Midterm Exam I Reference Solutions

Nov. 4, 2009

Instructor: *Chin-Liang Wang*

1.

(1) True.

$x[n+N]=x[n] \;\Rightarrow\; y[n+N_0]=y[n]$ where $N_0=2N$.

(2) True.

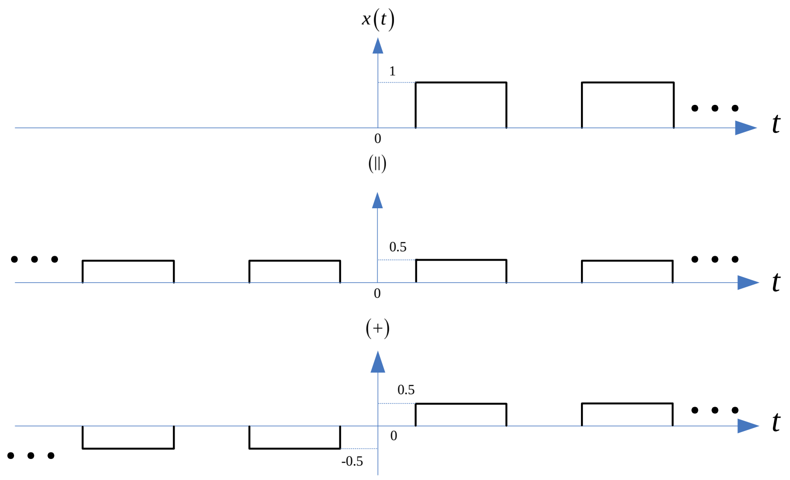
All memoryless systems are causal, since the output responds only to the current value of the input.

(3) False.

$tu(t)$ is neither energy signal nor power signal.

(4) False.

It is noted that any signal can be broken into a sum of an odd signal and an even signal.



(5) False.

Since $|h[n]|\le K$ => $\sum_{n=-\infty}^{\infty}|h[n]|=\infty$ => system is unstable.

2.

(1)

$$x(t)=\left[\sin\left(2t-\frac{\pi}{3}\right)\right]^2=\frac{1-\cos\left(4t-\frac{2\pi}{3}\right)}{2}$$

Periodic and period $=2\pi/4=\pi/2$

(2)

$$x[n] = \cos\left(\frac{1}{5}\pi n\right)\sin\left(\frac{1}{3}\pi n\right) = \frac{1}{2}\left\{\sin\left(\frac{8}{15}\pi n\right) + \sin\left(\frac{2}{15}\pi n\right)\right\}$$

$$\left.\begin{array}{l} \frac{8}{15}\pi N = 2\pi m \Rightarrow N = \frac{15}{4}m = 15, 30, \ldots \\ \frac{2}{15}\pi N = 2\pi l \Rightarrow N = 15l = 15, 30, \ldots \end{array}\right\} \Rightarrow N = 15 \text{ samples}$$

3.

Please refer to homework #1.

4.

$$y[n] = x[n] * h[n] = u[n] * h[n] - u[-n] * h[n]$$

$$u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$

$n \geq 0$,

$$\sum_{k=-\infty}^{n} h[k] = \sum_{k=-\infty}^{-1} 2^k + \sum_{k=0}^{n}\left(\frac{1}{4}\right)^k = \left(2^{-1} + 2^{-2} + \cdots\right) + \left[1 + \frac{1}{4} + \cdots + \left(\frac{1}{4}\right)^n\right]$$

$$= 1 + \frac{4}{3}\left[1 - \left(\frac{1}{4}\right)^{n+1}\right] = \frac{7}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^n$$

