Reference Solution of Midterm Exam II

1. $T = 6, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{6}\int_{-3}^{3} x(t)e^{-jk\omega_0 t}dt$$
$$= \frac{1}{6}\int_{-2}^{-1} e^{-jk\omega_0 t}dt - \frac{1}{6}\int_{1}^{2} e^{-jk\omega_0 t}dt$$
$$= \frac{1}{6}\cdot\frac{1}{-jk\omega_0}(e^{jk\omega_0} - e^{j2k\omega_0}) - \frac{1}{6}\cdot\frac{1}{-jk\omega_0}(e^{-j2k\omega_0} - e^{-jk\omega_0})$$
$$= \frac{j}{6k\omega_0}(e^{jk\omega_0} + e^{-jk\omega_0} - e^{j2k\omega_0} - e^{-j2k\omega_0})$$
$$= \frac{j}{\pi k}[\cos(\frac{\pi k}{3}) - \cos(\frac{2\pi k}{3})] \qquad (6\%)$$
$$a_0 = \frac{1}{6}\int_{-3}^{3} x(t)e^{-j0\omega_0 t}dt = 0 \qquad (4\%)$$

2. 3 points for each subproblem

(1) $\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0) = 0$

(2) $\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega = 2\pi\int_{-\infty}^{\infty}|x(t)|^2\, dt = 4\pi\int_{1}^{4}|t-1|^2\, dt = 4\pi\cdot 9 = 36\pi$

(3) $\int_{-\infty}^{\infty} X(j\omega)e^{j2\omega}d\omega = 2\pi x(2) = 2\pi$

(4)

$\because x(t)$ is real and even

$\therefore \mathrm{Im}(X(j\omega)) = 0 \Rightarrow \tan^{-1}\left\{\frac{\mathrm{Im}(X(j\omega))}{\mathrm{Re}(X(j\omega))}\right\} =$

3.

(1) (3%)

$$Y(e^{j\Omega}) - \frac{3}{4}e^{-j\Omega}Y(e^{j\Omega}) + \frac{1}{8}e^{-j2\Omega}Y(e^{j\Omega}) = 2X(e^{j\Omega})$$
$$Y(e^{j\Omega})(1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}) = 2X(e^{j\Omega})$$
$$H(e^{j\Omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}}$$

(2) (4%)

$$H(e^{j\Omega}) = \frac{2}{1-\frac{3}{4}e^{-j\Omega}+\frac{1}{8}e^{-j2\Omega}} = \frac{2}{(1-\frac{1}{2}e^{-j\Omega})(1-\frac{1}{4}e^{-j\Omega})} = \frac{4}{1-\frac{1}{2}e^{-j\Omega}} - \frac{2}{1-\frac{1}{4}e^{-j\Omega}}$$

$$h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$$

(3) (5%)

$$X(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\Omega})^2(1-\frac{1}{4}e^{-j\Omega})} = \frac{-4}{1-\frac{1}{2}e^{-j\Omega}} + \frac{4}{(1-\frac{1}{2}e^{-j\Omega})^2} + \frac{2}{1-\frac{1}{4}e^{-j\Omega}}$$

$$y[n] = -4(\frac{1}{2})^n u[n] + 4(n+1)(\frac{1}{2})^n u[n] + 2(\frac{1}{4})^n u[n] = 4n(\frac{1}{2})^n u[n] + 2(\frac{1}{4})^n u[n]$$

4. (10%)

$$x[n] = \sin(\frac{\pi n}{4}) + \cos(\frac{\pi n}{2}) = \frac{1}{2j}(e^{j\frac{\pi n}{4}} - e^{-j\frac{\pi n}{4}}) + \frac{1}{2}(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}})$$

$$y[n] = \frac{1}{2j}[H(e^{j\frac{\pi}{4}})e^{j\frac{\pi n}{4}} - H(e^{-j\frac{\pi}{4}})e^{-j\frac{\pi n}{4}}] + \frac{1}{2}[H(e^{j\frac{\pi}{2}})e^{j\frac{\pi n}{2}} - H(e^{-j\frac{\pi}{2}})e^{-j\frac{\pi n}{2}}]$$

$$= \frac{1}{2j}[2(1+j)e^{j\frac{\pi n}{4}} - 2(1-j)e^{-j\frac{\pi n}{4}}] + \frac{1}{2}[\frac{4}{3}e^{j\frac{\pi n}{2}} - \frac{4}{3}e^{-j\frac{\pi n}{2}}]$$

$$= 2\sin(\frac{\pi n}{4}) + 2\cos(\frac{\pi n}{4}) + \frac{4}{3}\cos(\frac{\pi n}{2})$$

