1-5 Systems

1. A system can be viewed as any process that results in a transformation of signals

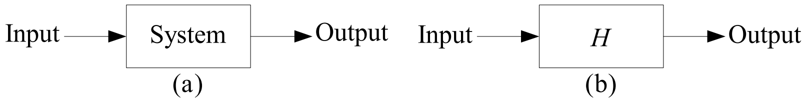


Figure 1.21 (a) Block diagram of a system; (b) representation of a system operator H.

(1) Continuous-time systems: continuous-time input and continuous-time output

$$x(t) \to y(t), \quad y(t) = H\{x(t)\} \tag{1.69}$$

(2) Discrete-time systems: discrete-time input and discrete-time output

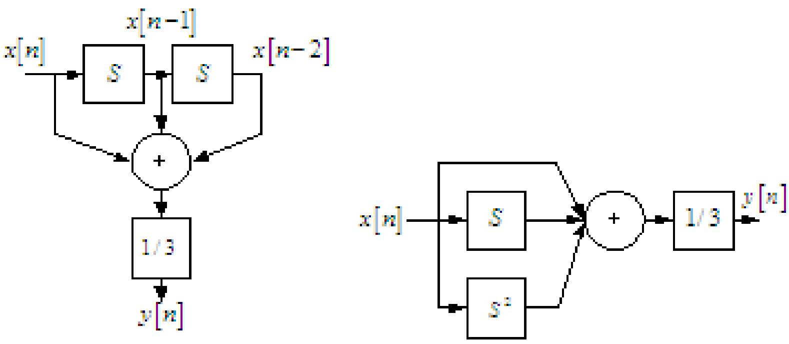
$$x[n] \to y[n], \quad y[n] = H\{x[n]\} \tag{1.70}$$

Example 1.18: Moving-average (MA) system

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

$y[n]$ is the average of three consecutive sample values $x[n-2]$, $x[n-1]$, and $x[n]$. The value of $y[n]$ changes as n moves along the discrete-time axis. Let the operator S^k represent a system that shifts the input $x[n]$ by k time units to produce an output equal to $x[n-k]$. Then, the overall operator for the moving average system can be expressed by

$$H = \frac{1}{3}\left(1 + S + S^2\right)$$



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2. Interconnection of systems:

(1) Series interconnection (or cascade interconnection)

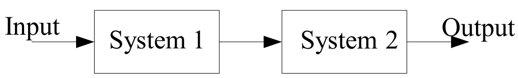


Figure 1.22 Series (cascade) interconnection [1].

(2) Parallel interconnection

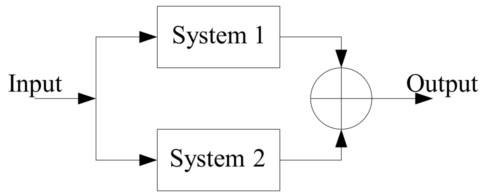


Figure 1.23 Parallel interconnection [1].

(3) Series/parallel interconnection

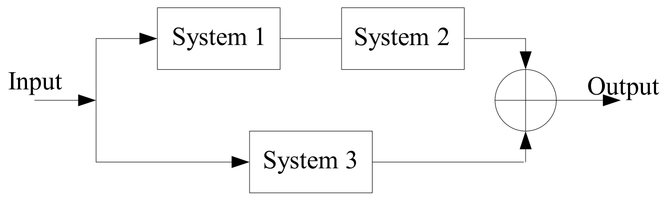
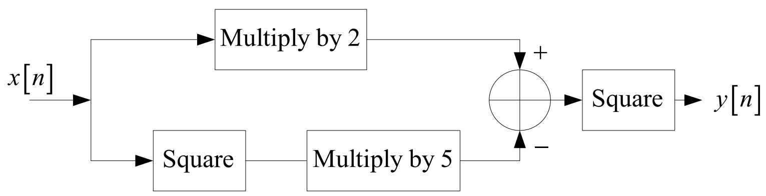


Figure 1.24 Series/parallel interconnection [1].