$n < 0$,

$$\sum_{k=-\infty}^{n} h[k] = \sum_{k=-\infty}^{n} 2^k = 2^n + 2^{n-1} + \cdots$$

$$= 2^n\left(1 + 2^{-1} + \cdots\right) = 2^{n+1}$$

$$u[-n] * h[n] = \sum_{k=n}^{\infty} h[k]$$

$n \geq 0$,

$$\sum_{k=n}^{\infty} h[k] = \sum_{k=n}^{\infty}\left(\frac{1}{4}\right)^k = \left(\frac{1}{4}\right)^n + \left(\frac{1}{4}\right)^{n+1} + \cdots = \left(\frac{1}{4}\right)^n\left(1 + \frac{1}{4} + \cdots\right) = \frac{4}{3}\left(\frac{1}{4}\right)^n$$

$n < 0$,

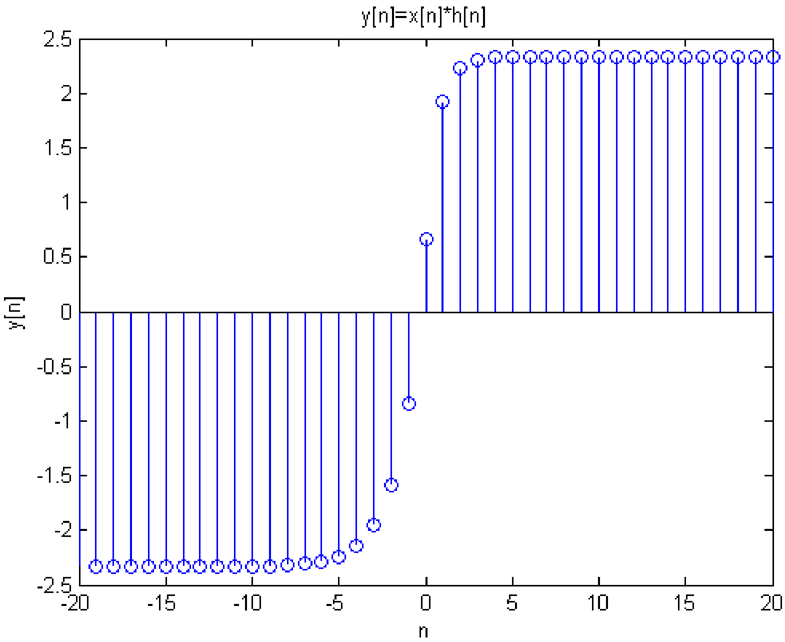
$$\sum_{k=n}^{\infty} h[k] = \sum_{k=n}^{-1} 2^k + \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = 2^{-1} + 2^{-2} + \cdots + 2^n + \left(1 + \frac{1}{4} + \cdots\right)$$

$$= 2^{-1}\left(1 + 2^{-1} + \cdots + 2^{n+1}\right) + \frac{4}{3} = 1 - 2^n + \frac{4}{3} = \frac{7}{3} - 2^n$$

$$y[n] = \left[\frac{7}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u[n] + 2^{n+1}u[-n-1] - \left\{\frac{4}{3}\left(\frac{1}{4}\right)^n u[n] + \left[\frac{7}{3} - 2^n\right]u[-n-1]\right\}$$

$$= \left[\frac{7}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^n - \frac{4}{3}\left(\frac{1}{4}\right)^n\right]u[n] + \left\{2^{n+1} - \left[\frac{7}{3} - 2^n\right]\right\}u[-n-1]$$

$$= \left[\frac{7}{3} - \frac{5}{3}\left(\frac{1}{4}\right)^n\right]u[n] + \left(3 \cdot 2^n - \frac{7}{3}\right)u[-n-1]$$



5.

