Reference Solution of Midterm Exam I

1.

(1) (6%)

$$r^2 - \frac{5}{6}r + \frac{1}{6} = 0 \Rightarrow r = \frac{1}{2}, \frac{1}{3}$$

We assume $y^{(h)}[n] = c_1(\frac{1}{2})^n + c_2(\frac{1}{3})^n$.

For $n < 0 \Rightarrow x[n] = 0$

$$\Rightarrow \begin{cases} y[-1] = 2c_1 + 3c_2 = 1 \\ y[-2] = 4c_1 + 9c_2 = 0 \end{cases} \Rightarrow c_1 = \frac{3}{2},\ c_2 = -\frac{2}{3}.$$

$$\Rightarrow y[n] = \frac{3}{2}(\frac{1}{2})^n - \frac{2}{3}(\frac{1}{3})^n \quad \text{for} \quad n < 0.$$

(2) For $n \geq 0$ (8%)

We assume $y^{(p)}[n] = kn(\frac{1}{3})^n u[n]$ (2%)

$$kn(\frac{1}{3})^n - \frac{5}{6}k(n-1)(\frac{1}{3})^{n-1} + \frac{1}{6}k(n-2)(\frac{1}{3})^{n-2} = (\frac{1}{3})^n \quad \Rightarrow k = -2.$$

$$\Rightarrow y[n] = c_1(\frac{1}{2})^n + c_2(\frac{1}{3})^n - 2n(\frac{1}{3})^n u[n]$$

Consider the initial condition $y[-1] = 1$, and $y[-2] = 0$, then we have

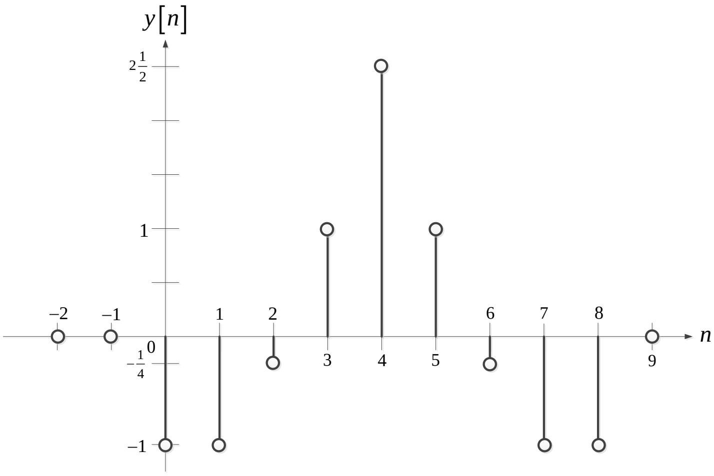
$$\begin{cases} y[0] = \frac{11}{6} \\ y[1] = \frac{61}{36} \end{cases} \Rightarrow \begin{cases} y[0] = c_1 + c_2 = \frac{11}{6} \\ y[1] = \frac{1}{2}c_1 + \frac{1}{3}c_2 = \frac{61}{36} \end{cases} \Rightarrow c_1 = \frac{21}{2},\ c_2 = -\frac{26}{3}.$$

Finally, we get

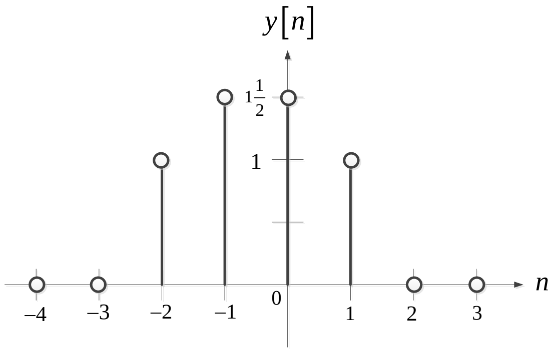
$$y[n] = \left[\frac{21}{2}(\frac{1}{2})^n - \frac{26}{3}(\frac{1}{3})^n - 2n(\frac{1}{3})^n\right]u[n]$$

2.

