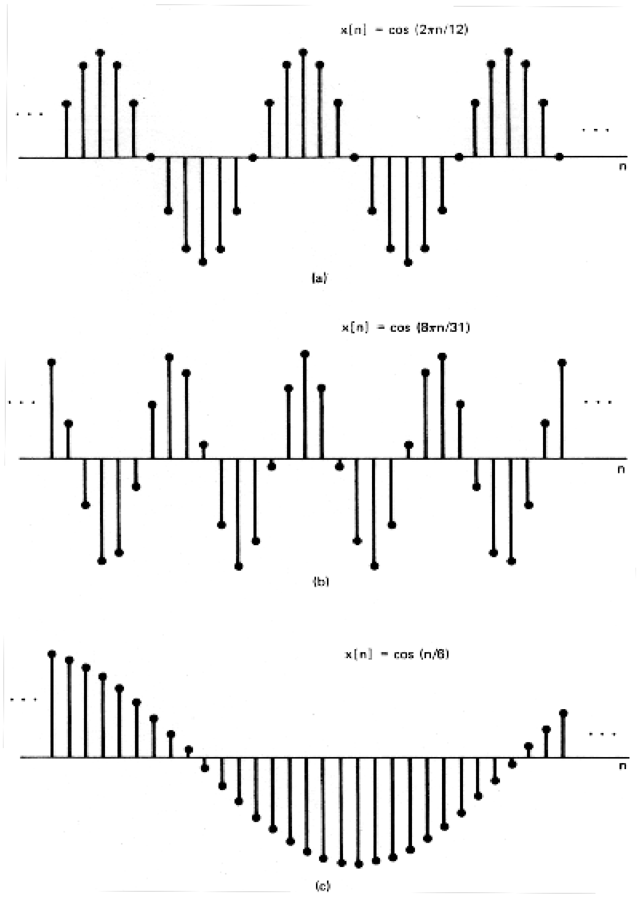

Figure 1.18 Discrete-time sinusoidal signals [1].

(3) General complex exponential

$$B = |B|e^{j\theta}, \quad r = |r|e^{j\Omega_0}$$
$$Br^n = |B||r|^n \cos(\Omega_0 n + \theta) + j|B||r|^n \sin(\Omega_0 n + \theta) \tag{1.61}$$

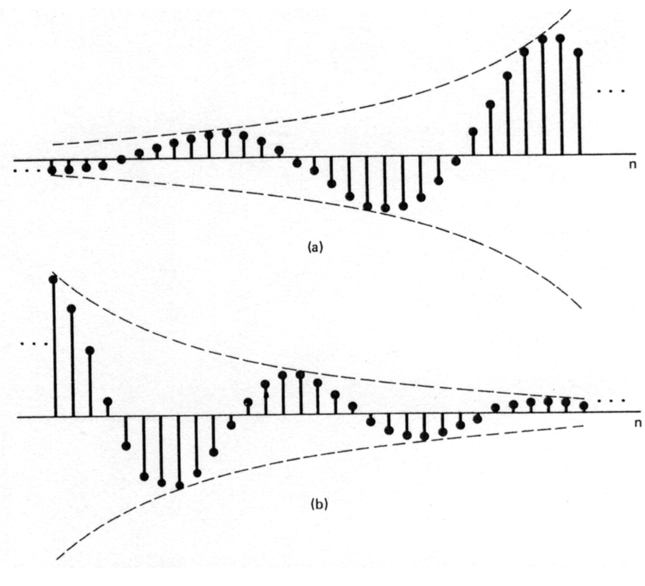

Figure 1.19 $x[n] = |B||r|^n \cos(\Omega_0 n + \theta)$: (a) $|r| > 1$ (growing); (b) $|r| < 1$ (decaying) [1].

(4) Periodicity properties of discrete-time complex exponentials

Continuous-time $e^{j\omega_0 t}$

(a) The larger the magnitude of ω_0, the higher the rate of oscillation in the signal.

(b) $e^{j\omega_0 t}$ is periodic for any value of ω_0.

Discrete-time $e^{j\Omega_0 n}$

(a) $e^{j(\Omega_0+2\pi)n} = e^{j\Omega_0 n}e^{j2\pi n} = e^{j\Omega_0 n}$ (1.62)

- The signal with frequency Ω_0 is identical to the signals with frequencies $\Omega_0 \pm 2\pi$, $\Omega_0 \pm 4\pi$, and so on.
- We only need to consider an interval of 2π in which to choose Ω_0. ($0 \le \Omega_0 < 2\pi$ or $-\pi \le \Omega_0 < \pi$).
- The signal $e^{j\Omega_0 n}$ does not have a continually increasing rate of oscillation as Ω_0 is increasing in magnitude.