Example 1.19: $y[n]=\left(2x[n]-5x^2[n]\right)^2$



Viewing a complex system in this manner is often useful in facilitating the analysis of the system. ■

(4) Feedback interconnection

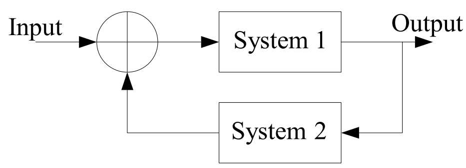


Figure 1.25 Feedback interconnection [1].

3. Properties of systems

   (1) Systems with or without memory

      (a) Memoryless systems: the output at a given time is dependent only on the input at the same time.

(b) Systems with memory: the output at a given time is dependent on the inputs at some previous and/or future time instants other than (or in addition to) the input at the same time.

Example 1.20:

- $y[n]=\sum_{k=-\infty}^{\infty} x[k]$ and $y(t)=x(t-1)$ have memory.
- A resistor is memoryless because of $i(t)=v(t)/R$.
- An inductor has memory because of $i(t)=\frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau$.
- The MA system $y[n]=\frac{1}{3}\left(x[n]+x[n-1]+x[n-2]\right)$ has memory.
- $y[n]=x^2[n]$ is memoryless. ■

(2) Invertibility

A system is said to be invertible if distinct inputs lead to distinct outputs. That is, there must be a one-to-one mapping between the input and output signals for a system to be invertible.

⇒ Observing the system output, we can determine the system input.

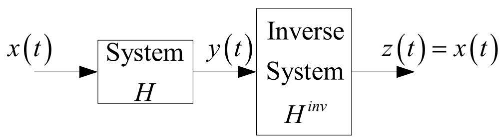


Figure 1.26 A system with the invertibility property.

$$H^{inv}\{y(t)\}=H^{inv}\{H\{x(t)\}\}=H^{inv}H\{x(t)\} \tag{1.71}$$

For this output signal to equal the original input $x(t)$, we require that

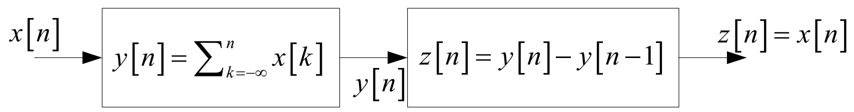
$$H^{inv}H=I \tag{1.72}$$

where I denotes the identity operator.

Example 1.21: $y[n]=\sum_{k=-\infty}^{n} x[k]$ (invertible system)

The difference between two successive values of the output is precisely the last input value.

$$z[n]=y[n]-y[n-1]=x[n]$$



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Example 1.22: $y(t)=x^2(t) \;\Rightarrow\; x(t)=\sqrt{y(t)}$ or $-\sqrt{y(t)}$

$\Rightarrow$ a non-invertible system ■

Example 1.23: $y(t)=x(t-t_0)=S^{t_0}\{x(t)\}$

$$S^{-t_0}\{y(t)\}=S^{-t_0}\{S^{t_0}\{x(t)\}\}=S^{-t_0}S^{t_0}\{x(t)\}=I\{x(t)\}$$

The inverse of the system is a time shift $-t_0$. ■

(3) Causality

A system is causal if the output at any time depends only on values of the input at the present time and in the past.

$\Rightarrow$ This is often referred to as the non-anticipative property.

- If two inputs to a causal system are identical up to some time t_0 or n_0, the corresponding outputs must also be equal up to this same time.

  Example 1.24:

$$\left.\begin{array}{l} y[n]=x[n]-x[n+1] \\ y(t)=x(t+1) \end{array}\right\}\text{ noncausal;}\quad \left.\begin{array}{l} y[n]=\sum_{k=-\infty}^{n} x[k] \\ y(t)=x(t-1) \end{array}\right\}\text{ causal}$$

■

- All memoryless systems are causal. (True)
  All systems with memory are causal. (False)
- Causality is not of fundamental importance in some applications, such as image processing, in which the independent variable is not time.
- The important point to note here is that causality is required for a system to be capable of operating in *real time*.

  Example 1.25:

  The MA system described by $y[n]=(x[n]+x[n-1]+x[n-2])/3$ is causal, while the MA system described by $(x[n+1]+x[n]+x[n-1])/3$ is noncausal.

