Midterm Exam II Reference Solutions

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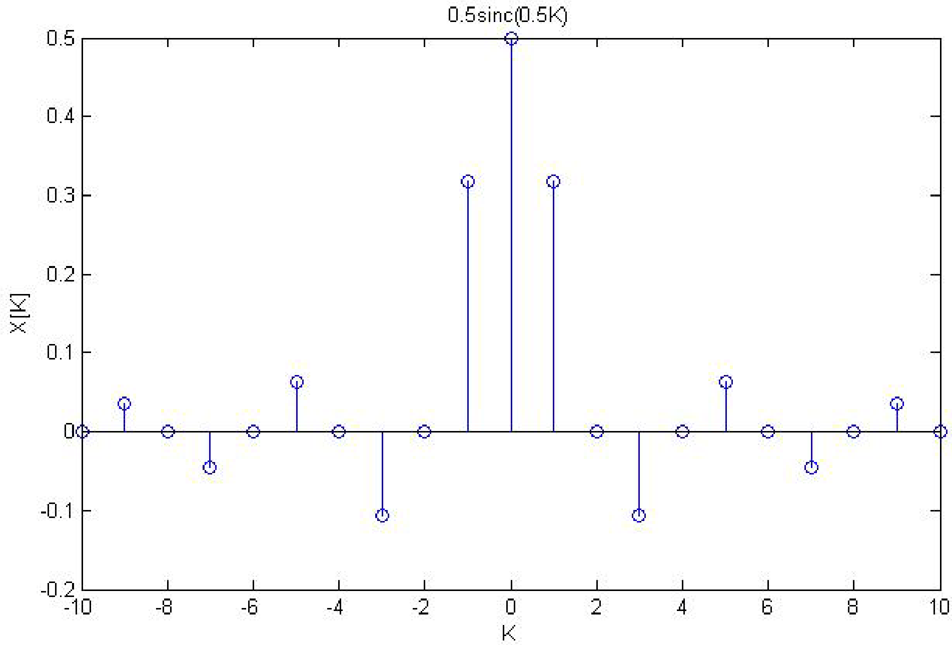
1. Fundamental period of $x(t) = T = 4 \Rightarrow \omega_0 = 2\pi/4 = \pi/2$

$$X[0] = \frac{1}{T}\int_T x(t)dt = \frac{1}{4}\int_{-1}^{1} x(t)dt = 0.5$$

$$X[K] = \frac{1}{4}\int_{-1}^{1} x(t)e^{-jK\omega_0 t}dt = \frac{1}{-j4K\omega_0}e^{-jK\omega_0 t}\Big|_{-1}^{1} = \frac{1}{jK2\pi}\left(e^{jK\omega_0 t} - e^{-jK\omega_0 t}\right) = \frac{\sin\left(\frac{K\pi}{2}\right)}{K\pi} = \frac{1}{2}\mathrm{sinc}\left(\frac{K}{2}\right)$$



2.

(1) For $a > 0,\ b > 0,$ and $a \neq b$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(a-j\omega)(b-j\omega)} = \frac{1}{b-a}\left(\frac{1}{(a-j\omega)} - \frac{1}{(b-j\omega)}\right)$$

$$\therefore y(t) = \frac{1}{b-a}\left(e^{at} - e^{bt}\right)u(-t)$$

(2) For $a > 0,\ b > 0,$ and $a = b$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(a-j\omega)^2} = -j\frac{d}{d\omega}\left(\frac{1}{(a-j\omega)}\right)$$

$$\therefore y(t) = -te^{at}u(-t)$$

3. By utilizing the concept of eigenfunction:

$$y(t)=\sum_{k=0}^{2}(0.5)^k\frac{e^{j2kt}H(2k)-e^{-j2kt}H(-2k)}{2j}=\frac{1}{2}\sin(2t-2)+\frac{1}{4}\sin(4t-4).$$

4.

(1) $\int_{-\infty}^{\infty}x(t)dt=X(0)=1.$

(2) $\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2d\omega$

$$=\frac{2}{2\pi}\left[\int_0^1(1+\omega)^2d\omega+\int_1^2 2^2d\omega+\int_2^4(4-\omega)^2d\omega\right]=\frac{9}{\pi}.$$

