EE3610 2009 Fall-Midterm Exam II Reference Solutions

1.

(1)

$$X(j\omega) = \frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{(a+j\omega)}\right]$$

$$\because e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{(a+j\omega)} \qquad \therefore te^{-at}u(t) \xleftrightarrow{\mathcal{F}} X(j\omega).$$

(2)

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt$$

$$= \left[\frac{e^{t(1-j\omega)}}{1-j\omega}\right]\Bigg|_{-\infty}^{0} - \left[\frac{e^{-t(1+j\omega)}}{1+j\omega}\right]\Bigg|_{0}^{\infty} = \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2}$$

2.

Since $X(j\omega) = 4\pi\delta(\omega) + 4\pi\left[\delta(\omega-3\pi)+\delta(\omega+3\pi)\right] - j\pi\left[\delta(\omega-6\pi)+\delta(\omega+6\pi)\right]$ and $Y(j\omega) = 2\pi\delta(\omega) + 3\pi\left[\delta(\omega-6\pi)+\delta(\omega+6\pi)\right]$, we can determine $H(j\omega)$ when $\omega = 0,\ \pm 3\pi,\ \pm 6\pi.$ Furthermore, $H(0) = 0.5,$ $H(j3\pi) = H(-j3\pi) = 0,$ $H(j6\pi) = 3j,$ and $H(-j6\pi) = -3j.$

3.

$$Y(\omega)\left[(j\omega)^2 + 7(j\omega) + 10\right] = X(\omega)(-j\omega+1)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega+1}{(j\omega)^2 + 7(j\omega)+10} = \frac{-2}{j\omega+5} + \frac{1}{j\omega+2}$$

$$h(t) = \left(-2e^{-5t} + e^{-2t}\right)u(t)$$

4.

(1) $\int_{-\infty}^{\infty} x(t)dt = X(0) = 1.$

(2) $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 9/\pi.$

(3) $\int_{-\infty}^{\infty} x(t)e^{j2t}dt = X(-2) = 2.$

(4) $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{-j\omega\cdot 0}d\omega = \frac{5}{\pi}.$

(5) $X(j\omega)$ is real and even => $\tan^{-1}\left\{\frac{\mathrm{Im}(x(t))}{\mathrm{Re}(x(t))}\right\} = \tan^{-1}\left\{\frac{0}{\mathrm{Re}(x(t))}\right\} = 0.$

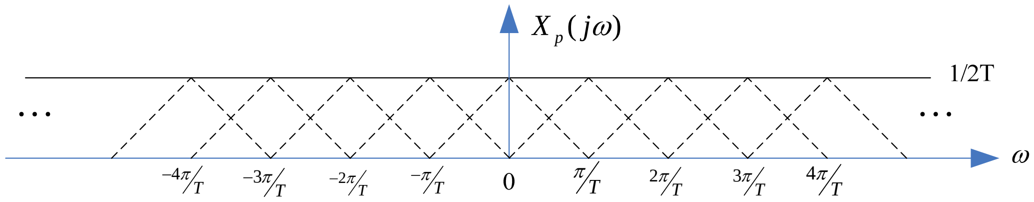
5.

$$p(t)=\sum_{n=-\infty}^{\infty}\delta\left(t-2nT\right)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}\ \left(Fourier\ Series\right)$$

where $\omega_0=\frac{2\pi}{T_0}=\frac{\pi}{T}$ and $a_k=\frac{1}{2T}\int_{-T}^{T}\delta(t)e^{-jk\pi t/T}dt=\frac{1}{2T}$

$$p(t)=\sum_{k=-\infty}^{\infty}\frac{1}{2T}e^{jk\pi t/T}\stackrel{\mathcal{F}}{\longleftrightarrow}P\left(j\omega\right)=\frac{1}{2T}\sum_{k=-\infty}^{\infty}2\pi\delta(\omega-k\pi/T)$$

$$\Rightarrow X_p\left(j\omega\right)=\frac{1}{2\pi}X\left(j\omega\right)*P\left(j\omega\right)=\frac{1}{2T}\sum_{k=-\infty}^{\infty}X\left(j(\omega-k\pi/T)\right)$$



6.

(1) The F.S. coefficients of $x[n]$ are $a_k=\frac{1}{4}\sum_{n=0}^{3}x[n]e^{-j2\pi kn/4}=\frac{1}{4}$ for all k. Thus the DTFS representation of $x[n]$ is $x\left[n\right]=\sum_{k=<4>}\frac{1}{4}e^{j2\pi kn/4}$.

