# Midterm Exam I Reference Solutions

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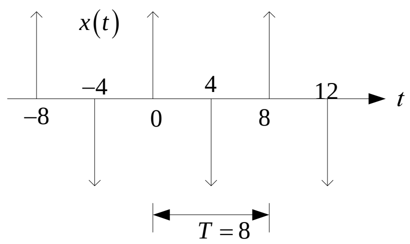
1. (9%)

(1) Periodic,

$$\left.\begin{aligned} 2t = 2\pi\frac{1}{T_1}t \Rightarrow T_1 = \pi \\ 3t = 2\pi\frac{1}{T_2}t \Rightarrow T_2 = \frac{2\pi}{3} \end{aligned}\right\} \Rightarrow T_0 = 2\pi$$

(2) Periodic, $T = 8$.




(3) Periodic,

$$x[n] = e^{j\frac{\pi}{16}n}\cos\left(\frac{\pi}{17}n\right) = \left\{\cos\left(\frac{\pi}{16}n\right) + j\sin\left(\frac{\pi}{16}n\right)\right\}\cos\left(\frac{\pi}{17}n\right)$$

$$= \cos\left(\frac{\pi}{16}n\right)\cos\left(\frac{\pi}{17}n\right) + j\sin\left(\frac{\pi}{16}n\right)\cos\left(\frac{\pi}{17}n\right)$$

$$= \frac{1}{2}\left\{\cos\left(\frac{33\pi}{272}n\right) + \cos\left(\frac{\pi}{272}n\right)\right\} + \frac{1}{2}j\left\{\sin\left(\frac{33\pi}{272}n\right) + \sin\left(\frac{\pi}{272}n\right)\right\}$$

$$\left.\begin{aligned} \frac{33\pi}{272}N_1 = 2\pi m \Rightarrow N_1 = \frac{544}{33}m \\ \frac{\pi}{272}N_2 = 2\pi k \Rightarrow N_2 = 544k \end{aligned}\right\} \Rightarrow N = 544$$

2.

(1) The step response $s(t) = h(t) * u(t)$ (6%)

For $t < 0$, $s(t) = 0$.

For $0 \le t < 1$, $s(t) = \int_0^t h(t)u(t-\tau)dt = \int_0^t 1 \cdot dt = t$.

For $1 \le t < 2$, $s(t) = \int_0^1 1 \cdot dt + \int_1^t (-1) \cdot dt = 2 - t$.

For $t \ge 2$, $s(t) = 0$

$$s(t)=\begin{cases} t, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & \text{otherwise} \end{cases}$$

(2) $x(t)=u(t)+u(t-1)-2u(t-2)$ (3%)

(3) (3%)

$$\begin{aligned} y(t) &= h(t)*x(t) \\ &= h(t)*\left[u(t)+u(t-1)-2u(t-2)\right] \\ &= h(t)*u(t)+h(t)*u(t-1)-2h(t)*u(t-2) \\ &= s(t)+s(t-1)-2s(t-2) \\ &= \begin{cases} t, & 0\le t<1 \\ 1, & 1\le t<2 \\ 7-3t, & 2\le t<3 \\ 2t-8, & 3\le t<4 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where

$$s(t-1)=\begin{cases} t-1, & 1\le t<2 \\ 3-t, & 2\le t<3 \\ 0, & \text{otherwise} \end{cases},\quad 2s(t-2)=\begin{cases} 2t-4, & 2\le t<3 \\ 8-2t, & 3\le t<4 \\ 0, & \text{otherwise} \end{cases}$$

3. (12%)

$$x_1[n]*x_2[n]=\sum_{k=\infty}^{-\infty}x_2[k]\,x_1[n-k]=\sum_{k=1}^{3}x_1[n-k]$$

$$x_1[n]*x_2[n]=\begin{cases} 0, & n\le 0 \\ 1, & n=1 \\ 2, & n=2 \\ 3, & n=3 \\ 3, & n=4 \\ 3, & n=5 \\ 3, & n=6 \\ 2, & n=7 \\ 1, & n=8 \\ 0, & n\ge 9 \end{cases}$$

4. (10%)

(1) **Memoryless**: $y[n]$ depends on the $n^{th}$ value of $x[n]$ only.