(1) (7%)



(2) (8%)



3. (15%)

(1) $y(t) = \left(\cos(\pi t)\right)x(t)$

(i) memoryless: 輸出只與當時的輸入有關

(ii) causal

(iii) stable: $\left|y(t)\right| = \left|\cos(\pi t)x(t)\right| \le M_x$

(iv) time-varying: $y(t) = H\{x(t)\} = \cos(\pi t)x(t)$
$H\{x(t-t_0)\} = \cos(\pi t)x(t-t_0) \ne y(t-t_0)$

(v) linear: $ay_1(t) + by_2(t) = H\{ax_1(t) + bx_2(t)\}$

(2) $y[n]=x[n^2]$

(i) memory: 輸出與未來輸入有關
(ii) non-causal: 輸出與未來輸入有關
(iii) stable: $|y[n]|=|x[n^2]|\le M_x$
(iv) time-varying: $y[n]=H\{x[n]\}=x[n^2]$
$H\{x[n-1]\}=x[n^2-1]\ne y[n-1]$
(v) linear: $ay_1[n]+by_2[n]=H\{ax_1[n]+bx_2[n]\}$

(3) $y[n]=x[n]\sum_{k=0}^{\infty}\delta[n-k]\Rightarrow \begin{array}{ll} y[n]=0 & \text{for } n<0 \\ y[n]=x[n] & \text{for } n\ge 0 \end{array}$

(i) memoryless: 輸出只與當時的輸入有關
(ii) causal
(iii) stable
(iv) time-varying
(v) linear: $ay_1[n]+by_2[n]=H\{ax_1[n]+bx_2[n]\}$

4. (12%)

(1) $x_1(t)=2e^{-4|t|}$

Energy signal and $E=1$.
This signal is non-periodic signal $\Rightarrow$ maybe an energy signal

$$E_{x_1(t)}=\lim_{T\to\infty}\int_{-T/2}^{T/2}x_1^2(t)\,dt=\int_{-\infty}^{\infty}2e^{-4|t|}dt=4\int_0^{\infty}e^{-4t}dt=1<\infty.$$

Therefore, it's proved that $x_1(t)$ is an energy signal and its corresponding energy is 1.

(2) $x_2(t)=5\cos(\pi t)+\sin(5\pi t)$

Power signal and $P=1$.
This signal is periodic signal $\Rightarrow$ maybe a power signal

$$P_{x_2(t)} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_2^2(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\big(5\cos(\pi t)+\sin(5\pi t)\big)^2\,dt$$
$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} 25\cos^2(\pi t)+\sin^2(5\pi t)+10\cos(\pi t)\sin(5\pi t)\,dt$$
$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{25}{2}\big(1+\cos(2\pi t)\big)+\frac{1}{2}\big(1-\cos(10\pi t)\big)+\frac{10}{2}\big(\sin(6\pi t)-\sin(-4\pi t)\big)\,dt$$
$$= \frac{25}{2}+\frac{1}{2} = 13 < \infty$$

Hence, it's proved that $x_2(t)$ is a power signal and its corresponding power is 13.

(3) $x_3(t) = tu(t)$

Neither energy signal nor power signal.

$$E_{x_3(t)} = \int_{-\infty}^{\infty}\big(tu(t)\big)^2\,dt = \int_0^{\infty} t^2 dt = \frac{1}{3}t^3\Big|_0^{\infty} \to \infty$$
$$P_{x_3(t)} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\big(tu(t)\big)^2\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} t^2 dt = \lim_{T\to\infty}\frac{\left(T/2\right)^3}{3T} \to \infty$$

5. (10%)

$$s^2+3s+2=0,\ s=-1,-2 \Rightarrow y^{(h)}(t) = c_1e^{-t}+c_2e^{-2t}$$

$$\because x(t) = e^{-3t}u(t).\quad y^{(p)}(t) = ke^{-3t} \quad \Rightarrow k = \frac{1}{2}$$

$$\Rightarrow y(t) = c_1e^{-t}+c_2e^{-3t}+\frac{1}{2}e^{-3t}u(t)$$

Zero-state response $\Rightarrow y(0^-)=0,\ y'(0)=0$

$$\Rightarrow c_1 = c_2 = 0 \ \Rightarrow y^{(f)}(t) = \frac{1}{2}e^{-3t}u(t)$$

6. (12%)