(1) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] = \sum_{k=-\infty}^{\infty} u[k-4] \cdot h[n-k]$

$y[n] = \sum_{k=4}^{\infty} h[n-k]$

Evaluation the above summation:

For $n < 4$: $y[n] = 0$

For $n = 4$: $y[n] = h[0] = 1$

For $n = 5$: $y[n] = h[1] + h[0] = 2$

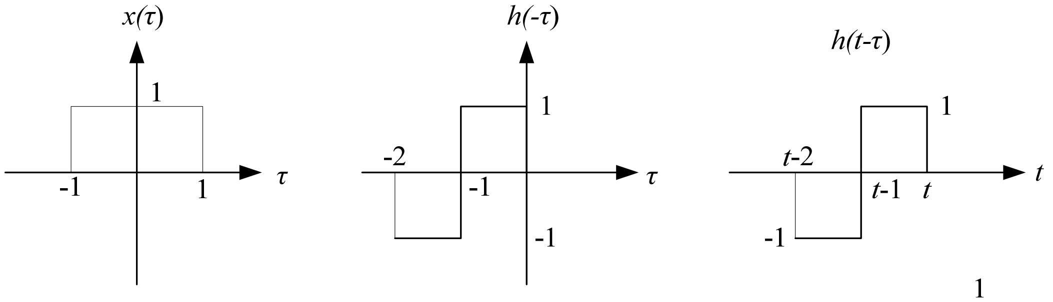
For $n = 6$: $y[n] = h[2] + h[1] + h[0] = 3$

For $n = 7$: $y[n] = h[3] + h[2] + h[1] + h[0] = 4$

For $n = 8$: $y[n] = h[4] + h[3] + h[2] + h[1] + h[0] = 2$

For $n \geq 9$: $y[n] = h[5] + h[4] + h[3] + h[2] + h[1] + h[0] = 0$

(2) $y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$



$$h(t-\tau)=\begin{cases}-1 & , \quad t-2\le\tau<t-1\\ 1 & , \quad t-1\le\tau<t\\ 0 & , \quad \text{otherwise}\end{cases}$$

when $t<-1$ $\longrightarrow$ $t<-1$: $y(t)=0$ (1%)

when $t\ge -1, t-1<-1$ $\longrightarrow$ $-1\le t<0$: $y(t)=\int_{-1}^{t}1\cdot d\tau=t+1$ (1%)

when $t-1\ge -1, t-2<-1$ $\longrightarrow$ $0\le t<1$: $y(t)=\int_{t-1}^{t}1\cdot d\tau+\int_{-1}^{t-1}(-1)\cdot d\tau=1-t$ (1%)

when $t-2\ge -1, t-1<1$ $\longrightarrow$ $1\le t<2$: $y(t)=\int_{t-1}^{1}1\cdot d\tau+\int_{t-2}^{t-1}(-1)\cdot d\tau=1-t$ (1%)

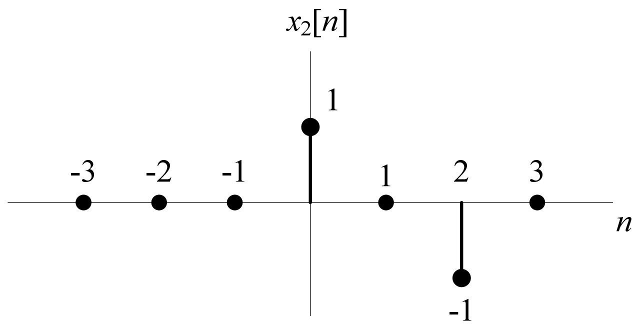
when $t-1\ge 1, t-2<1$ $\longrightarrow$ $2\le t<3$: $y(t)=\int_{t-2}^{1}(-1)\cdot d\tau=t-3$ (1%)

when $t-2\ge 1$ $\longrightarrow$ $3\le t$: $y(t)=0$ (1%)

6. Since convolution is commutative,

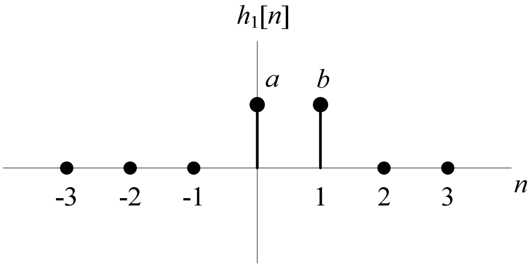
$$y[n]=x[n]*h_1[n]*h_2[n]\Rightarrow y[n]=(x[n]*h_2[n])*h_1[n]$$