**Stable**: $|y[n]| = 3|x[n]| + 5$, which is stable for finite $|x[n]|$.

**Causal**: This doesn't use future values of $x[n]$, so it is causal.

**Nonlinear**: $T\left(ax_1[n] + bx_2[n]\right) = 3ax_1[n] + 3bx_2[n] + 5 \neq ay_1[n] + by_2[n]$

**Time-invariant**: $T\left(x[n-n_0]\right) = 3x[n-n_0] + 5 = y[n-n_0]$

(2) **Memoryless**: $y[n]$ depends on the $n^{th}$ value of $x[n]$ only.

**Unstable**: $|x[n]| = 0$, $|y[n]| = |\log_{10}(0)| = \infty$.

**Causal**: This doesn't use future values of $x[n]$, so it is causal.

**Nonlinear**: $$y_1[n] = \log_{10}\left(|\alpha x_1[n]|\right);\ y_2[n] = \log_{10}\left(|\beta x_2[n]|\right)$$ $$y_3[n] = \log_{10}\left(|\alpha x_1[n] + \beta x_2[n]|\right) \neq y_1[n] + y_2[n]$$

**Time-invariant**: $$x_2[n] = x[n-n_0]$$ $$y_2[n] = \log_{10}(|x_2[n]|) = \log_{10}(|x[n-n_0]|) = y[n-n_0]$$

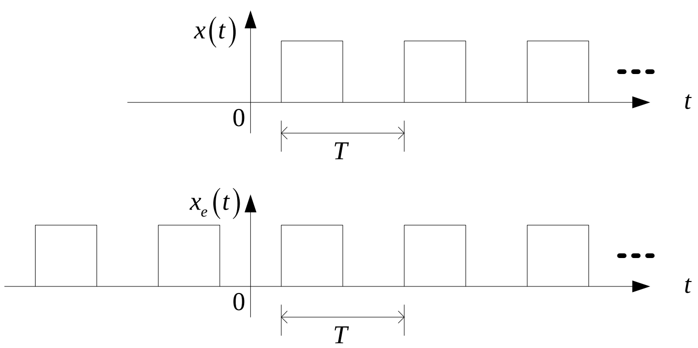
5. (10%)

(1) **True**.

$$y[n+N] = x[2n+2N] = x[2n] = y[n]$$
$$\left(\because x[2n] = x[2n+N] = x[2n+2N]\right)$$

(2) **False**.

It is noted that any signal can be broken into a sum of an odd signal and an even signal.




(3) **False**.

$tu(t)$ is neither energy signal nor power signal.

(4) **True**.

All memoryless systems are causal, since the output corresponds only to the current value of the input.

(5) **True**.

BIBO stable $\Leftrightarrow$ $h[n]$ is absolutely summable.

$h[n]$ is not absolutely summable. $\Rightarrow$ not BIBO stable

6.

(1) (5%)

Since the $y_3[n]=\frac{1}{2}(y_1[n]-y_2[n])$, $x_3[n]=\frac{1}{2}(x_1[n]-x_2[n])=\delta[n-1]$.

(2) (5%)

$\because y_3[n]=\delta[n-2]+\delta[n-3]$, and $x_3[n]=\delta[n-1]$ $\Rightarrow h[n]=\delta[n-1]+\delta[n-2]$

7.

$$\frac{d^2y(t)}{dt^2}-6\frac{dy(t)}{dt}+8y(t)=x(t),\ y(0)=0,\ \left.\frac{d}{dt}y(t)\right|_{t=0}=y'(0)=0.$$

The homogeneous solution is

$$y^{(h)}(t)=c_1e^{2t}+c_2e^{4t}.$$

(1) (6%)

$$\because x(t)=\sin(2t)\ \therefore y^{(p)}(t)=A\sin 2t+B\cos 2t.$$

$$y^{(p)'}(t)=2A\cos 2t-2B\sin 2t,$$
$$y^{(p)''}(t)=-4A\sin 2t-4B\cos 2t,$$
$$\because \frac{d^2y^{(p)}(t)}{dt^2}-6\frac{dy^{(p)}(t)}{dt}+8y^{(p)}(t)=\sin(2t)$$
$$\therefore A=\frac{1}{40},\ B=\frac{3}{40}\Rightarrow y^{(p)}(t)=\frac{1}{40}\sin 2t+\frac{3}{40}\cos 2t.$$