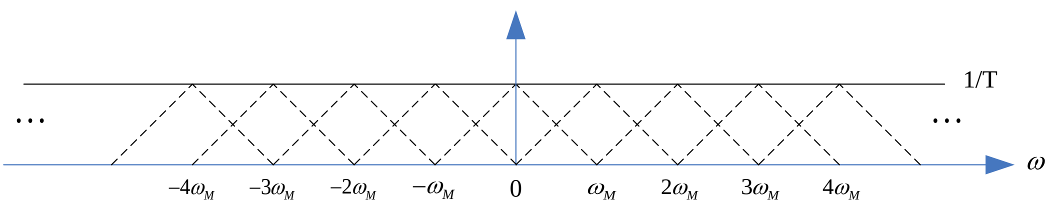
(3) $\int_{-\infty}^{\infty}x(t)e^{j2t}dt=X(-2)=2.$

(4) $x(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{-j\omega\cdot 0}d\omega=\frac{11}{2\pi}.$

(5) $X(\omega)$ is real and even => $\tan^{-1}\left\{\frac{\mathrm{Im}(x(t))}{\mathrm{Re}(x(t))}\right\}=\tan^{-1}\left\{\frac{0}{\mathrm{Re}(x(t))}\right\}=0.$

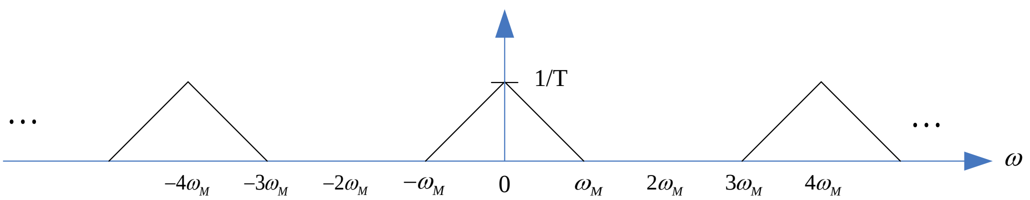
5.

(1) $\omega_s=\frac{2\pi}{T}=\omega_M$



$x(t)$ can't be reconstructed from $x_p(t)$ due to the aliasing in $X_p(\omega)$.

(2) $\omega_s=\frac{2\pi}{T}=4\omega_M$



$x(t)$ can be reconstructed from $x_p(t)$ since there is no aliasing in $X_p(\omega)$.

6.

(1)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\Omega n},$$

Therefore

$$X(e^{\frac{2\pi k}{N}}) = \sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\frac{2\pi k}{N}n},$$

Now, we may write the expression for the Fourier coefficients of $\tilde{x}[n]$ as

$$a_k = \frac{1}{N}\sum_{<N>} \tilde{x}[n]e^{-j\frac{2\pi k}{N}n} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\frac{2\pi k}{N}n},$$

$\because x[n] = \tilde{x}[n]$ in a range $n_0 \le n \le n_0 + N - 1$,

$$a_k = \frac{1}{N} X(e^{\frac{2\pi k}{N}}).$$

(2)

From the given information,

$$X(\Omega) = 1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega}$$

$$= e^{-j(\frac{3}{2})\Omega}\left\{e^{j(\frac{3}{2})\Omega} + e^{-j(\frac{3}{2})\Omega}\right\} + e^{-j(\frac{3}{2})\Omega}\left\{e^{j(\frac{1}{2})\Omega} + e^{-j(\frac{1}{2})\Omega}\right\},$$

$$= 2e^{-j(\frac{3}{2})\Omega}\left\{\cos(3\tfrac{\Omega}{2}) + \cos(\tfrac{\Omega}{2})\right\},$$

$$a_k = \frac{1}{N} X(e^{j\frac{2\pi k}{N}}) = \frac{1}{N} 2e^{-j\left(\frac{3}{2}\right)\frac{2\pi k}{N}}\left\{\cos(\tfrac{6\pi k}{2N}) + \cos(\tfrac{\pi k}{N})\right\}.$$

7.

(1)

$$X[k] = 2 + e^{-j\frac{2\pi}{5}k} + e^{-j3\frac{2\pi}{5}k}.$$

(2)

$$Y[k] = X^2[k] = 4 + 4e^{-j\frac{2\pi}{5}k} + e^{-j2\frac{2\pi}{5}k} + 4e^{-j3\frac{2\pi}{5}k} + 2e^{-j4\frac{2\pi}{5}k} + e^{-j6\frac{2\pi}{5}k}$$

$$= 4 + 5e^{-j\frac{2\pi}{5}k} + e^{-j2\frac{2\pi}{5}k} + 4e^{-j3\frac{2\pi}{5}k} + 2e^{-j4\frac{2\pi}{5}k}.$$

(3) N ≧ 4+4-1=7.

8.

$$x[n] = |n|\left(\tfrac{1}{3}\right)^{|n|} \xleftrightarrow{DTFT} X(\Omega)$$

(1) $Y(\Omega) = \mathrm{Im}\{X(\Omega)\}$,

Since $x[n]$ is real and even, $X(\Omega)$ is also real and even, $y[n] = 0$.

