Homework 3 Solution

1. (20%)

(1) $X(\omega)=\dfrac{1}{1+j\omega}$, and $Y(\omega)=\dfrac{1}{2+j\omega}+\dfrac{1}{3+j\omega}=\dfrac{5+2j\omega}{(2+j\omega)(3+j\omega)}$

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{5+7j\omega-2\omega^2}{(2+j\omega)(3+j\omega)}=2-\frac{1}{2+j\omega}-\frac{2}{3+j\omega}$$

$$\therefore h(t)=2\delta(t)-(e^{-2t}+2e^{-3t})u(t)$$

(2) $X(\omega)=\dfrac{1}{2+j\omega}$, and $Y(\omega)=\dfrac{2}{(2+j\omega)^2}e^{-j2\omega}$

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{2}{2+j\omega}e^{-j2\omega} \quad \therefore h(t)=2e^{-2(t-2)}u(t-2)$$

2. (30%)

(1)

$$\begin{aligned}
e^{-3|t|} &\xleftrightarrow{FT} \frac{6}{9+\omega^2}\\
s(t-1) &\xleftrightarrow{FT} e^{-j\omega}S(j\omega)\\
tw(t) &\xleftrightarrow{FT} j\frac{d}{d\omega}W(j\omega)\\
X(j\omega) &= j\frac{d}{d\omega}\left[e^{-j\omega}\frac{6}{9+\omega^2}\right]\\
&= \frac{6e^{-j\omega}}{9+\omega^2}-\frac{12j\omega^{-j\omega}}{(9+\omega^2)^2}
\end{aligned}$$

(2)

$$\begin{aligned}
x(t) &= \sin(2\pi t)e^{-t}u(t)\\
&= \frac{1}{2j}e^{j2\pi t}e^{-t}u(t)-\frac{1}{2j}e^{-j2\pi t}e^{-t}u(t)
\end{aligned}$$

$$\begin{aligned}
e^{-t}u(t) &\xleftrightarrow{FT} \frac{1}{1+j\omega}\\
e^{j2\pi t}s(t) &\xleftrightarrow{FT} S(j(\omega-2\pi))\\
X(j\omega) &= \frac{1}{2j}\left[\frac{1}{1+j(\omega-2\pi)}-\frac{1}{1+j(\omega+2\pi)}\right]
\end{aligned}$$

(3)

$$\frac{\sin(Wt)}{\pi t} \quad \overset{FT}{\longleftrightarrow} \quad \begin{cases} 1 & \omega \le W \\ 0, & \text{otherwise} \end{cases}$$

$$s_1(t)s_2(t) \quad \overset{FT}{\longleftrightarrow} \quad \frac{1}{2\pi} S_1(j\omega) * S_2(j\omega)$$

$$X(j\omega) = \begin{cases} 5 - \frac{|\omega|}{\pi} & \pi < |\omega| \le 5\pi \\ 4 & |\omega| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

3. (20%)

(1)

$$\text{Since } \frac{1}{(1+j\omega)^2} \overset{F.T.}{\longleftrightarrow} te^{-t}u(t) \quad \text{and} \quad j\omega S(\omega) \overset{F.T.}{\longleftrightarrow} \frac{d}{dt}s(t)$$

$$\therefore x(t) = \frac{d}{dt}\left[te^{-t}u(t)\right] = (1-t)e^{-t}u(t)$$

(2)

$$\because \frac{2\sin(\omega)}{\omega} \overset{F}{\longleftrightarrow} \text{rect}(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{let } S(j\omega) = 2 \cdot \frac{2\sin(2\omega)}{2\omega} \overset{F}{\longleftrightarrow} s(t) = \text{rect}\left(\frac{t}{2}\right) = \begin{cases} 1, & |t| \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$S_1(j\omega) = 2\sin(4\omega) \cdot S(j\omega) \overset{F}{\longleftrightarrow} s_1(t) = -js(t+4) + js(t-4)$$

$$X(j\omega) = \frac{d}{d\omega} S_1(j\omega) \overset{F}{\longleftrightarrow} x(t) = -jts_1(t)$$

$$x(t) = -t\text{rect}\left(\frac{t+4}{2}\right) + t\text{rect}\left(\frac{t-4}{2}\right)$$

4. (20%)

(1)

$$jwY(jw) + 3Y(jw) = X(jw)$$

$$H(jw) = \frac{Y(jw)}{X(jw)}$$

$$= \frac{1}{jw+3}$$

$$h(t) = e^{-3t}u(t)$$

(2)

$$\frac{d^3}{dt^3}y(t)-3\frac{d}{dt}y(t)-2y(t)=3\frac{d^2}{dt^2}x(t)+8\frac{d}{dt}x(t)-10x(t)$$

$$\Rightarrow [(jw)^3-3jw-2]Y(jw)=[3(jw)^2+8jw-10]X(jw)$$

$$\Rightarrow H(jw)=\frac{Y(jw)}{X(jw)}=\frac{-3w^2+8jw-10}{-jw^3-3jw-2}$$

$$=\frac{-3w^2+8jw-10}{(jw+1)^2(jw-2)}$$

$$=\frac{A}{(jw+1)^2}+\frac{B}{(jw+1)}+\frac{C}{(jw-2)}$$

$$A=\left\{\frac{A}{(jw+1)^2}(jw+1)^2+\frac{B}{(jw+1)}(jw+1)^2+\frac{C}{(jw-2)}(jw+1)^2\right\}\Big|_{w=j}$$

$$=\left\{(jw+1)^2H(w)\right\}\Big|_{w=j}$$

$$=\left\{(jw+1)^2\times\frac{-3w^2+8jw-10}{(jw+1)^2(jw-2)}\right\}\Big|_{w=j}$$

$$=5$$

$$B=\frac{1}{j}\frac{d}{dw}\left\{\frac{A}{(jw+1)^2}(jw+1)^2+\frac{B}{(jw+1)}(jw+1)^2+\frac{C}{(jw-2)}(jw+1)^2\right\}\Big|_{w=j}$$

$$=\left\{\frac{1}{j}\frac{d}{dw}(jw+1)^2H(w)\right\}\Big|_{w=j}$$

$$=\left\{\frac{1}{j}\frac{d}{dw}\left(\frac{-3w^2+8jw-10}{(jw-2)}\right)\right\}\Big|_{w=j}$$

$$=1$$

$$C=\left\{(jw-2)H(w)\right\}\Big|_{w=-2j}$$

$$=2$$

$$\Rightarrow H(jw)=\frac{5}{(jw+1)^2}+\frac{1}{(jw+1)}+\frac{2}{(jw-2)}$$

$$\Rightarrow h(t)=5te^{-t}u(t)+e^{-t}u(t)-2e^{2t}u(-t)$$

5. (10%)

$$X[k]=\frac{\sin(k\frac{\pi}{8})}{\pi k}\xleftrightarrow{FS;\pi}x(t)=\begin{cases}1 & ,|t|\le\dfrac{\pi}{8w_0}\\ 0 & ,\dfrac{\pi}{8w_0}<|t|\le\dfrac{2\pi}{w_0}\end{cases}$$

$$\pi^2\sum_{k=-\infty}^{\infty}\frac{\sin^2(k\pi/8)}{\pi^2k^2}=\frac{\pi^2}{T}\int_{-0.5T}^{0.5T}|x(t)|^2dt$$

$$=\frac{\pi w_0}{2}\int_{-\frac{\pi}{8w_0}}^{\frac{\pi}{8w_0}}|1|^2dt$$

$$=\frac{2\pi^2w_0}{16w_0}$$

$$=\frac{\pi^2}{8}$$