$y(t)=c_1e^{2t}+c_2e^{4t}+\frac{1}{40}\sin 2t+\frac{3}{40}\cos 2t,$ and $y(0)=0, y'(0)=0\Rightarrow$

$c_1=-\frac{1}{8},\ c_2=\frac{1}{20}\Rightarrow y(t)=-\frac{1}{8}e^{2t}+\frac{1}{20}e^{4t}+\frac{1}{40}\sin 2t+\frac{3}{40}\cos 2t.$

(2) (6%)

$\because x(t)=e^{-2t}\ \therefore y^{(p)}(t)=Ae^{-2t}.$

$$y^{(p)'}(t) = -2Ae^{-2t}$$
$$y^{(p)''}(t) = 4Ae^{-2t},$$
$$\because \frac{d^2y^{(p)}(t)}{dt^2} - 6\frac{dy^{(p)}(t)}{dt} + 8y^{(p)}(t) = e^{-2t}$$
$$\therefore A = \frac{1}{24} \Rightarrow y^{(p)}(t) = \frac{1}{24}e^{-2t}.$$

$$y(t) = c_1e^{2t} + c_2e^{4t} + \frac{1}{24}e^{-2t}, \text{ and } y(0) = 0, y'(0) = 0 \Rightarrow$$
$$c_1 = -\frac{1}{8},\ c_2 = \frac{1}{12} \Rightarrow y(t) = -\frac{1}{8}e^{2t} + \frac{1}{12}e^{4t} + \frac{1}{24}e^{-2t}.$$

8.

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = \frac{1}{2}x[n],\ y[-1] = 1, y[-2] = 0.$$
$$\Rightarrow y^{(h)}[n] = c_1(\frac{1}{2})^n + c_2n(\frac{1}{2})^n$$

(1) (7%)

$$\because x[n] = u[n] \therefore\ y^{(p)}[n] = Au[n]$$
$$\because y^{(p)}[n] - y^{(p)}[n-1] + \frac{1}{4}y^{(p)}[n-2] = \frac{1}{2}u[n]$$
$$\therefore A = 2 \Rightarrow y[n] = c_1(\frac{1}{2})^n + c_2n(\frac{1}{2})^n + 2u[n].$$

$$\because y[0] = \frac{3}{2},\ y[1] = \frac{7}{4} \therefore c_1 = \frac{-1}{2},\ c_2 = 0.$$
$$\Rightarrow y[n] = \frac{-1}{2}(\frac{1}{2})^n + 2u[n].$$

(2) (6%)

$$y^{(n)}[n] = c_1(\frac{1}{2})^n + c_2n(\frac{1}{2})^n,\ y[-1] = 1, y[-2] = 0 \Rightarrow$$
$$c_1 = 1,\ c_2 = \frac{1}{2} \Rightarrow y^{(n)}[n] = (\frac{1}{2})^n + \frac{1}{2}n(\frac{1}{2})^n.$$

$$y^{(f)}[n] = c_1(\frac{1}{2})^n + c_2n(\frac{1}{2})^n + 2u[n],\ y[-1] = y[-2] = 0 \Rightarrow$$
$$y[0] = \frac{1}{2},\ y[1] = 1 \Rightarrow c_1 = \frac{-3}{2},\ c_2 = \frac{-1}{2} \Rightarrow$$
$$y^{(f)}[n] = \frac{-3}{2}(\frac{1}{2})^n + \frac{-1}{2}n(\frac{1}{2})^n + 2u[n].$$

9.

$$y[n]-\frac{1}{4}y[n-2]=x[n]-x[n-1].$$

$$y^{(h)}[n]=c_1(\frac{1}{2})^n+c_2(-\frac{1}{2})^n.$$

(1) (5%)

$$\because x[n]=n^2 \therefore x[n]-x[n-1]=2n-1 \Rightarrow y^{(p)}[n]=an+b,$$
$$y^{(p)}[n]-\frac{1}{4}y^{(p)}[n-2]=2n-1 \Rightarrow a=\frac{8}{3}, b=\frac{-28}{9} \Rightarrow$$
$$y^{(p)}[n]=\frac{8}{3}n-\frac{28}{9}.$$

