Midterm Exam I Reference Solutions

April 7, 2009

Instructor: *Chin-Liang Wang*

1.

(1) False.

$x[n] = \begin{cases} 2 & n = 1 \\ 1 & n \neq 1 \end{cases}$ is not periodic, but $y[n] = 1$ for all n, which is periodic.

(2) False.

$y[n] = x[n] + n.$

or

An incrementally linear system can't tell us whether it is time invariant.

(3) False.

H: input $=$ output is causal and memoryless.

(4) False.

$\left. \begin{aligned} y[n] &= (x[n] + x[n-1])^2 \\ y[n] &= \max(x[n], x[n-1]) \end{aligned} \right\}$ with the same impulse response.

or

There are many nonlinear systems with $h[n]$ as its impulse response.

(5) False.

$tu(t)$ is neither energy signal nor power signal.

2.

(1) **Memory**: $y(t)$ depends on $x(t+1)$.

Stable: Bounded $x(t)$ will result in bounded $y(t)$.

Non-causal: $y(t)$ depends on $x(t+1)$.

Linear:
$$
\begin{aligned}
y_1(t) &= x_1(t+1)\sin(\omega t + 1) \\
y_2(t) &= x_2(t+1)\sin(\omega t + 1) \\
x_3(t) &= a x_1(t) + b x_2(t) \\
y_3(t) &= x_3(t+1)\sin(\omega t + 1) \\
&= a x_1(t+1)\sin(\omega t + 1) + b x_2(t+1)\sin(\omega t + 1) \\
&= a y_1(t) + b y_2(t)
\end{aligned}
$$

Not T.I.: The output has time varying gain.

(2) **Memoryless**: $y[n]$ depends only on the current value of $x[n]$.

Unstable: $|y[n]| \to \infty$ when $n \to -\infty$.

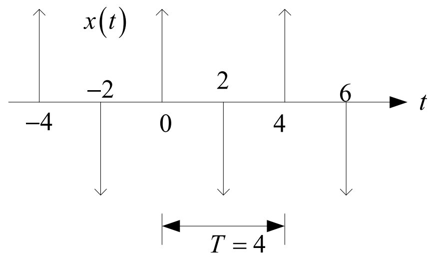
Causal: $y[n]$ depends only on the currently value of $x[n]$.

Non-linear: If $x[n]=0$, then $y[n] \neq 0$.

Not T.I.: The output has time varying gain.

3.

(1)



Periodic,

(2)

We have to find the smallest integer N ($N \neq 0$) such that

$$\cos\left[4(n+N)+\frac{\pi}{4}\right] = \cos\left[4n+\frac{\pi}{4}\right]$$

For the above to be hold, the following has to be true for some integer(s) k.

$$4(n+N)+\frac{\pi}{4} = 4n+\frac{\pi}{4}+2\pi k \quad ==> N = \frac{\pi}{2}k$$

However, since π isn't a rational number, we can't find an integer N that satisfied this. Thus, the function is not periodic.

4.

(1) $y[n] = 2x[n] + 3x[n-1] - 4x^2[n-2]$

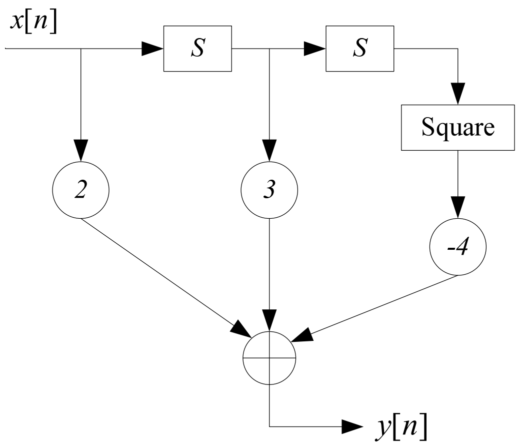
$= 2x[n] + 3S\{x[n]\} - 4S^2\{x^2[n]\}$

$= (2x[n] + 3S)\{x[n]\} - 4S^2\{x^2[n]\}$

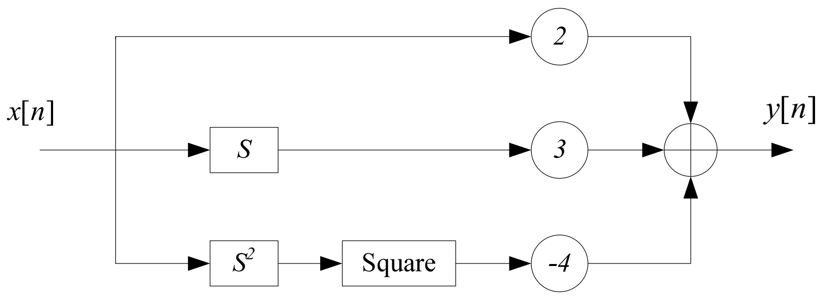
$H: y[n] = (2x[n] + 3S)\{x[n]\} - 4S^2\{x^2[n]\}$

(2)

(a) Cascade implementation of operator H:



(b) Parallel implementation of operator H:



5.

