Reference Solutions of Homework # 2

1.

$$y[n]-y[n-1]+\frac{1}{4}y[n-2]=\frac{1}{2}x[n],\ y[-1]=1, y[-2]=0.$$

$$\Rightarrow y^{(h)}[n]=c_1(\frac{1}{2})^n+c_2n(\frac{1}{2})^n$$

(1) (10%)

$$\because x[n]=u[n] \therefore\ y^{(p)}[n]=Au[n]$$

$$\because y^{(p)}[n]-y^{(p)}[n-1]+\frac{1}{4}y^{(p)}[n-2]=\frac{1}{2}u[n]$$

$$\therefore A=2 \Rightarrow y[n]=c_1(\frac{1}{2})^n+c_2n(\frac{1}{2})^n+2u[n].$$

$$\because y[0]=\frac{3}{2},\ y[1]=\frac{7}{4} \therefore c_1=\frac{-1}{2},\ c_2=0.$$

$$\Rightarrow y[n]=\frac{-1}{2}(\frac{1}{2})^n+2u[n].$$

(2) (10%)

$\because r_1=\frac{1}{2}$, and $r_2=\frac{1}{2} \Rightarrow$ double roots, and $x[n]=(\frac{1}{2})^n u[n]$

We set the particular solution as $y^p[n]=kn^2(\frac{1}{2})^n$.

Substituting $y^p[n]$ into the difference equation, we have

$$kn^2(\frac{1}{2})^n-k(n-1)^2(\frac{1}{2})^{n-1}+\frac{1}{4}k(n-2)^2(\frac{1}{2})^{n-2}=\frac{1}{2}(\frac{1}{2})^n u[n].$$

After rearranging the above equation, we get

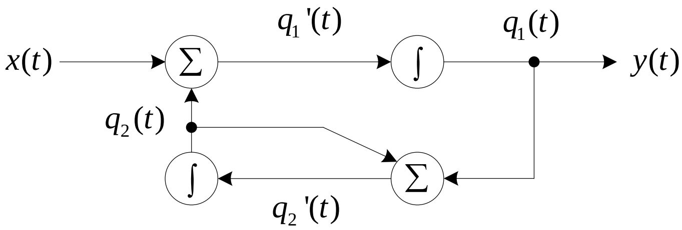
$$kn^2-2k(n-1)^2+k(n-2)^2=\frac{1}{2}. \quad \Rightarrow k=\frac{1}{4}$$

Hence, the particular solution of the difference equation is

$$y^{(p)}[n]=\frac{n^2}{4}(\frac{1}{2})^n=n^2(\frac{1}{2})^{n+2}, n\ge 0.$$

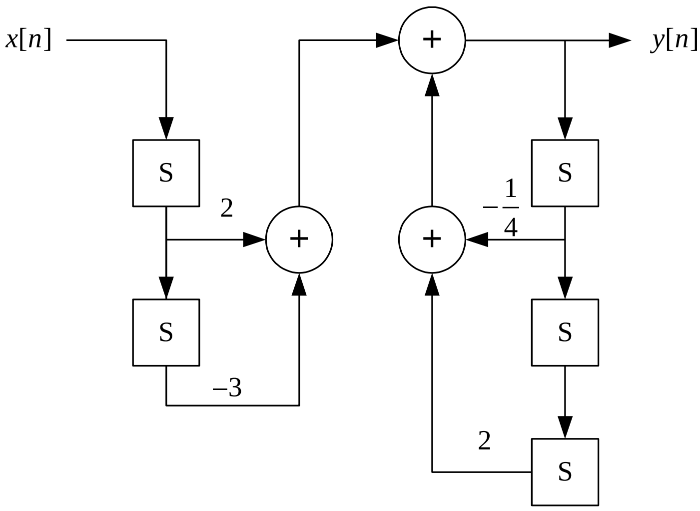
2.

(1) (7%)

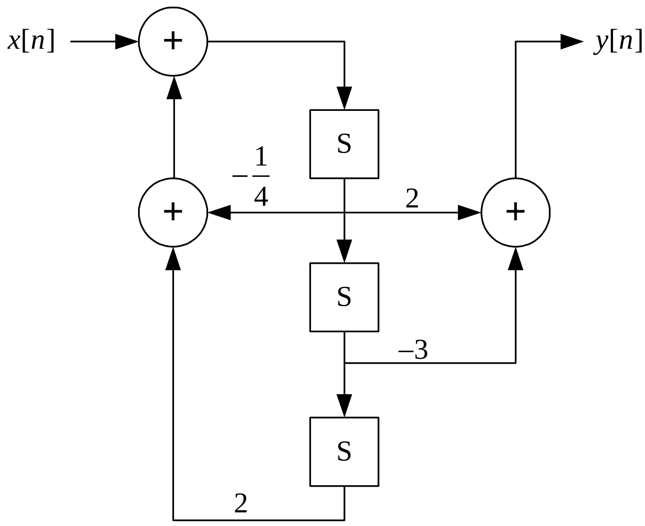


$$q_1'(t) = x(t) + q_2(t)$$
$$q_2'(t) = q_1(t) + q_2(t)$$
$$y(t) = q_1(t)$$
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ c = [1, 0],\ d = 0$$

(2) Direct form I: (4%)



Direct form II: (4%)



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3.

