Homework 2 Solution

1.

(30%) $x[n] = (-1)^n (u[n] - u[n-7])$ and $h[n] = u[n+3]$.

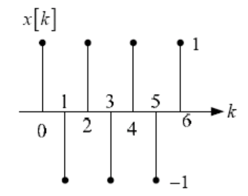
$n+3<0,\ n<-3,\ w_n[k]=0,\ y[n]=0$

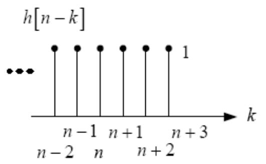
$0 \le n+3 \le 6,\ -3 \le n \le 3,\quad w_n[k] = (-1)^k,\ 0 \le k \le n+3$

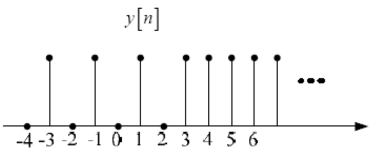
$$y[n] = \sum_{k=0}^{n+3} (-1)^k = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$6 < n+3,\ 3<n,\quad w_n[k] = (-1)^k,\ 0 \le k \le 6$

$$y[n] = \sum_{k=0}^{6} (-1)^k = 1$$







2.

(30%) Evaluate the following continuous-time convolution integrals:

$$y(t) = \cos(\pi t)[u(t+1) - u(t-1)] * [u(t+1) - u(t-1)]$$

Let $x(t) = [u(t+1) - u(t-1)]$ and $h(t) = \cos(\pi t)[u(t+1) - u(t-1)]$. Then

$h(-\tau) = [u(-\tau+1) - u(-\tau-1)] = [u(\tau+1) - u(\tau-1)]$ ($\because$ symmetric property)

$h(t-\tau) = [u(\tau-t+1) - u(\tau-t-1)]$

$w_t(\tau) = x(\tau)h(t-\tau)$

For $t+1<-1,\ t<-2,\ w_t(\tau)=0,\ y(t)=0$

For $t+1<1,\ -2 \le t < 0,\ -1<\tau<t+1,\ w_t(\tau) = \cos(\pi\tau)$,

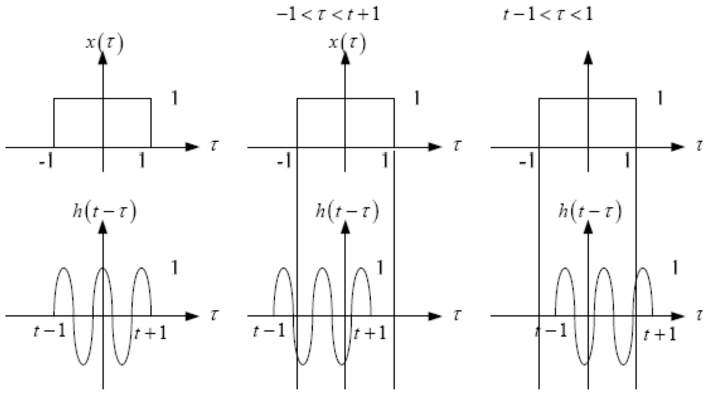
$$y(t) = \int_{-1}^{t+1} \cos(\pi\tau)d\tau = \frac{1}{\pi}\sin(\pi(t+1))$$

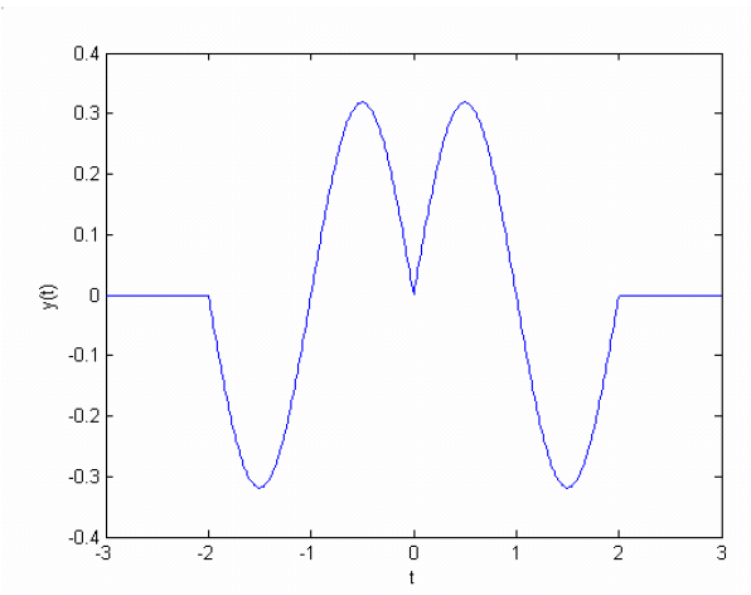
For $t-1<1,\ 0 \le t < 2,\ t-1<\tau<1,\ w_t(\tau) = \cos(\pi\tau)$

$$y(t) = \int_{t-1}^{1} \cos(\pi\tau)d\tau = -\frac{1}{\pi}\sin(\pi(t-1))$$

For $1<t-1,\ 2 \le t,\ w_t(\tau)=0,\ y(t)=0$

$$y(t)=\begin{cases} 0 & , t<-2 \\ \frac{1}{\pi}\sin\left(\pi(t+1)\right) & , -2\le t<0 \\ -\frac{1}{\pi}\sin\left(\pi(t-1)\right) & , 0\le t<2 \\ 0 & , t\ge 2 \end{cases}$$





3.