(2) The output signal $y[n]$ can be expressed as:

$$y[n]=\sum_{k=0}^{3}a_k H\left(e^{j2\pi k/4}\right)e^{j2\pi kn/4}$$

$$=\frac{1}{4}\left(H\left(e^{j0}\right)e^{j0}+H\left(e^{j\pi/2}\right)e^{jn\pi/2}+H\left(e^{j\pi}\right)e^{jn\pi}+H\left(e^{j3\pi/2}\right)e^{j3n\pi/2}\right)$$

$$=\cos\left(\frac{\pi}{2}n\right)=\frac{e^{j\left(\frac{\pi}{2}n\right)}+e^{-j\left(\frac{\pi}{2}n\right)}}{2}$$

$$=\frac{e^{j\left(\frac{\pi}{2}n\right)}+e^{j\left(\frac{3\pi}{2}n\right)}}{2}\quad\left(\because e^{-j\left(\frac{\pi}{2}n\right)}=e^{j\left(\left(2\pi-\frac{\pi}{2}\right)n\right)}\right)$$

$$\Rightarrow H\left(e^{j0}\right)=H\left(e^{j\pi}\right)=0,\ H\left(e^{j\pi/2}\right)=H\left(e^{j3\pi/2}\right)=2.$$

(3)

$$H_2\left(e^{j\Omega}\right)=1+e^{-j\Omega}+e^{-2j\Omega}-e^{j\Omega}-e^{2j\Omega}$$

$$Y_2(e^{j\Omega})=\frac{1}{4}(1+e^{-j\Omega}+e^{-2j\Omega}-e^{j\Omega}-e^{2j\Omega})$$

$$\to b_k=\frac{1}{4}(1+e^{jk\pi/2}+e^{-jk\pi/2})$$

7.

(1) $y[n] = x_1[n] \circledast x_2[n] = \begin{cases} 2 & 0 \le n \le 3 \\ 0 & o.w \end{cases}$

(2) $1 \le n \le 3$

(3) Taking the N points periodic convolution of $\hat{x}_1[n]$ and $\hat{x}_2[n]$ where N is larger than (4+2-2), then the result will be the same as the linear convolution of $x_1[n]$ and $x_2[n]$.

8.

(1)

$$Y(e^{j\Omega}) - \frac{3}{4}e^{-j\Omega}Y(e^{j\Omega}) + \frac{1}{8}e^{-j2\Omega}Y(e^{j\Omega}) = 2X(e^{j\Omega})$$

$$Y(e^{j\Omega})(1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}) = 2X(e^{j\Omega})$$

$$H(e^{j\Omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}}$$

(2)

$$H(e^{j\Omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}} = \frac{2}{(1 - \frac{1}{2}e^{-j\Omega})(1 - \frac{1}{4}e^{-j\Omega})}$$

$$= \frac{4}{(1 - \frac{1}{2}e^{-j\Omega})} - \frac{2}{(1 - \frac{1}{4}e^{-j\Omega})}$$

$$h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$$

(3)

$$x[n] = (\frac{1}{4})^n u[n]$$

$$X(e^{j\Omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\Omega})(1 - \frac{1}{4}e^{-j\Omega})^2}$$

$$= \frac{-4}{(1 - \frac{1}{4}e^{-j\Omega})} + \frac{-2}{(1 - \frac{1}{4}e^{-j\Omega})^2} + \frac{8}{(1 - \frac{1}{2}e^{-j\Omega})}$$

$$y[n] = -4(\frac{1}{4})^n u[n] - 2(n+1)(\frac{1}{4})^n u[n] + 8(\frac{1}{2})^n u[n]$$

9.

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

(1) $$H(e^{-j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{j\Omega n} = \left(\sum_{n=-\infty}^{\infty} h^*[n]e^{-j\Omega n}\right)^* = \left(\sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}\right)^* = H^*(e^{j\Omega})$$

(2) $$\sum_{n=-\infty}^{\infty} h^*[n]e^{-j\Omega n} = \left(\sum_{n=-\infty}^{\infty} h[n]e^{j\Omega n}\right)^* = H^*(e^{-j\Omega})$$

10.

(1)

$$H_1(e^{j\Omega}) = H_{lp}(e^{j(\Omega-\pi)})$$

$$H_1(e^{j\Omega}) = \begin{cases} 0, & |\Omega| < 0.8\pi \\ 1, & 0.8\pi \le |\Omega| \le \pi \end{cases}.$$

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(2)

$$H_2(e^{j\Omega}) = H(e^{j\Omega}) * (\delta(\Omega - 0.5\pi) + \delta(\Omega + 0.5\pi))$$

$$H_2(e^{j\Omega}) = \begin{cases} 0, & |\Omega| < 0.3\pi \\ 1, & 0.3\pi \le |\Omega| \le 0.7\pi \\ 0, & 0.7\pi < |\Omega| \le \pi \end{cases}$$

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(3) NO! The reasons are infinite length and non-causal property of $h_{lp}[n]$.