  $0 \to \pi$: signal with increasing rates of oscillation

  $\pi \to 2\pi$: signal with decreasing rates of oscillation

Example 1.15: Magnitude of the Fourier transform of a discrete-time signal ■

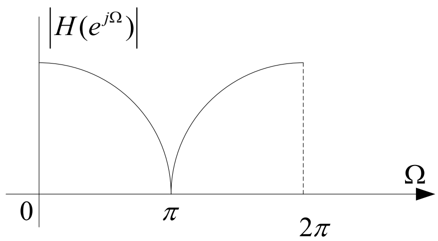

Figure 1.20 The magnitude of $H(e^{j\Omega})$ from $\Omega = 0$ to $\Omega = 2\pi$ radians.

(b) $e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} \Rightarrow e^{j\Omega_0 N} = 1 \Rightarrow \Omega_0 N = 2\pi m \Rightarrow \dfrac{\Omega_0}{2\pi} = \dfrac{m}{N}$ (1.63)

The signal $e^{j\Omega_0 n}$ is periodic with period N only if $\Omega_0/2\pi$ is a rational number. (For example, Figs. 18(a) and (b) are periodic, T = 12 and 31; Fig. 18(c) is not periodic.)

- If $x[n]$ is periodic with fundamental period N, its fundamental frequency is $2\pi/N$ (radians/sample).
- N: the number of samples contained in a single cycle of $x[n]$.
- If N and m have no factors in common, the fundamental period of $x[n]$ is N.

- $\frac{2\pi}{N} = \frac{\Omega_0}{m}$ and $N = m \cdot \frac{2\pi}{\Omega_0}$.
- Constant discrete-time signal: fundamental frequency = 0; fundamental period is undefined.

Example 1.16: Periods of discrete-time sinusoidal signals

A pair of sinusoidal signals with a common angular frequency is defined by $x_1[n] = \sin[5\pi n]$ and $x_2[n] = \sqrt{3}\cos[5\pi n]$. Find their fundamental period and express the composite sinusoidal signal

$$y[n] = x_1[n] + x_2[n].$$

$\Omega = 5\pi$ rad/sample $\Rightarrow N = 2\pi m/\Omega = 2m/5$

For $x_1[n]$ and $x_2[n]$ to be period, N must be an integer. This can be so only for m = 5, 10, 15,…, which results in N = 2, 4, 6,….

$$\begin{aligned} y[n] &= \sin(5\pi n) + \sqrt{3}\cos(5\pi n) \\ &= \sqrt{1+3}\left[\frac{1}{2}\sin(5\pi n) + \frac{\sqrt{3}}{2}\cos(5\pi n)\right] \\ &= 2\left[\sin(5\pi n)\cos\left(\frac{\pi}{3}\right) + \cos(5\pi n)\sin\left(\frac{\pi}{3}\right)\right] \\ &= 2\sin\left(5\pi n + \frac{\pi}{3}\right) \end{aligned}$$

■

(c) Differences between the signals $e^{j\omega_0 t}$ and $e^{j\Omega_0 n}$

$e^{j\omega_0 t}$	$e^{j\Omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for exponentials at frequencies separated by 2π
Periodic for any choice of ω_0	Periodic only if $\Omega_0 = m\cdot(2\pi / N)$ $N > 0$; m and N are integers.
Fundamental frequency ω_0	Fundamental frequency: $(\Omega_0 / m) = (2\pi / N)$; m and N have no factors in common.
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $2\pi / \omega_0$	Fundamental period $\Omega_0 = 0$: undefined $\Omega_0 \neq 0$: $N = m\cdot(2\pi / \Omega_0)$

(d) Harmonically related periodic exponentials

$$\phi_k[n] = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \ldots \tag{1.64}$$

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} = e^{j2\pi n}\cdot e^{jk(2\pi/N)n} = \phi_k[n] \tag{1.65}$$

There are only N distinct periodic exponentials in the set given in the above equation.

(e) A discrete-time signal obtained by taking samples of a continuous-time signal

$$x[n] = e^{j\omega_0 nT} = e^{j(\omega_0 T)n} \Rightarrow \Omega_0 = \omega_0 T \tag{1.66}$$

$x[n]$ is periodic only if $(\Omega_0 / 2\pi) = (\omega_0 T/2\pi)$ is a rational number.
Similarly,

$$x(t) = \cos(2\pi t) \tag{1.67}$$

$$\begin{aligned} x[n] = x[nT] = \cos(2\pi nT) = \cos(\Omega_0 n) \\ \Rightarrow \Omega_0 = 2\pi T \end{aligned} \tag{1.68}$$

Example 1.17:

Fig. 1.18(a): $T = 1/12 \Rightarrow \Omega_0 = \pi/6 \Rightarrow \Omega_0/2\pi = 1/12 \Rightarrow N = 12$
Fig. 1.18(b): $T = 4/31 \Rightarrow \Omega_0 = 8\pi/31 \Rightarrow \Omega_0/2\pi = 4/31 \Rightarrow N = 31$
Fig. 1.18(c): $T = 1/12\pi \Rightarrow \Omega_0 = 1/6 \Rightarrow \Omega_0/2\pi = 1/12\pi$ ■