5.

$$x[n] = n(\frac{1}{2})^{|n|} \xleftrightarrow{DTFT} X(e^{j\Omega})$$

real, odd pure imaginary

(1) (3%)

$$Y(e^{j\Omega}) = \mathrm{Re}\{X(e^{j\Omega})\} = 0 \Rightarrow y[n] = 0$$

(2) (3%)

$$y[n] = -jnx[n] = -jn^2(1/2)^{|n|}.$$

(3) (4%)

$$y[n] = 2\pi\left[x[n](e^{j\frac{\pi}{2}n}x[n])\right] = 2\pi n^2(1/2)^{2|n|}e^{j\frac{\pi}{2}n}.$$

6.

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3]$$

(1) (4%)

$$X[k] = 2 + e^{-j\frac{2\pi}{5}k} + e^{-j3\frac{2\pi}{5}k}.$$

(2) (4%)

$$Y[k] = X^2[k] = 4 + 4e^{-j\frac{2\pi}{5}k} + e^{-j2\frac{2\pi}{5}k} + 4e^{-j3\frac{2\pi}{5}k} + 2e^{-j4\frac{2\pi}{5}k} + e^{-j6\frac{2\pi}{5}k}$$

$$= 4 + 5e^{-j\frac{2\pi}{5}k} + e^{-j2\frac{2\pi}{5}k} + 4e^{-j3\frac{2\pi}{5}k} + 2e^{-j4\frac{2\pi}{5}k}$$

$$\Rightarrow y[n] = 4\delta[n] + 5\delta[n-1] + \delta[n-2] + 4\delta[n-3] + 2\delta[n-4]$$

(3) (4%)

Not equal, N should be more than or equal to 7 such that

$$z[n] = x[n] * x[n] = IDFT\{X^2[k]\} = x[n] \circledast x[n]$$

7.

(1) (4%)

$$H_1(e^{j\Omega}) = \left|A(e^{j\Omega})\right| e^{-j\alpha\Omega},\ H_2(e^{j\Omega}) = \left|B(e^{j\Omega})\right| e^{-j\beta\Omega}$$

$$H_3(e^{j\Omega}) = H_1(e^{j\Omega}) H_2(e^{j\Omega}) = \left|A(e^{j\Omega})\right|\left|B(e^{j\Omega})\right| e^{-j(\alpha+\beta)\Omega}$$

$$\angle H_3(e^{j\Omega}) = -(\alpha+\beta)\Omega \ \Rightarrow \text{ is linearly proportional to } \Omega$$

A cascade system of two linear phase systems is linear phase.

(2) (4%)

$$H_4(e^{j\Omega}) = H_1(e^{j\Omega}) + H_2(e^{j\Omega}) = \left|A(e^{j\Omega})\right| e^{-j\alpha\Omega} + \left|B(e^{j\Omega})\right| e^{-j\beta\Omega}$$

$$\angle H_4(e^{j\Omega}) \neq -(\alpha+\beta)\Omega$$

A parallel system of two linear phase systems is not linear phase.

8. (10%)

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-j\frac{2\pi k}{N}n}, \quad N = 7$$

$$a_k = \frac{1}{7}\sum_{n=0}^{6} (\delta[n] + \delta[n-3] - \delta[n-4]) e^{-j\frac{2\pi k}{7}n}$$

$$= \frac{1}{7}(1 + e^{-j\frac{6\pi k}{7}} - e^{-j\frac{8\pi k}{7}})$$

9.

(1)

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{j\omega+4}{6+5j\omega+(j\omega)^2}$$
$$\Rightarrow 6Y(j\omega)+5(j\omega)Y(j\omega)+(j\omega)^2Y(j\omega)=(j\omega)X(j\omega)+4X(j\omega)$$
$$\Rightarrow \frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{dx(t)}{dt}+4x(t)$$

(2)

$$H(j\omega)=\frac{j\omega+4}{6+5j\omega+(j\omega)^2}=\frac{A}{j\omega+2}+\frac{B}{j\omega+3}$$
$$A=\left.\frac{j\omega+4}{j\omega+3}\right|_{j\omega=-2}=2,\quad B=\left.\frac{j\omega+4}{j\omega+2}\right|_{j\omega=-3}=-1$$
$$\Rightarrow H(j\omega)=\frac{2}{j\omega+2}+\frac{-1}{j\omega+3}$$
$$\Rightarrow h(t)=2e^{-2t}u(t)-e^{-3t}u(t)$$

