Homework No. 7 Solution

1.

(1)

$$e^{-t}e^{jt}u(t) \xrightarrow{L} \frac{1}{s+1-j}, \mathrm{Re}\{s\} > -1$$

$$e^{-t}e^{-jt}u(t) \xrightarrow{L} \frac{1}{s+1+j}, \mathrm{Re}\{s\} > -1$$

$$e^{-t}\sin(t)u(t) = \frac{1}{2j}[e^{-t}e^{jt} - e^{-t}e^{-jt}]u(t) \xrightarrow{L} \frac{1}{(s+1)^2+1}$$

ROC: Re{s}> -1

(2)

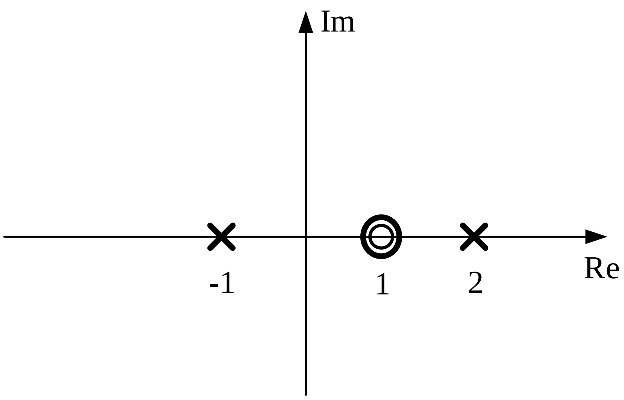
$$\underset{\text{right-sided}}{A(s) = \frac{1}{s}} \overset{\mathcal{L}}{\longleftrightarrow} a(t) = u(t)$$

$$B(s) = e^{-3s}A(s) \overset{\mathcal{L}}{\longleftrightarrow} b(t) = a(t-3) = u(t-3)$$

$$X(s) = \frac{d}{ds}B(s) \overset{\mathcal{L}}{\longleftrightarrow} x(t) = -tb(t) = -tu(t-3)$$

2.

(1)



The possible ROCs are:

- ROC$_1$: $\mathrm{Re}\{s\} > 2$
- ROC$_2$: $-1 < \mathrm{Re}\{s\} < 2$
- ROC$_3$: $\mathrm{Re}\{s\} < -1$

(2)

- ROC$_1$: $\mathrm{Re}\{s\} > 2 \Rightarrow$ causal, unstable.

- ROC2: $-1 < \text{Re}\{s\} < 2 \Rightarrow$ noncausal, stable.
- ROC3: $\text{Re}\{s\} < -1 \Rightarrow$ noncausal, unstable.

(3)

The inverse is $H_{inv}(s) = \dfrac{s^2 - s - 2}{s^2 - 2s + 1}$.

By long division, we obtain $H_{inv}(s) = 1 + \dfrac{s-3}{(s-1)^2}$.

By the partial-fraction expansion method, $H_{inv}(s) = 1 + \dfrac{1}{s-1} + \dfrac{(-2)}{(s-1)^2}$.

The inverse system is known to be stable, thus

$h_{inv}(t) = \delta(t) - e^t u(-t) + 2te^t u(-t)$.

3. Taking the unilateral Laplace transform of both sides of the given differential equation, we get

$$s^3Y(s) - s^2 y(0^-) - sy'(0^-) - y''(0^-) + 6s^2Y(s) - 6sy(0^-)$$
$$-6y(0^-) + 11sY(s) - 11y(0^-) + 6Y(s) = X(s).$$

(1) For the zero state response, assume that all the initial conditions are zero. Furthermore, from the given $z(t)$ we may determine

$$X(\text{s}) = \frac{1}{s+4}, \qquad \text{Re}\{s\} > -4.$$

Then we have

$$Y(s)\{s^3 + 6s^2 + 11s + 6\} = \frac{1}{s+4}$$

Therefore,

$$Y(s) = \{s^3 + 6s^2 + 11s + 6\} = \frac{1}{(s+4)(s^3 + 6s^2 + 11s + 6)}$$

Taking the inverse unilateral Laplace transform of the partial fraction expansion of the above equation, we get

$$y(t) = \frac{1}{6}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t).$$

(2) For the zero-input response, we assume that $X(s) = 0$. Assuming that the initial conditions are as given, we obtain

$$Y(s) = \frac{s^2 + 5s + 6}{s^3 + 6s^2 + 11s + 6} = \frac{1}{s+1}.$$

Taking the inverse unilateral Laplace transform of the above equation, we get

$$y(t) = e^{-t}u(t).$$

(3) The total response is the sum of the zero-state and zero-input responses.