(1) Periodic,

$$x(t) = \left|\cos\left(2t-\frac{\pi}{3}\right)\sin\left(3t-\frac{\pi}{2}\right)\right| = \frac{1}{2}\left|\sin\left(5t-\frac{5\pi}{6}\right)-\sin\left(-t+\frac{\pi}{6}\right)\right|$$
$$T = lcm\left(2\pi,\frac{2\pi}{5}\right)\Big/2 = \pi$$

(2) Periodic,

$$x[n] = e^{j\frac{\pi}{16}n}\cos\left(\frac{\pi}{17}n\right) = \left\{\cos\left(\frac{\pi}{16}n\right) + j\sin\left(\frac{\pi}{16}n\right)\right\}\cos\left(\frac{\pi}{17}n\right)$$

$$= \cos\left(\frac{\pi}{16}n\right)\cos\left(\frac{\pi}{17}n\right) + j\sin\left(\frac{\pi}{16}n\right)\cos\left(\frac{\pi}{17}n\right)$$

$$= \frac{1}{2}\left\{\cos\left(\frac{33\pi}{272}n\right) + \cos\left(\frac{\pi}{272}n\right)\right\} + \frac{1}{2}j\left\{\sin\left(\frac{33\pi}{272}n\right) + \sin\left(\frac{\pi}{272}n\right)\right\}$$

$$\left.\begin{aligned} &\frac{33\pi}{272}N_1 = 2\pi m \Rightarrow N_1 = \frac{544}{33}m \\ &\frac{\pi}{272}N_2 = 2\pi k \Rightarrow N_2 = 544k \end{aligned}\right\} \Rightarrow N = 544$$

(3) Aperiodic

7. (10%)

$$y[n] - \rho y[n-1] = x[n], \ |\rho| < 1, \ y[-1] = 0.$$

(1) Determine the impulse response $\Rightarrow x[n] = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$

$$y[0] = x[0] + \rho y[-1] = 1 + 0 = 1$$
$$y[1] = x[1] + \rho y[0] = 0 + \rho = \rho$$
$$y[2] = x[2] + \rho y[1] = 0 + \rho \cdot \rho = \rho^2$$
$$y[3] = x[3] + \rho y[2] = 0 + \rho \cdot \rho^2 = \rho^2$$
$$\vdots \qquad\qquad \vdots$$
$$y[n] = x[n] + \rho y[n-1] = 0 + \rho \cdot \rho^{n-1} = \rho^n \ , n \geq 0$$

and $y[n] = 0, n < 0$

So we have the impulse response represented as

$y[\ n]= \ h[\ n]= \rho^n \quad u[$

(2) Determine the step response $\Rightarrow x[n] = u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & otherwise \end{cases}$

$$
\begin{aligned}
y[0] &= x[0] + \rho y[-1] = 1 + 0 = 1 \\
y[1] &= x[1] + \rho y[0] = 1 + \rho \\
y[2] &= x[2] + \rho y[1] = 1 + \rho(1+\rho) = 1 + \rho + \rho^2 \\
y[3] &= x[3] + \rho y[2] = 1 + \rho \cdot (1 + \rho + \rho^2) = 1 + \rho + \rho^2 + \rho^3 \\
&\ \vdots \qquad\qquad \vdots \\
y[n] &= x[n] + \rho y[n-1] = 1 + \rho \cdot (1 + \rho + \rho^2 + \cdots + \rho^{n-1}) \\
&= \sum_{k=0}^{n} \rho^k \ , n \geq 0
\end{aligned}
$$

and $y[n] = 0, n < 0$

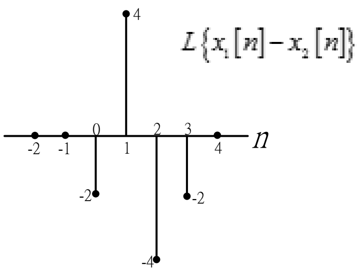
Therefore, we can find the step response $s[n] = \sum_{k=0}^{n} \rho^k u[n]$.