(2) $Y(\Omega) = X(\Omega) \circledast X(\Omega + \tfrac{\pi}{2})$,

$$y[n] = 2\pi x[n]\left(e^{j\frac{\pi}{2}n} x[n]\right) = 2\pi n^2 \left(1/3\right)^{2|n|} e^{j\frac{\pi}{2}n}.$$

(3) $Y(\Omega) = e^{-j4\Omega}\left[X(\Omega + \tfrac{\pi}{4}) + X(\Omega - \tfrac{\pi}{4})\right]$,

$$y[n] = e^{-j\frac{\pi}{4}(n-4)} x[n-4] + e^{j\frac{\pi}{4}(n-4)} x[n-4]$$

$$= 2\cos(\tfrac{\pi}{4}(n-4))x[n-4] = 2\cos(\tfrac{\pi}{4}(n-4))\,|n-4|\left(\tfrac{1}{3}\right)^{|n-4|}$$

$$= -2\cos(\tfrac{\pi}{4}n)\,|n-4|\left(\tfrac{1}{3}\right)^{|n-4|}.$$

$(e^{-j4\Omega} \Rightarrow n-4;\ \Omega + \tfrac{\pi}{4} \Rightarrow e^{-j\frac{\pi}{4}n};\ \Omega - \tfrac{\pi}{4} \Rightarrow e^{j\frac{\pi}{4}n}\ )$

9.

$$x[n] = \left(\tfrac{1}{2}\right)^2 u[n] + \left(\tfrac{1}{4}\right)^2 u[n],\ \ y[n] = 6\left(\tfrac{1}{2}\right)^2 u[n] + 6\left(\tfrac{3}{4}\right)^2 u[n],$$

(1)

$$X(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{1-\frac{1}{4}e^{-j\Omega}},$$

$$Y(\Omega) = \frac{6}{1-\frac{1}{2}e^{-j\Omega}} + \frac{6}{1-\frac{3}{4}e^{-j\Omega}},$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\frac{6}{1-\frac{1}{2}e^{-j\Omega}} + \frac{6}{1-\frac{3}{4}e^{-j\Omega}}}{\frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{1-\frac{1}{4}e^{-j\Omega}}} = 6\cdot\frac{\frac{1-\frac{5}{8}e^{-j\Omega}}{1-\frac{3}{4}e^{-j\Omega}}}{\frac{1-\frac{3}{8}e^{-j\Omega}}{1-\frac{1}{4}e^{-j\Omega}}} = 6\cdot\frac{(1-\frac{5}{8}e^{-j\Omega})(1-\frac{1}{4}e^{-j\Omega})}{(1-\frac{3}{4}e^{-j\Omega})(1-\frac{3}{8}e^{-j\Omega})}$$

$$= 6\cdot\frac{1-\frac{7}{8}e^{-j\Omega}+\frac{5}{32}e^{-j2\Omega}}{(1-\frac{3}{4}e^{-j\Omega})(1-\frac{3}{8}e^{-j\Omega})} = 6\cdot\frac{1-\frac{7}{8}e^{-j\Omega}+\frac{5}{32}e^{-j2\Omega}}{1-\frac{9}{8}e^{-j\Omega}+\frac{9}{32}e^{-j2\Omega}}.\ \ (1)_{\#}$$

(2)

$$H(\Omega) = 6\cdot\frac{1-\frac{7}{8}e^{-j\Omega}+\frac{5}{32}e^{-j2\Omega}}{(1-\frac{3}{4}e^{-j\Omega})(1-\frac{3}{8}e^{-j\Omega})} = \frac{6}{(1-\frac{3}{4}e^{-j\Omega})(1-\frac{3}{8}e^{-j\Omega})} + \frac{-\frac{21}{4}e^{-j\Omega}}{(1-\frac{3}{4}e^{-j\Omega})(1-\frac{3}{8}e^{-j\Omega})} + \frac{\frac{15}{16}e^{-j2\Omega}}{(1-\frac{3}{4}e^{-j\Omega})(1-\frac{3}{8}e^{-j\Omega})}$$

$$= 6\left[\frac{2}{(1-\frac{3}{4}e^{-j\Omega})} - \frac{1}{(1-\frac{3}{8}e^{-j\Omega})}\right] - \tfrac{21}{4}e^{-j\Omega}\left[\frac{2}{(1-\frac{3}{4}e^{-j\Omega})} - \frac{1}{(1-\frac{3}{8}e^{-j\Omega})}\right] + \tfrac{15}{16}e^{-j2\Omega}\left[\frac{2}{(1-\frac{3}{4}e^{-j\Omega})} - \frac{1}{(1-\frac{3}{8}e^{-j\Omega})}\right]$$