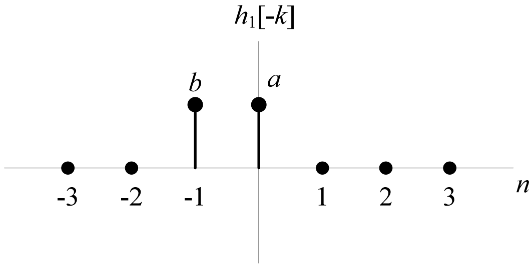
Let $x_2[n]=x[n]*h_2[n]=\delta[n]-\delta[n-2]$



$$y[n]=x_2[n]*h_1[n]$$

Suppose $h_1[n]$ shown below,





when $n = 0$: $y[0] = 1 = 1 \cdot a + 0 \cdot b + (-1) \cdot 0 = a$

when $n = 1$: $y[1] = 2 = 1 \cdot b + 0 \cdot a + (-1) \cdot 0 = b$

when $n = 2$: $y[2] = -1 = 0 \cdot b + (-1) \cdot a = -a$

when $n = 3$: $y[3] = -2 = (-1) \cdot b + 0 \cdot a = -b$

Therefore $h_1[n] = \delta[n] + 2\delta[n-1]$.

7.

$$3y[n] - y[n-1] = x[n] - x[n-1],$$

$$y_h[n]: 3r - 1 = 0, r = \frac{1}{3}, y_h[n] = E(\frac{1}{3})^n$$

$$x[n] = (\frac{1}{2})^n u[n] + n^2 u[n]$$

$$y_p[n] = A(\frac{1}{2})^n + Bn^2 + Cn + D$$

$$3y_p[n] = 3A(\frac{1}{2})^n + 3Bn^2 + 3Cn + 3D$$

$$y_p[n-1] = A(\frac{1}{2})^{n-1} + B(n-1)^2 + C(n-1) + D$$

$$= 2A(\frac{1}{2})^2 + Bn^2 - 2Bn + B + Cn - C + D$$

$$3y[n] - y[n-1] = x[n] - x[n-1]$$

$$A(\frac{1}{2})^n + 2Bn^2 + (2C - 2B)n + (2D - B + C) = -(\frac{1}{2})^n + 2n - 1$$

$A = -1,\ B = 0,\ C = 1,$ and $D = -1$

$$y_p[n] = -(\frac{1}{2})^n + n - 1$$

$$y[n] = E(\frac{1}{3})^n - (\frac{1}{2})^n + n - 1$$

$$y[0] = \frac{1}{3} = E - 1 - 1, E = \frac{7}{3}$$

$$y[n] = \frac{7}{3}(\frac{1}{3})^n - (\frac{1}{2})^n + n - 1$$

8.

$$r^2+6r+8=0 \Rightarrow r=-4,-2$$
$$y_h(t)=c_1e^{-2t}+c_2e^{-4t}$$
$$y_p(t)=ke^{-t}u(t)=\frac{2}{3}e^{-t}u(t)$$
$$y(t)=\frac{2}{3}e^{-t}u(t)+c_1e^{-2t}+c_2e^{-4t}$$
$$y(0^-)=-1=\frac{2}{3}+c_1+c_2$$
$$\frac{d}{dt}y(0)\bigg|_{t=0^-}=1=-\frac{2}{3}-2c_1-4c_2$$
$$c_1=\frac{5}{2},\ c_2=\frac{5}{6}$$
$$y(t)=\frac{2}{3}e^{-t}u(t)-\frac{5}{2}e^{-2t}+\frac{5}{6}e^{-4t}$$
$$y_n(t)=c_1e^{-4t}+c_2e^{-2t}$$
$$y(0^-)=-1=c_1+c_2$$
$$\frac{d}{dt}y(t)\bigg|_{t=0^-}=1=-4c_1-2c_2$$
$$y_n(t)=\frac{1}{2}e^{-4t}-\frac{3}{2}e^{-2t}$$
$$y_f(t)=\frac{2}{3}e^{-t}u(t)+c_1e^{-2t}u(t)+c_2e^{-4t}u(t)$$
$$y(0)=0=\frac{2}{3}+c_1+c_2$$
$$\frac{d}{dt}y(t)\bigg|_{t=0^-}=0=-\frac{2}{3}-2c_1-4c_2$$
$$y_f(t)=\frac{2}{3}e^{-t}u(t)-e^{-2t}u(t)+\frac{1}{3}e^{-4t}u(t)$$

9.