(3) $\because |\rho| < 1 \ \Rightarrow$ BIBO stable

8. (11%)

(1)

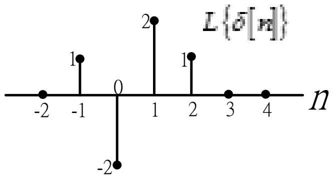
$$L\{x_1[n] - x_2[n]\} = y_1[n] - y_2[n]$$



(2)

$$\delta[n] = \frac{1}{2} x_1[n] - \frac{1}{2} x_2[n] + x_3[n]$$

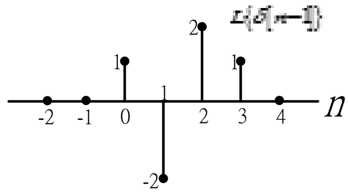
$$L\{\delta[n]\} = \frac{1}{2} y_1[n] - \frac{1}{2} y_2[n] + y_3[n]$$



(3)

$$\delta[n-1] = -\frac{1}{2}\left(x_1[n] - x_2[n]\right)$$

$$L\{\delta[n-1]\} = -\frac{1}{2}\left(y_1[n] - y_2[n]\right)$$

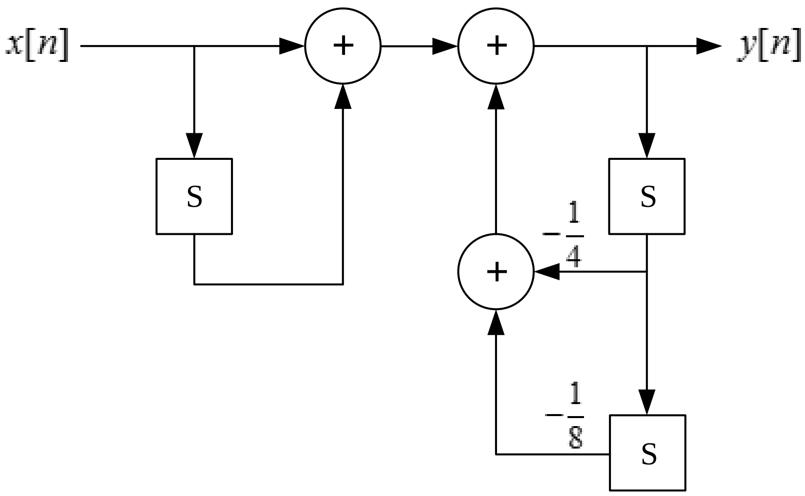


If the input $\delta[n]$ delay 1 unit, the output $L\{\delta[n]\}$ also delay 1 unit.

The system is time-invariant.

9. (12%)

(1)

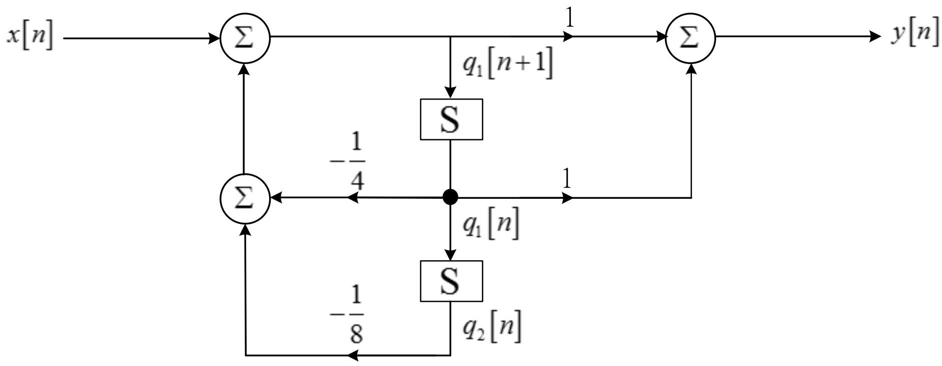


(I) $y[n] + \frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] + x[n-1]$

or

(II) $y[n] = x[n] + x[n-1] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2]$

(2)



$$q_1[n+1] = -\frac{1}{4}q_1[n] - \frac{1}{8}q_2[n] + x[n]$$

$$q_2[n+1] = q_1[n]$$

$$y[n] = \frac{3}{4}q_1[n] - \frac{1}{8}q_2[n] + x[n]$$

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{8} \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \left[\frac{3}{4}, -\frac{1}{8}\right], \quad D = 1$$