$$y(t) = \frac{7}{6}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t).$$

4. If $x(t) = e^{2t}$ produces $y(t) = (1/6)e^{2t}$, then $H(2)=1/6$. Also, by taking the Laplace transform of both sides of the given differential equation we get

$$H(s) = \frac{s + b(s+4)}{s(s+4)(s+2)}.$$

Since $H(2)=1/6$, we may deduce that b=1. Therefore

$$H(s) = \frac{2(s+2)}{s(s+4)(s+2)} = \frac{2}{s(s+4)}.$$

5.

(1) The Laplace transform of the relation between $x(t)$ and $y(t)$ can be written as

$$Y(s) = \frac{s^2 - s - 6}{s^2 + 2s + 1}X(s)$$

Taking the inverse Laplace transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 2\frac{d^2y(t)}{dt^2} + y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} - 6x(t).$$

(2) The two poles of the system are at -1. Since the system is causal, the ROC must be to the right of s = -1. Therefore, the ROC must include the $j\omega -$axis . Hence, the system is stable.

6.

(1)

$$Z(s) = X(s) - H_B(s)Y(s)$$

$$Y(s) = W(s) = H_A(s)Z(s) = H_A(s)X(s) - H_A(s)H_B(s)Y(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{H_A(s)}{1 + H_A(s)H_B(s)}$$

(2)

$$\frac{d^2w(t)}{dt^2} + \frac{dw(t)}{dt} + aw(t) = \frac{d^2z(t)}{dt^2} - z(t) \overset{F}{\longleftrightarrow} s^2W(s) + sW(s) + aW(s) = s^2Z(s) - Z(s)$$

$$\Rightarrow (s^2 + s + a)W(s) = (s^2 - 1)Z(s)$$

$$\Rightarrow H_A(s) = \frac{W(s)}{Z(s)} = \frac{s^2 - 1}{(s^2 + s + a)} = \frac{(s+1)(s-1)}{(s^2 + s + a)}, \text{ and we have } H_B(s) = \frac{1}{s+1}.$$

$$H(s)=\frac{H_A(s)}{1+H_A(s)H_B(s)}=\frac{\frac{s^2-1}{s^2+s+a}}{1+\frac{(s+1)(s-1)}{s^2+s+a}\cdot\frac{1}{s+1}}=\frac{\frac{s^2-1}{s^2+s+a}}{1+\frac{s-1}{s^2+s+a}}$$

$$=\frac{\frac{s^2-1}{s^2+s+a}}{\frac{s^2+2s+a-1}{s^2+s+a}}=\frac{s^2-1}{s^2+2s+(a-1)}$$

and we have $h(t)=\delta(t)-2e^{-t}u(t)\overset{F}{\longleftrightarrow}H(s)=1-\frac{2}{s+1}=\frac{s-1}{s+1}$

$$\Rightarrow H(s)=\frac{s^2-1}{s^2+2s+(a-1)}=\frac{s-1}{s+1}=\frac{(s-1)(s+1)}{(s+1)(s+1)}=\frac{s^2-1}{s^2+2s+1}$$

$\Rightarrow a-1=1$

$\Rightarrow a=2$

(3) Since $y(t)$ is right-sided,

=> $Y(s)=\frac{1}{s+2}$, ROC: $\text{Re}\{s\}>-2$.

=> $X(s)=\frac{Y(s)}{H(s)}=\frac{s+1}{(s-1)(s+2)}=\frac{2/3}{(s-1)}+\frac{1/3}{(s+2)}$.

For the causal input, the ROC of $x(t)$ should be $\text{Re}\{s\}>-2$.

we get $x(t)=\frac{2}{3}e^{t}u(t)+\frac{1}{3}e^{-2t}u(t)$.