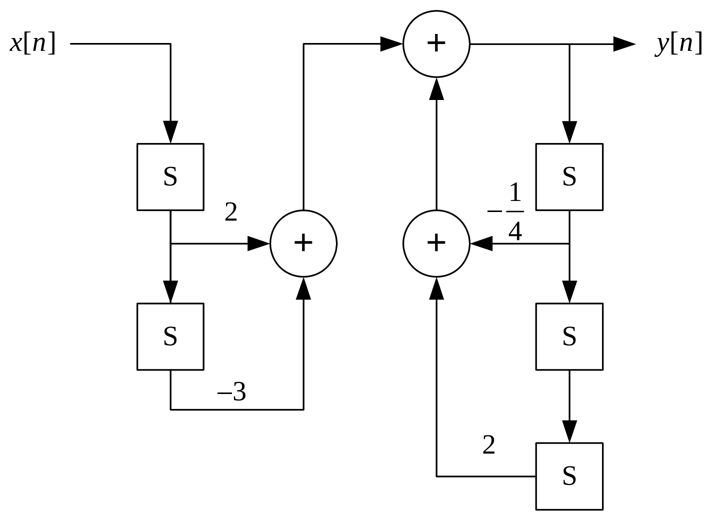
(2) (5%)

$$\because x[n]=(-\frac{1}{2})^n \therefore y^{(p)}[n]=An(-\frac{1}{2})^n$$
$$y^{(p)}[n]-\frac{1}{4}y^{(p)}[n-2]=(-\frac{1}{2})^n-(-\frac{1}{2})^{n-1} \Rightarrow A=\frac{3}{2} \Rightarrow$$
$$y^{(p)}[n]=\frac{3}{2}n(-\frac{1}{2})^n.$$

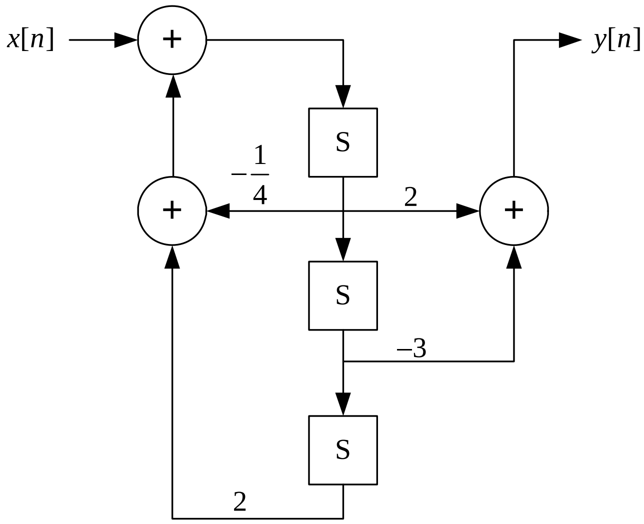
10.

(1) (5%)

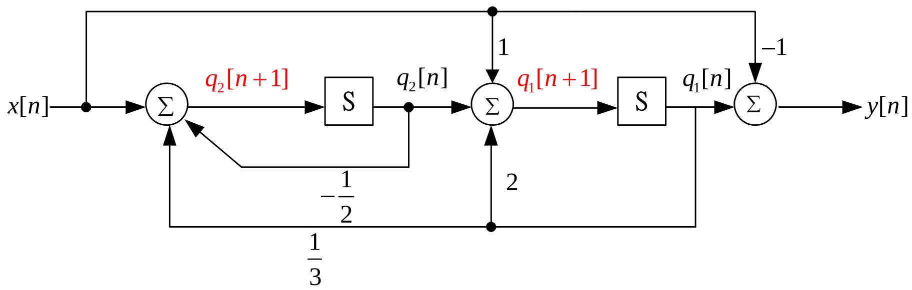
Direct form I



Direct form II




(2) (7%)




$$q_1[n+1] = 2q_1[n] + q_2[n] + x[n]$$

$$q_2[n+1] = \frac{1}{3}q_1[n] - \frac{1}{2}q_2[n] + x[n]$$

$$y[n] = q_1[n] - x[n]$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D = -1 .$$