Homework No. 3 Solution

1. (25%)

Fundamental period of $x(t) = T = 2 \Rightarrow \omega_0 = 2\pi/2 = \pi$

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{2}\int_0^2 x(t)dt = -0.5$$

$$a_k = \frac{1}{2}\int_0^2 x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_0^2 \left[\delta(t) - 2\delta(t-1)\right]e^{-jk\omega_0 t}dt$$

$$= \frac{1}{2} - e^{-jk\pi} = \frac{1}{2} - (-1)^k, k \neq 0.$$

2. (25%)

(1) (6%)

Fundamental period of $x(t) = T = \frac{1}{2}$

Fundamental period of $y(t) = T = \frac{1}{4}$

Fundamental period of $z(t) = T = \frac{1}{2}$

(2) (6%)

Note: if m, n integers $T = 2\pi / w$

$$\int_0^T \cos(mw_0 t)\cos(nw_0 t)\,dt = \begin{cases} T/2, m = n \\ 0 \quad ,m \neq n \end{cases}$$

$$\int_0^T \sin(mw_0 t)\sin(nw_0 t)\,dt = \begin{cases} T/2, m = n \\ 0 \quad ,m \neq n \end{cases}$$

$$\int_0^T \cos(mw_0 t)\sin(nw_0 t)\,dt = 0.$$

$$x(t) = \cos(4\pi t), T = \frac{1}{2}, w = 4\pi$$

$$a_0 = \frac{1}{T}\int_T \cos(4\pi t)dt = 0$$

$$a_k = \frac{1}{T}\int_T \cos(4\pi t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T \left(\cos(w_0 t)\cos(kw_0 t) - j\cos(w_0 t)\sin(kw_0 t)\right)dt$$

$$= 2\int_0^{\frac{1}{2}} \cos(w_0 t)\cos(kw_0 t)dt - \underbrace{2j\int_0^{\frac{1}{2}} \cos(w_0 t)\sin(kw_0 t)dt}_{0}$$

$$\rightarrow a_k = 2\int_0^{\frac{1}{2}} \cos(w_0 t)\cos(kw_0 t)dt = \begin{cases} \frac{1}{2}, k = \pm 1 \\ 0 \ , o.w. \end{cases}$$

(3) (6%)

$$y(t)=\sin(8\pi t), T=\frac{1}{4}, w=8\pi$$

$$a_0=\frac{1}{T}\int_T \sin(w_0t)dt=0$$

$$a_k=\frac{1}{T}\int_T\left(\sin(w_0t)\cos(kw_0t)-j\sin(w_0t)\sin(kw_0t)\right)dt$$

$$=\frac{-j}{T}\int_0^T \sin(w_0t)\sin(kw_0t)dt=\begin{cases}\frac{-j}{T}\frac{T}{2}=-\frac{j}{2}, k=1\\ \frac{j}{T}\frac{T}{2}=\frac{j}{2} \quad ,k=-1\\ 0 \quad ,o.w.\end{cases}$$

(4) (7%)

$$z(t)=\frac{1}{2}\sin(12\pi t)+\frac{1}{2}\sin(4\pi t), T=\frac{1}{2}, w=4\pi$$

$$a_0=\frac{1}{T}\int_T\frac{1}{2}\sin(12\pi t)dt+\frac{1}{T}\int_T\frac{1}{2}\sin(4\pi t)dt=0$$

$$a_k=\int_0^{\frac{1}{2}}\left[\sin(3w_0t)\cos(kw_0t)-j\sin(3w_0t)\sin(kw_0t)\right]dt$$

$$=\int_0^{\frac{1}{2}}\left[\sin(w_0t)\cos(kw_0t)-j\sin(w_0t)\sin(kw_0t)\right]dt$$

$$=-j\int_0^{\frac{1}{2}}\sin(3w_0t)\sin(kw_0t)dt-j\int_0^{\frac{1}{2}}\sin(w_0t)\sin(kw_0t)dt$$

$$=\begin{cases}-\frac{j}{4} \quad ,k=1,3\\ \frac{j}{4} \quad ,k=-1,-3\\ 0 \quad ,o.w.\end{cases}$$

3. (25%)

(1) $X_0(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-jwt}dt=\int_0^1 e^{-t}e^{-jwt}dt=\frac{1}{1+jw}\left(1-e^{-(1+jw)}\right)$

$$x(t)=x_0(t)+x_0(-t)$$

$$X(j\omega)=X_0(j\omega)+X_0(-j\omega)$$

$$=\frac{1-e^{-(1+jw)}}{1+jw}+\frac{1-e^{-(1-jw)}}{1-jw}=\frac{2-2e^{-1}\cos w+2we^{-1}\sin w}{1+w^2}$$

4. (25%)

令 $X_1(jw)=\begin{cases}0 & ,0<w<1\\ w-1 & ,1<w<2\\ 1 & ,2<w<3\\ 0 & ,o.w.\end{cases}$

$$X(jw)=X_1(jw)-X_1(-jw)\ \Rightarrow\ x(t)=x_1(t)-x_1(-t)$$

$$x_1(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(jw)e^{jwt}dw=\frac{1}{2\pi}\int_1^2(w-1)e^{jwt}dw+\frac{1}{2\pi}\int_2^3 e^{jwt}dw$$

$$=\frac{1}{2\pi}\left[\frac{1}{t^2}\left(e^{j2t}-e^{jt}\right)+\frac{1}{jt}e^{j3t}\right]$$

$$x(t)=x_1(t)-x_1(-t)=\frac{1}{2\pi}\left[\frac{1}{t^2}\left(e^{j2t}-e^{jt}\right)+\frac{1}{jt}e^{j3t}\right]-\frac{1}{2\pi}\left[\frac{1}{t^2}\left(e^{-j2t}-e^{-jt}\right)+\frac{1}{-jt}e^{-j3t}\right]$$

$$=\frac{\cos(3t)}{j\pi t}+\frac{\sin(t)-\sin(2t)}{j\pi t^2}$$