Let $x(t)=\left[u(t+1)-u(t-1)\right]$ and $h(t)=\cos(\pi t)\left[u(t+1)-u(t-1)\right]$. Then

$h(-\tau)=\left[u(-\tau+1)-u(-\tau-1)\right]=\left[u(\tau+1)-u(\tau-1)\right]$ ($\because$ symmetric property)

$h(t-\tau)=\left[u(\tau-t+1)-u(\tau-t-1)\right]$

$w_t(\tau)=x(\tau)h(t-\tau)$

For $t+1<-1$, $t<-2$, $w_t(\tau)=0$, $y(t)=0$

For $t+1<1$, $-2\le t<0$, $-1<\tau<t+1$, $w_t(\tau)=\cos(\pi\tau)$,

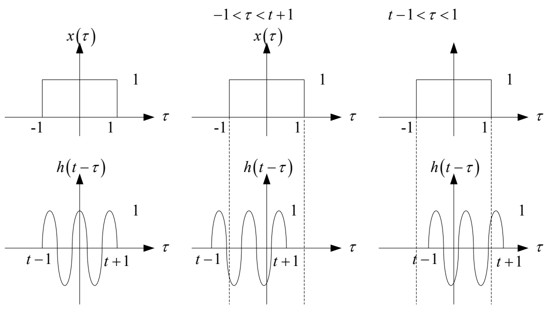
$$y(t)=\int_{-1}^{t+1}\cos(\pi\tau)d\tau=\frac{1}{\pi}\sin\left(\pi(t+1)\right)$$

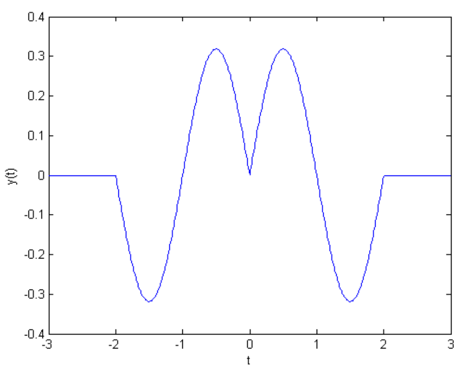
For $t-1<1$, $0\le t<2$, $t-1<\tau<1$, $w_t(\tau)=\cos(\pi\tau)$

$$y(t)=\int_{t-1}^{1}\cos(\pi\tau)d\tau=-\frac{1}{\pi}\sin\left(\pi(t-1)\right)$$

For $1<t-1$, $2\le t$, $w_t(\tau)=0$, $y(t)=0$

$$y(t)=\begin{cases}0 & , t<-2\\ \frac{1}{\pi}\sin\left(\pi(t+1)\right) & , -2\le t<0\\ -\frac{1}{\pi}\sin\left(\pi(t-1)\right) & , 0\le t<2\\ 0 & , t\ge 2\end{cases}$$





6.

(1) According to the properties of an LTI system,

$x_1[n]=x[n-1]+2x[n-3]+x[n-5]$,

$y_1[n]=y[n-1]+2y[n-3]+y[n-5]$,

$= -\delta[n-1]-2\delta[n-2]+(a-3)\,\delta[n-3]+(2a-4)\delta[n-4]+(3a-3)\,\delta[n-5]$
$+(4a-2)\,\delta[n-6]+(3a-1)\,\delta[n-7]+2a\,\delta[n-8]+a\,\delta[n-9]$.

(2) $\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k-\infty}^{\infty}\{\delta[k]+\delta[k-1]+\delta[k-2])\}h[n-k]$

$h[n]+2h[n-1]+h[n-2] = -\delta[n]-2\delta[n-1]+(a-1)\delta[n-2]+2a\delta[n-3]+a\delta[n-4]$

for $n=0$, $h[0]=-\,\delta[0]=-1$;

For $n>0$, $h[n]=-\,\delta[n]-2\delta[n-1]+(a-1)$

$\delta[n-2]+2a\delta[n-3]+a\delta[n-4]-h[n-2]-2h[n-1]$;

$h[1]=0-2+0+0+0-0+2=0$

$h[2]=0-0+(a-1)+0+0+1-0=a$

$h[3]=0+0+0+2a+0-0-2a=0$

$h[n]=0$, for $n>3$.