$$h[n] = 12\left(\tfrac{3}{4}\right)^n u[n] - 6\left(\tfrac{3}{8}\right)^n u[n] - \tfrac{21}{2}\left(\tfrac{3}{4}\right)^{n-1} u[n-1] + \tfrac{21}{4}\left(\tfrac{3}{8}\right)^{n-1} u[n-1] + \tfrac{15}{8}\left(\tfrac{3}{4}\right)^{n-2} u[n-2] - \tfrac{15}{16}\left(\tfrac{3}{8}\right)^{n-2} u[n-2].\ \ (2)_{\#}$$

or

$$H(\Omega) = 6\cdot\frac{1-\frac{7}{8}e^{-j\Omega}+\frac{5}{32}e^{-j2\Omega}}{1-\frac{9}{8}e^{-j\Omega}+\frac{9}{32}e^{-j2\Omega}} = 6\left(\tfrac{5}{9} + \frac{\frac{4}{9}}{(1-\frac{3}{4}e^{-j\Omega})(1-\frac{3}{8}e^{-j\Omega})} - \frac{\frac{1}{4}e^{-j\Omega}}{(1-\frac{3}{4}e^{-j\Omega})(1-\frac{3}{8}e^{-j\Omega})}\right)$$

$$= \frac{1}{3} + \frac{8}{3}\left[\frac{2}{(1-\frac{3}{4}e^{-j\Omega})} - \frac{1}{(1-\frac{3}{8}e^{-j\Omega})}\right] - \frac{3}{2}e^{-j\Omega}\left[\frac{2}{(1-\frac{3}{4}e^{-j\Omega})} - \frac{1}{(1-\frac{3}{8}e^{-j\Omega})}\right]$$

$$h[n] = \frac{10}{3}\delta[n] + \frac{16}{3}\left(\tfrac{3}{4}\right)^n u[n] - \frac{8}{3}\left(\tfrac{3}{8}\right)^n u[n] - 3\left(\tfrac{3}{4}\right)^{n-1} u[n-1] + \frac{3}{2}\left(\tfrac{3}{8}\right)^{n-1} u[n-1].\ \ (2_{\#})$$

(3)

$$6\cdot\frac{1-\frac{7}{8}e^{-j\Omega}+\frac{5}{32}e^{-j2\Omega}}{1-\frac{9}{8}e^{-j\Omega}+\frac{9}{32}e^{-j2\Omega}} = \frac{Y(\Omega)}{X(\Omega)},\ \ \Rightarrow 6(1-\tfrac{7}{8}e^{-j\Omega}+\tfrac{5}{32}e^{-j2\Omega})X(\Omega) = (1-\tfrac{9}{8}e^{-j\Omega}+\tfrac{9}{32}e^{-j2\Omega})Y(\Omega),$$

$$y[n] - \tfrac{9}{8}y[n-1] + \tfrac{5}{32}y[n-2] = 6x[n] - \tfrac{21}{4}x[n-1] + \tfrac{27}{16}x[n-2].\ \ (3)_{\#}$$

10.

$$x[n]=\sin(\frac{\pi n}{8})-2\cos(\frac{\pi n}{4}),$$

$$X(\Omega)=\frac{\pi}{j}\delta(\Omega-\frac{\pi}{8})-\frac{\pi}{j}\delta(\Omega+\frac{\pi}{8})-2\pi\delta(\Omega-\frac{\pi}{4})-2\pi\delta(\Omega-\frac{\pi}{4}),\ \text{as } -\pi\le\Omega\le\pi$$

($X(\Omega)$ is periodec, we only show one here.)

$$h[n]=\frac{\sin(\pi n/6)\sin(\pi n/2)}{\pi^2 n^2},$$

$$H(\Omega)=\frac{1}{2\pi}\left(H_1(\Omega)\circledast H_2(\Omega)\right),\ \text{wh re } H_1(\Omega)=\begin{cases}1, -\frac{\pi}{6}\le\Omega\le\frac{\pi}{6}\\ 0, \quad otherwise\end{cases},\ \text{as } -\pi\le\Omega\le\pi,$$

$$H_2(\Omega)=\begin{cases}1, -\frac{\pi}{2}\le\Omega\le\frac{\pi}{2}\\ 0, \quad otherwise\end{cases},\ \text{as } -\pi\le\Omega\le\pi,$$

Thus,