■

(4) Stability

Bounded input → bounded output (BIBO): "stable system"

Bounded input → unbounded output (the magnitude grows without bound): "unstable system"

A system is BIBO stable if the output signal $y(t)$ satisfies the condition

$$|y(t)| \le M_y < \infty \text{ for all } t \tag{1.73}$$

whenever the input signal $x(t)$ satisfies the condition

$$|x(t)| \le M_x < \infty \text{ for all } t\,. \tag{1.74}$$

Both M_x and M_y represent some finite positive numbers.

Example 1.26: $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$

$$\begin{aligned} |y[n]| &= \frac{1}{3}\left|\left(x[n] + x[n-1] + x[n-2]\right)\right| \\ &\le \frac{1}{3}\left(|x[n]| + |x[n-1]| + |x[n-2]|\right) \\ &\le \frac{1}{3}\left(M_x + M_x + M_x\right) = M_x \end{aligned}$$

The MA system is stable. ■

Example 1.27: $y[n] = \frac{1}{2M+1}\sum_{k=-M}^{M} x[n-k]$

$x[n]$ is bounded → $y[n]$ is bounded ⇒ "stable system" ■

Example 1.28: $y[n] = r^n x[n],\ r > 1$

$$|y[n]| = |r^n||x[n]| \to \infty \ (\because r > 1)$$ ■

Example 1.29: $y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n]$

$y[-1] = 0,\ y[0] = 1,\ y[1] = 2,\ y[2] = 3, \ldots, y[n]$ grows without bound.

⇒ The output signal grows without bound. ⇒ This is an unstable system. ■

(5) Time invariance

A system is time-invariant if a time shift in the input signal causes an identical time shift in the output signal. Put another way, the characteristics of a time-invariant system do not change with time. Otherwise, the system is said to be time-variant or time-varying.

$$\begin{aligned} x(t) &\rightarrow y(t) \\ x(t-t_0) &\rightarrow y(t-t_0) \end{aligned} \tag{1.75}$$

Example 1.30: $y(t)=\sin(x(t))$

$y_1(t)=\sin(x_1(t)) \Rightarrow y_1(t-t_0)=\sin(x_1(t-t_0))$

Let $x_2(t)=x_1(t-t_0)$. Then $y_2(t)=\sin(x_2(t))=\sin(x_1(t-t_0))=y_1(t-t_0)$.

$\Rightarrow$ The system is time-invariant. ■

Example 1.31: $y(n)=nx(n)$

$y_1[n]=nx_1[n] \Rightarrow y_1[n-n_0]=(n-n_0)x_1[n-n_0]$

Let $x_2[n]=x_1[n-n_0]$. Then $y_2[n]=nx_2[n]=nx_1[n-n_0]\neq y_1[n-n_0]$

$\Rightarrow$ The system is time-varying. ■

Example 1.32: Inductor

$$y_1(t)=i(t)=\frac{1}{L}\int_{-\infty}^{t} x_1(\tau)\,d\tau$$

$$y_1(t-t_0)=\frac{1}{L}\int_{-\infty}^{t-t_0} x_1(\tau)\,d\tau$$

$$y_2(t)=\frac{1}{L}\int_{-\infty}^{t} x_1(\tau-t_0)\,d\tau \overset{\tau'=\tau-t_0}{=} \frac{1}{L}\int_{-\infty}^{t-t_0} x_1(\tau')\,d\tau' = y_1(t-t_0)$$

$\Rightarrow$ The system is time-invariant. ■

Example 1.33: Thermistor

A thermistor, $y_1(t)=x_1(t)/R(t)$, has a resistance that varies with time due to temperature changes.

$y_2(t)=x_1(t-t_0)/R(t)$; $y_1(t-t_0)=x_1(t-t_0)/R(t-t_0)$.

In general, $R(t)\neq R(t-t_0)$ for $t_0\neq 0$. So $y_1(t-t_0)\neq y_2(t)$ for $t_0\neq 0$.

$\Rightarrow$ The system is time-varying. ■

(6) Linearity

A linear system is one that possesses the two important properties of additivity and homogeneity (i.e., the superposition principle). That is, if the input is formed by a weighted sum of several signals, then the output is simply the weighted sum (or the superposition) of the responses corresponding to those signals.