$h[n]=-\,\delta[n]+a\delta[n-2]$.

(3) $h[n]*h^{\text{inv}}[n]=\delta[n]$,

$\sum_{k=-\infty}^{\infty} h[k]h^{inv}[n-k] = \sum_{-\infty}^{\infty}\{-\delta[k]+a\delta[k-2])\}h[n-k]$

$-h^{inv}[n]+ah^{inv}[n-2]=\delta[n]$

For $n<0$, $h^{\text{inv}}[n]=0$, ∵Causal.

For $n=0$, $-\,h^{\text{inv}}[0]+a\,h^{\text{inv}}[-2]=1$, $h^{\text{inv}}[0]=-1$;

For $n>0$, $h^{\text{inv}}[n]=ah^{\text{inv}}[n-2]$;

$h^{\text{inv}}[1]=0$,

$h^{\text{inv}}[2]=-a$.

$h^{\text{inv}}[3]=0$,

$h^{\text{inv}}[4]=-a^2$,…….

$|a|<1$, ∵stable.

7.

(1)

$$q[n]=\sum_{k=-\infty}^{\infty} v[k]w[n-k]$$
$$=\sum_{k-\infty}^{\infty}\left\{\begin{array}{l}(\delta[k]+\delta[k-1]+\delta[k-2]+\delta[k-3]+\delta[k-4]+\delta[k-5])\\ \cdot(\delta[n-k]-2\delta[n-k-2]+3\delta[n-k-4])\end{array}\right\}$$

$q[0]=1$; $q[1]=1$; $q[2]=-1$; $q[3]=-1$; $q[4]=2$; $q[5]=2$; $q[6]=1$; $q[7]=1$;
$q[8]=3$; $q[9]=3$.

(2)

$$y[n]=\sum_{k=-\infty}^{n-1}s[k]$$
$$=\begin{cases}\delta[n-1]+2\delta[n-2]+5\delta[n-3]+8\delta[n-4]+12\delta[n-5]+16\delta[n-6]\\ 19\delta[n-7]+22\delta[n-8]+23\delta[n-9]+24\delta[n-10].....\end{cases}$$

$$y[n]=r[n]*v[n]=\sum_{k=-\infty}^{\infty}v[k]r[n-k]$$
$$=r[n]+r[n-1]+r[n-2]+r[n-3]+r[n-4]+r[n-5]=y[n]$$

For $n<0$, $r[n]=0$,
For $n\geqq 0$, $r[0]=0$,
$y[1]=1=r[1]+r[0]$, $r[1]=1$
$y[2]=2=r[2]+r[1]+r[0]$, $r[2]=1$
$y[3]=5=r[3]+r[2]+r[1]+r[0]$, $r[3]=3$
$y[4]=8=r[4]+r[3]+r[2]+r[1]+r[0]$, $r[4]=3$
$y[5]=12=r[5]+r[4]+r[3]+r[2]+r[1]+r[0]$, $r[5]=4$
$y[6]=16=r[6]+r[5]+r[4]+r[3]+r[2]+r[1]$, $r[6]=4$
$y[7]=19=r[7]+r[6]+r[5]+r[4]+r[3]+r[2]$, $r[7]=4$
$y[8]=22=r[8]+r[7]+r[6]+r[5]+r[4]+r[3]$, $r[8]=4$
$y[9]=23=r[9]+r[8]+r[7]+r[6]+r[5]+r[4]$, $r[9]=4$
$y[10]=24=r[10]+r[9]+r[8]+r[7]+r[6]+r[5]$, $r[10]=4$
for $n>10$, r[n]=4.

8.

(1)

find $y^{(h)}$

$$r^2-2r=1=0, r=1,1 \quad \Rightarrow y^{(h)}(t)=\left(c_1+c_2t\right)e^{-t}$$

find $y^{(p)}$

$$y^{(p)}(t)=t^2(At^2+Bt+c)e^t$$

$$y^{(p)'}(t)=(2At+(A+3B)t^2+(B+4C)t^3+Ct^4)e^t$$

$$y^{(p)''}(t)=(2A+(4A+6B)t \;\; +(A+6B+12C)t^2+(B+8C)t^3+Ct^4)e^t$$

Use $\dfrac{d^2y^{(p)}}{dt^2}-2\dfrac{dy^{(p)}}{dt}+y^{(p)}=t^2e^t \Rightarrow A=\dfrac{1}{12}, B=0, C=0$

$$\therefore y^p(t)=\frac{1}{12}t^4e^t \Rightarrow y(t)=(c_1+c_2t)e^t+\frac{1}{12}t^4e^t$$

find y(t)

