Homework No.4 Solution

1. (20%)

(1) $\int_{-\infty}^{\infty} x(t)\,dt = X(0) = 2$

(2) $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega = \frac{10}{\pi}$

(3) $\int_{-\infty}^{\infty} x(t)e^{j2t}\,dt = X(-2) = 1$

(4) $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega\cdot 0}\,d\omega = \frac{6}{\pi}$

(5) $\left.\frac{dx(t)}{dt}\right|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{-j\omega\cdot 0}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\text{odd function}}{j\omega} \cdot \underbrace{X(j\omega)}_{\text{even function}}\,d\omega = 0$

2. (20%)

(1)

$$x(t) = \sum_{k=0}^{2}(-1)^k \sin\left(\frac{2\pi k}{3}t\right) = \frac{1}{2j}\sum_{k=0}^{2}(-1)^k\left(e^{j\frac{2\pi k}{3}t} - e^{-j\frac{2\pi k}{3}t}\right)$$

$$\Rightarrow X(j\omega) = \frac{\pi}{j}\sum_{k=0}^{2}(-1)^k\left[\delta\left(\omega - \frac{2\pi k}{3}\right) - \delta\left(\omega + \frac{2\pi k}{3}\right)\right]$$

(2) From $te^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(a+j\omega)^2}$ and the property mentioned in lecture note, We have

$$-jtx(t) = -jt^2e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d}{d\omega}\left(X(\omega)\right) = \frac{d}{d\omega}\left(\frac{1}{(a+j\omega)^2}\right) = \frac{-2j}{(a+j\omega)^3}$$

$$\Rightarrow \frac{t^2}{2}e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(a+j\omega)^3}$$

$$\Rightarrow x(t) = \frac{t^2}{2}e^{-at}u(t),\ a > 0.$$

3. (20%)

(1)

$$(j\omega)^2 Z(j\omega) - (j\omega)Z(j\omega) - 6Z(j\omega) = X(j\omega)$$

$$\Rightarrow H_A(j\omega) = \frac{Z(j\omega)}{X(j\omega)} = \frac{1}{(j\omega)^2 - j\omega - 6} = \frac{1}{(-3+j\omega)(2+j\omega)} = \frac{0.2}{(-3+j\omega)} + \frac{(-0.2)}{(2+j\omega)}$$

$$\Rightarrow h_A(t) = \frac{1}{5}\left[-e^{3t}u(-t) - e^{-2t}u(t)\right]$$

(2)

$$\frac{dy(t)}{dt} + 6y(t) = \frac{dz(t)}{dt} + bz(t)$$

$$\Rightarrow H_B(j\omega) = \frac{Y(j\omega)}{Z(j\omega)} = \frac{(b+j\omega)}{(6+j\omega)}$$

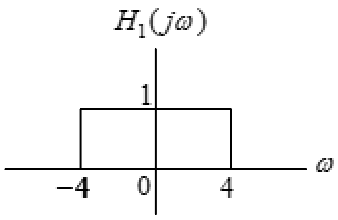
In order to make the system causal, we have to cancel out the non-causal term, i.e. the first term of $H_A(j\omega)$, yields $b = -3$.

4. (20%)

Note that $h(t) = h_1(t-1)$, where

$$h_1(t) = \frac{\sin(4t)}{\pi t}$$

The Fourier transform $H_1(j\omega)$ of $h_1(t)$ is as shown in the following figure.



From the above figure, we can see that the $h_1(t)$ is an ideal low-pass filter with passband $|\omega| \le 4$. Therefore, $h(t)$ can be seen as the impulse response of an ideal low-pass filter shifted by one to the right. Using the shift property, we have

$$H(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| \le 4 \\ 0 \quad, & o.w. \end{cases}$$

(1) $X_1(j\omega) = \pi e^{\frac{\pi}{12}}\delta(\omega-6) + \pi e^{\frac{\pi}{12}}\delta(\omega+6)$.

It is clear that $Y_1(j\omega) = X_1(j\omega)H(j\omega) = 0 \Rightarrow y_1(t) = 0$.

(2) $X_2(j\omega) = \frac{\pi}{j}\left[\sum_{k=0}^{\infty}(\frac{1}{2})^k\{\delta(\omega-3k) - \delta(\omega+3k)\}\right]$

$$\Rightarrow Y_2(j\omega) = X_2(j\omega)H(j\omega) = \frac{\pi}{2j}\{\delta(\omega-3k)-\delta(\omega+3k)\}\,\mathrm{e}^{-j\omega}$$

$$\Rightarrow y_2(t) = \frac{1}{2}\sin(3t-1)$$

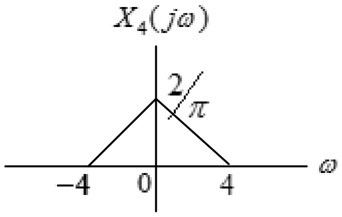
(3) We have

$$X_3(j\omega) = \begin{cases} e^{j\omega}, & |\omega| \le 4 \\ 0 \ , & o.w. \end{cases} \quad \Rightarrow Y_3(j\omega) = X_3(j\omega)H(j\omega) = X_3(j\omega)e^{-j\omega}.$$

This implies that

$$y_3(t) = x_3(t-1) = \frac{\sin(4t)}{\pi t}.$$

(4) $X_4(j\omega)$ is shown in the following figure.



Therefore,

$$Y_4(j\omega) = X_4(j\omega)H(j\omega) = X_4(j\omega)e^{-j\omega}$$

This implies that

$$y_4(t) = x_4(t-1) = \left(\frac{\sin(2(t-1))}{\pi(t-1)}\right)^2.$$

5. (20%)

(1) The frequency response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}. \quad (5\%)$$

(2) Finding the partial fraction expansion of the answer of part (1) and taking its inverse Fourier transform, we have

$$h(t) = \frac{3}{2}[e^{-4t} + e^{-2t}]u(t). \quad (10\%)$$

(3) We have

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{9+3j\omega}{8+6j\omega-\omega^2} \Rightarrow Y(j\omega)(8+6j\omega-\omega^2) = X(j\omega)(9+3j\omega).$$

Taking the inverse Fourier transform we obtain

$$\frac{d^2y(\mathrm{t})}{dt^2} + 6\frac{dy(\mathrm{t})}{dt} + 8y(t) = 3\frac{dy(\mathrm{t})}{dt} + 9x(t). \quad (5\%)$$