Homework No. 5 Solution

1. Let $x[n]$ be a periodic signal with period N and Fourier coefficients a_k.

(1) Express the Fourier coefficients b_k of $\left|x[n]\right|^2$ in terms of a_k. (10%)

Since $x[n] \xleftrightarrow{F.S.} a_k$ and $x[n] \xleftrightarrow{F.S.} a_{-k}^*$. By using the convolution property, we have: $x[n]x^*[n] = \left|x[n]\right|^2 \xleftrightarrow{F.S.} b_k = \sum_{l=<N>} a_l a_{l+k}^*$.

(2) If the coefficients a_k are real, is it guaranteed that the coefficients b_k are also real? (10%)

From (1), it is clear that the answer is yes.

2. When the impulse train $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$ is the input to a particular LTI system with frequency response $H\left(e^{j\Omega}\right)$, the output of the system is found to be $y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right)$. Determine the values of $H\left(e^{jk\pi/2}\right)$ for k = 0, 1, 2, and 3. (20%)

The F.S. of $x[n]$ are $a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j2\pi kn/4} = \frac{1}{4}$ for all k. The output signal $y[n]$ can be express as:

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{3} a_k H\left(e^{j2\pi k/4}\right) e^{j2\pi kn/4} \\
&= \frac{1}{4}\left(H\left(e^{j0}\right)e^{j0} + H\left(e^{j\pi/2}\right)e^{jn\pi/2} + H\left(e^{j\pi}\right)e^{jn\pi} + H\left(e^{j3\pi/2}\right)e^{j3n\pi/2}\right) \\
&= \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) = \frac{e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}}{2} \\
&= \frac{e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} + e^{j\left(\frac{3\pi}{2}n-\frac{\pi}{4}\right)}}{2} \quad \left(\because e^{-j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} = e^{j\left(\left(2\pi-\frac{\pi}{2}\right)n-\frac{\pi}{4}\right)}\right)
\end{aligned}
$$

$$\Rightarrow H\left(e^{j0}\right) = H\left(e^{j\pi}\right) = 0,\ H\left(e^{j\pi/2}\right) = 2e^{j\pi/4},\ \text{and } H\left(e^{j3\pi/2}\right) = 2e^{-j\pi/4}.$$

3. You are given $x[n]=n(1/2)^{|n|} \xleftrightarrow{DTFT} X(\Omega)$. Without evaluating $X(\Omega)$, find $y[n]$ if

(1) $Y(\Omega)=\mathrm{Re}\{X(\Omega)\}$ (5%)

$\Rightarrow$ Since $x[n]$ is real and odd, $X(\Omega)$ is pure imaginary, thus $y[n]=0$.

(2) $Y(\Omega)=dX(\Omega)/d\Omega$ (5%)

$\Rightarrow y[n]=-jnx[n]=-jn^2(1/2)^{|n|}$.

(3) $Y(\Omega)=X(\Omega)+X(-\Omega)$ (5%)

$\Rightarrow y[n]=x[n]+x[-n]=n(1/2)^{|n|}-n(1/2)^{|n|}=0$

(4) $Y(\Omega)=e^{-4j\Omega}X(\Omega)$ (5%)

$\Rightarrow y[n]=x[n-4]=(n-4)(1/2)^{|n-4|}$

4. Let $x[n]$ and $h[n]$ be the signals with the following Fourier transforms:

$$X\left(e^{j\Omega}\right)=3e^{-j\Omega}+1-e^{j\Omega}+2e^{j3\Omega}$$
$$H\left(e^{j\Omega}\right)=2e^{-j2\Omega}-e^{-j\Omega}+e^{j4\Omega}$$

Determine $y[n]=x[n]*h[n]$. (15%)

$$\begin{aligned}y[n]&=x[n]*h[n]\\&=\left(3\delta[n-1]+\delta[n]-\delta[n+1]+2\delta[n+3]\right)*\left(2\delta[n-2]-\delta[n-1]+\delta[n+4]\right)\\&=6\delta[n-3]-\delta[n-2]-3\delta[n-1]+\delta[n]+4\delta[n+1]-2\delta[n+2]+3\delta[n+3]\\&\quad+\delta[n+4]-\delta[n+5]+2\delta[n+7]\end{aligned}$$

5. Consider the finite-length sequence $x[n]=2\delta[n]+\delta[n-1]+\delta[n-3]$.

(1) Compute the five-point DFT $X[k]$. (10%)

$\Rightarrow X[k]=2+e^{-j\frac{2\pi}{5}k}+e^{-j3\frac{2\pi}{5}k}$.

(2) If $Y[k]=X^2[k]$, determine the sequence $y[n]$ with five-point inverse DFT for $n = 0 \sim 4$. (10%)

$$\begin{aligned}Y[k]=X^2[k]&=4+4e^{-j\frac{2\pi}{5}k}+e^{-j2\frac{2\pi}{5}k}+4e^{-j3\frac{2\pi}{5}k}+2e^{-j4\frac{2\pi}{5}k}+e^{-j6\frac{2\pi}{5}k}\\&=4+5e^{-j\frac{2\pi}{5}k}+e^{-j2\frac{2\pi}{5}k}+4e^{-j3\frac{2\pi}{5}k}+2e^{-j4\frac{2\pi}{5}k}\end{aligned}$$

$\therefore y[n]=4\delta[n]+5\delta[n-1]+\delta[n-2]+4\delta[n-3]+2\delta[n-4]$

(3) If N-point DFTs are used here, how should we choose N such that $y[n]=x[n]*x[n]$, for $0\le n\le N-1$. (5%)

$\Rightarrow N \geqq 4+4-1=7$.