(a) Additivity property:

$$\begin{aligned} &H\{x_1(t)\} = y_1(t) \\ &H\{x_2(t)\} = y_2(t) \\ &\Rightarrow H\{x_1(t)+x_2(t)\} = H\{x_1(t)\} + H\{x_2(t)\} = y_1(t)+y_2(t) \end{aligned} \tag{1.76}$$

Then for a system to be linear, it is necessary that the composite input $x_1(t)+x_2(t)$ produces the corresponding output $y_1(t)+y_2(t)$.

(b) Homogeneity or scaling property:

$$\begin{aligned} &H\{x(t)\} = y(t) \\ &\Rightarrow H\{ax(t)\} = aH\{x(t)\} = ay(t) \text{ for an arbitrary scalar } a \end{aligned} \tag{1.77}$$

(c) Superposition principle:

$$\begin{aligned} &H\{x_1(t)\} = y_1(t) \\ &H\{x_2(t)\} = y_2(t) \\ &\Rightarrow H\{ax_1(t)+bx_2(t)\} = H\{ax_1(t)\} + H\{bx_2(t)\} = ay_1(t)+by_2(t) \end{aligned} \tag{1.78}$$

(d) General superposition principle:

$$\begin{aligned} &H\{x_i(t)\} = y_i(t),\ i = 1,2,\ldots,N \\ &H\left\{\sum_{i=1}^{N} a_i x_i(t)\right\} = \sum_{i=1}^{N} a_i H\{x_i(t)\} = \sum_{i=1}^{N} a_i y_i(t) \end{aligned} \tag{1.79}$$

- The output of a linear system is zero when the input is zero. From the scaling property,

$$H\{ax(t)\} = y_2(t) = aH\{x(t)\} \Rightarrow \ y_2(t) = 0 \text{ when } a = 0. \tag{1.80}$$

  This means that “zero input” produces “zero output” for a linear system.

- An incrementally linear system is one that responds linearly to changes in the input, i.e., the superposition principle holds for changes in the input.
- When a system violates either the additivity property or the homogeneity property, the system is said to be nonlinear.

Example 1.34: $y[n]=2x[n]+3$

$x[n]=0 \Rightarrow y[n]=3\neq 0$. So the system is not linear.

$y_1[n]-y_2[n]=2x_1[n]+3-\{2x_2[n]+3\}=2\{x_1[n]-x_2[n]\}$

$\Rightarrow$ The system is incrementally linear. ■

- An incrementally linear system can be visualized as follows:

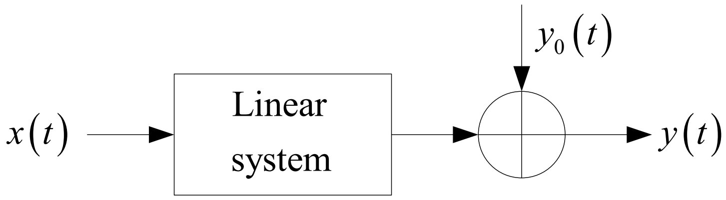

Figure 1.27 Structure of an incrementally linear system [1]. $y_0(t)$ is the zero-input response of the overall system.

Example 1.35: $y[n]=nx[n]$ is linear.

$$x[n]=\sum_{i=1}^{N}a_i x_i[n]$$
$$y[n]=n\sum_{i=1}^{N}a_i x_i[n]=\sum_{i=1}^{N}a_i n x_i[n]=\sum_{i=1}^{N}a_i y_i[n]$$ ■

Example 1.36: $y(t)=x(t)x(t-1)$ is nonlinear.

$$x(t)=\sum_{i=1}^{N}a_i x_i(t)$$
$$y(t)=\sum_{i=1}^{N}a_i x_i(t)\sum_{j=1}^{N}a_j x_j(t-1)$$
$$=\sum_{i=1}^{N}a_i\left[\sum_{j=1}^{N}a_j x_i(t)x_j(t-1)\right]\neq\sum_{i=1}^{N}a_i y_i(t)$$ ■

References

[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, *Signals and Systems*, 2nd Ed., Prentice-Hall, 1997.

[2] S. Haykin and B. Van Veen, *Signals and Systems*, 2nd Ed., Hoboken, NJ: John Wiley & Sons, 2003.

[3] H. P. Hsu, *Schaum's Outline of Theory and Problems of Signals and Systems*, McGraw-Hill, 1995.