Since $y(0)=1, \frac{dy}{dt}|_{t=0}=2$. We can get $c_1=1, c_2=1 \Rightarrow y(t)=(1+t)e^t+\frac{1}{12}t^4e^t$

(2)

$$y^{(h)}(t)=e^{-t}(A\cos t+B\sin t) \Rightarrow y^{(p)}(t)=te^{-t}(C\cos t+D\sin t)+(E\cos 2t+F\sin 2t)$$

(3)

$$y^{(h)}(t)=(A\cos 2t+B\sin 2t) \Rightarrow y^{(p)}(t)=t(ct^2+dt+e)(\cos 2t+\sin 2t)$$

9.

(1)

find $y^{(h)}[n]$

$$6r^2-r-1=0 \Rightarrow r=\frac{1}{2},-\frac{1}{3}$$

$$y^{(h)}[n]=c_1\left(\frac{1}{2}\right)^n+c_2\left(-\frac{1}{3}\right)^n$$

(2)

find $y^{(p)}[n]$

$$\because x[n]=u[n] \Rightarrow \therefore y^{(p)}[n]=Au[n]$$

Use $6y[n]-y[n-1]-y[n-2]=2x[n]-2x[n-1]+4x[n-2]$

$$6A-A-A=2-2+4 \Rightarrow A=1$$

$$\therefore y^{(p)}[n]=u[n]$$

(3)

find $y^{(n)}[n]$

Use $y^{(n)}[n]=c_1\left(\frac{1}{2}\right)^n+c_2\left(-\frac{1}{3}\right)^n, y[-1]=y[-2]=0$

$$c_1=0, c_2=0 \Rightarrow y^{(n)}[n]=0$$

find $y^{(f)}[n]$

$$y[-1]=y[-2]=0$$

$$6y[0]-y[-1]-y[-2]=2\Rightarrow y[0]=\frac{1}{3}$$

$$6y[1]-y[0]-y[-1]=0\Rightarrow y[1]=\frac{1}{18}$$

$$y^{(f)}[n]=c_3\left(\frac{1}{2}\right)^n+c_4\left(-\frac{1}{3}\right)^n+1, n\ge 0, y[0]=\frac{1}{3}, y[1]=\frac{1}{18}$$

$$\begin{cases} c_3+c_4=-\frac{2}{3} \\ \frac{1}{2}c_3-\frac{1}{3}c_4=-\frac{17}{18} \end{cases}\Rightarrow c_3=-\frac{7}{5}, c_4=\frac{11}{15}$$

$$\Rightarrow\therefore y^{(f)}[n]=\left(-\frac{7}{5}\left(\frac{1}{2}\right)^n+\frac{11}{5}\left(-\frac{1}{3}\right)^n+1\right)u[n]$$

10.

(1)

find $y^{(h)}[n]$

$$r^2-r+0.25=0\Rightarrow r=\frac{1}{2},\frac{1}{2}\Rightarrow y^{(h)}[n]=(c_1+c_2n)\left(\frac{1}{2}\right)^n$$

find $y^{(p)}[n]$

$$\because x[n]=n\left(\frac{1}{4}\right)^n\Rightarrow\therefore y^{(p)}[n]=(a+bn)\left(\frac{1}{4}\right)^n$$

Use $y[n]-y[n-1]+0.25y[n-2]=x[n]$

We get $a=28, b=7$

$$\therefore y^{(p)}[n]=(28+7n)\left(\frac{1}{4}\right)^n$$

$$\therefore y[n]=(c_1+c_2n)\left(\frac{1}{2}\right)^n+(28+7n)\left(\frac{1}{4}\right)^n$$

Use $\therefore y[n]=(c_1+c_2n)\left(\frac{1}{2}\right)^n+(28+7n)\left(\frac{1}{4}\right)^n, y[-1]=1, y[-2]=2$

We get $\begin{cases} 2c_1 - 4c_2 = -111 \\ 2c_1 - 2c_2 = -83 \end{cases} \Rightarrow c_1 = -\frac{55}{2}, c_2 = 14$

$$\therefore y[n] = (-\frac{55}{2} + 14n)\left(\frac{1}{2}\right)^n + (28 + 7n)\left(\frac{1}{4}\right)^n$$

(2)

$$\because x[n] = n^2\left(\frac{1}{2}\right)^n \Rightarrow \therefore y^{(p)}[n] = n^2(an^2 + bn + c)\left(\frac{1}{2}\right)^n$$