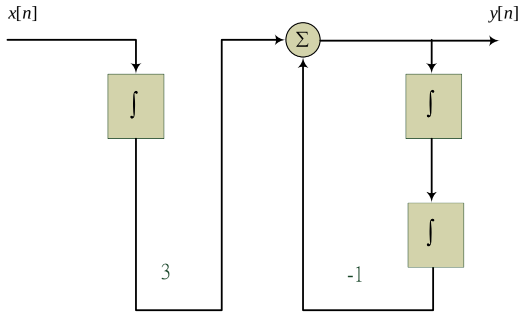
$$x(t)=-2\sin(2\pi t)+4\cos^2(2\pi t)=2-2\sin(2\pi t)+2\cos(4\pi t)$$
$$\therefore T_0=1 \text{ and } \omega_0=2\pi$$
$$\Rightarrow x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_0 t}=2+j\left(e^{j\omega_0 t}-e^{-j\omega_0 t}\right)+\left(e^{j2\omega_0 t}+e^{-j2\omega_0 t}\right)$$
$$\Rightarrow X[k]=\begin{cases}2 & k=0\\ j & k=1\\ -j & k=-1\\ 1 & k=\pm 2\\ 0 & k=o.w.\end{cases}$$

10.

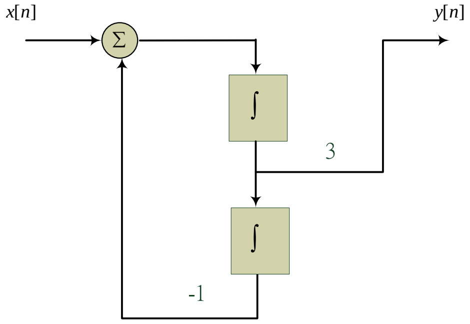
(1)

$$y(t)+\int\int y(t)\ \mathrm{d}t\mathrm{d}t=3\int x(t)\ \mathrm{d}t$$

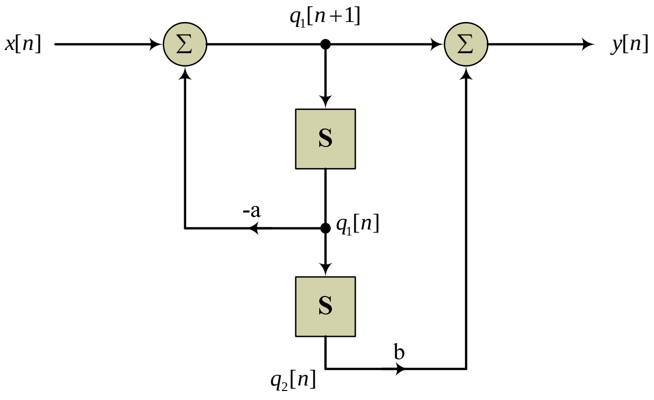
Direct form I



Direct form II



(2)



$$q_1[n+1]=-aq_1[n]+x[n]$$
$$q_2[n+1]=q_1[n]$$
$$y[n]=-aq_1[n]+x[n]+bq_2[n]$$
$$\mathbf{A}=\begin{bmatrix}-a & 0\\ 1 & 0\end{bmatrix}$$
$$\mathbf{b}=\begin{bmatrix}1\\ 0\end{bmatrix}$$
$$\mathbf{c}=[-a \quad b]$$
$$D=1$$