(i) If $y_h[n]=A(1/2)^n$, then we need to verify

$$A\left(\frac{1}{2}\right)^n - \frac{1}{2}A\left(\frac{1}{2}\right)^{n-1} = 0$$

it's true.

(ii) For n>=0

$$B\left(\frac{1}{3}\right)^n - \frac{1}{2}B\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n$$

There for B=-2.

(iii)
From eq. (1) we know that y[0]=x[0]+(1/2)y[-1]=z[0]=1,
now we also have

$$y[0]=A+B$$
$$A=1-B=3.$$

4.

$$x(t)=e^{3t}u(t)$$

and

$$y(t)=y_p(t)+y_h(t)$$

$y_h(t)$ is a solution of the homogeneous differential equation

$$\frac{d}{dt}y(t) + 2y(t) = 0$$

A common method for finding the particular solution for an exponential input signal is to look for a so-called forced response, i.e. a signal of the same form as the input. Since $x(t)=e^{3t}u(t)$ for t>0, we hypothesize a solution for t>0 of the form

$$y_p(t)=Ye^{3t}(t)$$

Where Y is a number that we must determine.
For t>0 yields

$$3Ye^{3t}+2Ye^{3t} = e^{3t}$$
$$Y=1/5$$
$$y_p(t)=(1/5)e^{3t}(t) \qquad t>0$$
$$y_h(t)=Ae^{st}$$
$$Ase^{st}+2Ae^{et} = Ae^{et}(s+2) = 0$$
$$s=-2$$
$$y(t)=Ae^{-2t}+(1/5)e^{3t},\ t>0$$

and set y(0)=0

$$A=-(1/5)$$

Thus for t>0

$$y(t)=(1/5)(e^{3t}-e^{-2t})$$

or

$$y(t)=(1/5)(e^{3t}-e^{-2t})u(t).$$

5.

(1)

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{dt}{dt} = 1$$

$y_p(t)=C=1$

$y(t)=Ae^{-t}+Bte^{-t}+1$

$y'(t)=-Ae^{-t}+Be^{-t}-Bte^{-t}$

$y(0)=-1$

$y'(0)=1$

$\Rightarrow A=-2\ ,\ B=1$

$y(t)=-2e^{-t}-te^{-t}+1$

(2)

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{d[\sin(t) + \cos(t)]}{dt} = \cos(t) - \sin(t)$$

$y_p(t)=A\cos(t) + B\sin(t) - 2A\sin(t) + 2B\cos(t) - A\cos(t) - B\sin(t) = \cos(t)-\sin(t)$

$A=B=(1/2)$

$y(t)=$

$y'(t)=-Ce^{-t} + De^{-t} - Dte^{-t} - (1/2)\sin(t) + (1/2)\cos(t)$

$y(0)=-1$

$y'(0)=1$

$\Rightarrow C=-(3/2)\ ,\ D=-1$

$y(t)= -(3/2)e^{-t} - te^{-t} + (1/2)\cos(t) + (1/2)\sin(t)$

6.

Natural response:

$$r-\frac{1}{2}=0\Rightarrow r=\frac{1}{2}\Rightarrow y^{(h)}[n]=c\left(\frac{1}{2}\right)^{n}$$

$$y[-1]=3=c\left(\frac{1}{2}\right)^{-1}\Rightarrow c=\frac{3}{2}\Rightarrow y^{(n)}[n]=\frac{3}{2}\left(\frac{1}{2}\right)^{n}$$

Forced response:

$$y^{(p)}[n]=k\left(\frac{-1}{2}\right)^{n}u[n]$$

$$k\left(\frac{-1}{2}\right)^{n}-k\frac{1}{2}\left(\frac{-1}{2}\right)^{n-1}=2\left(\frac{-1}{2}\right)^{n}\Rightarrow\left(\frac{-1}{2}\right)k-k\frac{1}{2}=2\left(\frac{-1}{2}\right)\Rightarrow k=1$$

$$\therefore y^{(p)}[n]=\left(\frac{-1}{2}\right)^{n}u[n]$$

$$y^{(f)}[n]=c\left(\frac{1}{2}\right)^{n}+\left(\frac{-1}{2}\right)^{n},\ n\geq 0$$

Translate initial condition

$$y[n]=\frac{1}{2}y[n-1]+2x[n]$$

$$y[0]=\frac{1}{2}y[-1]+2x[0]=\frac{1}{2}0+2=2$$

$$y[0]=2=c+1\Rightarrow c=1$$

$$\therefore y^{(f)}[n]=\left(\frac{1}{2}\right)^{n}+\left(\frac{-1}{2}\right)^{n},\ n\geq 0$$