(3)

$$x(t)=e^{-4t}u(t)-te^{-4t}u(t)\leftrightarrow X(j\omega)=\frac{j\omega+3}{(j\omega+4)^2}$$
$$Y(j\omega)=H(j\omega)X(j\omega)=\frac{j\omega+4}{(j\omega+2)(j\omega+3)}\cdot\frac{j\omega+3}{(j\omega+4)^2}$$
$$=\frac{1}{(j\omega+2)(j\omega+4)}=\frac{C}{j\omega+2}+\frac{D}{j\omega+4}$$
$$C=\left.\frac{1}{j\omega+4}\right|_{j\omega=-2}=\frac{1}{2},\quad D=\left.\frac{1}{j\omega+2}\right|_{j\omega=-4}=-\frac{1}{2}$$
$$\Rightarrow Y(j\omega)=\frac{0.5}{j\omega+2}+\frac{-0.5}{j\omega+4}$$
$$\Rightarrow y(t)=\frac{1}{2}e^{-2t}u(t)-\frac{1}{2}e^{-4t}u(t)$$

10.

(1) $\textit{Impulse train in time domain}\xleftrightarrow{F}\textit{Impulse train in frequency domain}$

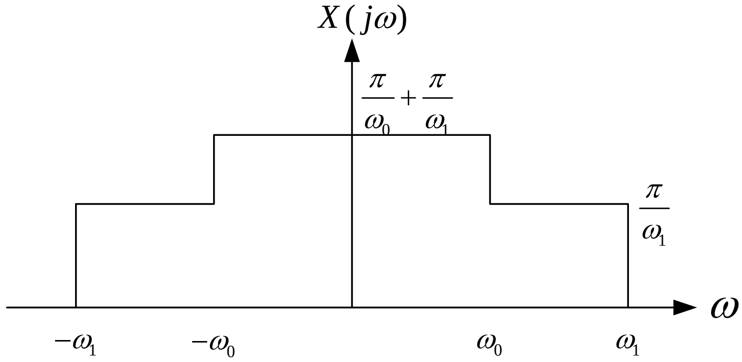
$$p(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)=\sum_{k=-\infty}^{\infty}a_k e^{j\omega_0 t}$$
$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-j\omega t}dt=\frac{1}{T}\Rightarrow p(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{j\omega_0 t}$$
$$\Rightarrow P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{2\pi k}{T})$$

(2) From the given table

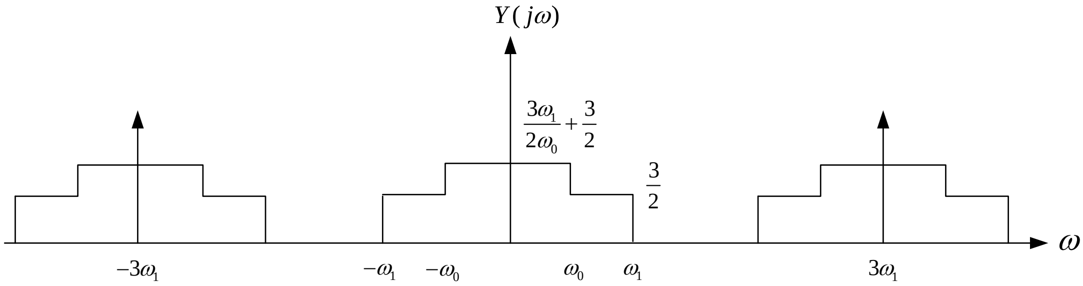
$$x(t) = x_0(t) + x_1(t) = \frac{\sin(\omega_0 t)}{\omega_0 t} + \frac{\sin(\omega_1 t)}{\omega_1 t}$$

$$X_0(j\omega) = \begin{cases} \frac{\pi}{|\omega_0|}, & |\omega| < \omega_0 \\ 0, & o.w. \end{cases}, \quad X_1(j\omega) = \begin{cases} \frac{\pi}{|\omega_1|}, & |\omega| < \omega_1 \\ 0, & o.w. \end{cases}$$

$$\Rightarrow X(j\omega) = X_0(j\omega) + X_1(j\omega)$$



(3)



Yes, it can be recovered from $y(t)$ because there is no aliasing in the spectrum.

(4)

$$T_{\max} = T_1 / 2$$