(1) (7%)

$$\because y_3[n] = \frac{1}{2}(y_1[n] - y_2[n]) \Rightarrow x_3[n] = \frac{1}{2}(x_1[n] - x_2[n]) = \delta[n-2].$$

(2) (8%)

$\because y_3[n] = \delta[n-2] + \delta[n-3]$, and $x_3[n] = \delta[n-2] \Rightarrow h[n] = \delta[n] + \delta[n-1]$

4.

(1) (5%) The step response $s(t) = h(t) * u(t)$

For $t < 0$, $s(t) = 0$.

For $0 \le t < 1$, $s(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t 1 \cdot d\tau = t$.

For $1 \le t < 2$, $s(t) = \int_0^1 1 \cdot d\tau + \int_1^t (-1) \cdot d\tau = 2 - t$.

For $t \ge 2$, $s(t) = 0$

$$s(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & \text{otherwise} \end{cases}$$. (6%)

(2) (5%)

$$x(t) = u(t) + u(t-1) + u(t-2) - 3u(t-3).$$

(3) (5%)

$$\begin{aligned} y(t) &= h(t) * x(t) = h(t) * \left[u(t) + u(t-1) + u(t-2) - 3u(t-3)\right] \\ &= h(t) * u(t) + h(t) * u(t-1) + h(t) * u(t-2) - 3h(t) * u(t-3) \\ &= s(t) + s(t-1) + s(t-2) - 3s(t-3). \end{aligned}$$

5.

(1) (10%) Note that

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau = \int_{-\infty}^{t-2} e^{-(t-2-\tau')} x(\tau')\, d\tau'$$

Therefore,

$$h(t) = e^{-(t-2)} u(t-2).$$

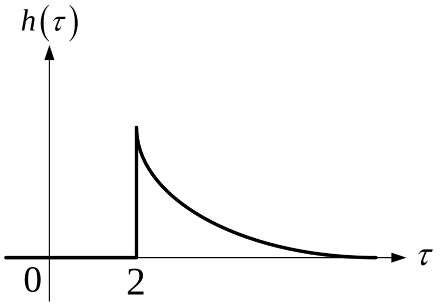
(2) (10%) We have

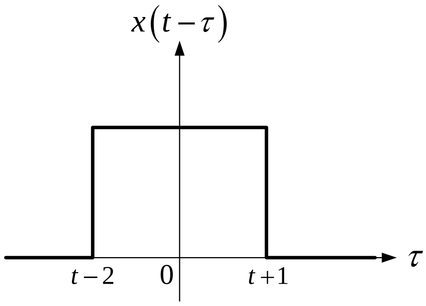
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{2}^{\infty} e^{-(\tau-2)}\left[u(t-\tau+1)-u(t-\tau-2)\right] \end{aligned}$$

$h(\tau)$ and $x(t-\tau)$ are as shown in the figure below.

Using this figure, we may write

$$y(t) = \begin{cases} 0, & t<1 \\ \int_{2}^{t+1} e^{-(\tau-2)}d\tau = 1-e^{-(t-1)}, & 1<t<4 \\ \int_{t-2}^{t+1} e^{-(\tau-2)}d\tau = e^{-(t-4)}\left[1-e^{-3}\right], & t>4 \end{cases}$$





6.

(1) (7%)

$$\begin{aligned} &y_p(t) = c_1\cos(3t)+c_2\sin(3t) \\ &y_p'(t) = -3c_1\sin(3t)+3c_2\cos(3t) \\ &y_p''(t) = -9c_1\cos(3t)-9c_2\sin(3t) \\ &y_p''(t)-5y_p'(t)+6y_p(t) = 2x(t) = 2\sin(3t) \\ &\begin{cases} -3c_1-15c_2=0 \\ 15c_1-3c_2=2 \end{cases} \Rightarrow c_1=\frac{5}{39},\ c_2=\frac{-1}{39}, \\ &\Rightarrow \therefore y_p(t) = \frac{5}{39}\cos(3t)-\frac{1}{39}\sin(3t) \end{aligned}$$

(2) (8%)

$$y_p(t) = Ae^{-3t} + Be^{-2t}$$

$$y_p'(t) = -3Ae^{-3t} - 2Be^{-2t}$$

$$y_p''(t) = 9Ae^{-3t} + 4Be^{-2t}$$

$$y_p''(t) - 5y_p'(t) + 6y_p(t) = 2x(t) = 2e^{-3t} + 2e^{-2t}$$

$$\begin{cases} 30Ae^{-3t} = 2e^{-3t} \\ 20Be^{-2t} = 2e^{-2t} \end{cases} \Rightarrow A = \frac{1}{15},\ B = \frac{1}{10},$$

$$\Rightarrow \therefore y_p(t) = \frac{1}{15}e^{-3t} + \frac{1}{10}e^{-2t}$$