Homework No. 4 Solution

1. $\frac{d^2}{dt^2}y(t)+2\frac{d}{dt}y(t)+y(t)=\frac{d}{dt}x(t)$, $y(0^-)=-1$, $\left.\frac{d}{dt}y(t)\right|_{t=0^-}=1$

$$r^2+2r+1=0 \Rightarrow r=-1,-1 \Rightarrow y^{(h)}(t)=c_1e^{-t}+c_2te^{-t}$$

(1) $x(t)=2e^{-t}u(t)$

Since e^{-t} and te^{-t} are in the homogeneous solution, the particular solution takes the form of $y^{(p)}(t)=kt^2e^{-t}$.

$$\frac{d}{dt}y^{(p)}(t)=2kte^{-t}-kt^2e^{-t}$$

$$\frac{d^2}{dt^2}y^{(p)}(t)=2ke^{-t}-2kte^{-t}-2kte^{-t}+kt^2e^{-t}=2ke^{-t}-4kte^{-t}+kt^2e^{-t}$$

$$\frac{d^2}{dt^2}y(t)+2\frac{d}{dt}y(t)+y(t)=\frac{d}{dt}x(t)$$

$$\Rightarrow 2ke^{-t}-4kte^{-t}+kt^2e^{-t}+4kte^{-t}-2kt^2e^{-t}+kt^2e^{-t}=-2e^{-t}$$

$$\Rightarrow 2ke^{-t}=-2e^{-t} \Rightarrow k=-1$$

$$\therefore y^{(p)}(t)=-t^2e^{-t}$$

$$y(t)=c_1e^{-t}+c_2te^{-t}-t^2e^{-t}$$

$$y(0^-)=c_1=-1$$

$$\frac{d}{dt}y(t)=-c_1e^{-t}+c_2e^{-t}-c_2te^{-t}-2te^{-t}+t^2e^{-t}$$

$$=e^{-t}+c_2e^{-t}-c_2te^{-t}-2te^{-t}+t^2e^{-t}$$

$$\left.\frac{d}{dt}y(t)\right|_{t=0^-}=1+c_2=1 \Rightarrow c_2=0$$

$$\therefore y(t)=-e^{-t}-t^2e^{-t}$$

(2) $x(t)=2\sin(t)$

$$y^{(p)}(t)=A\cos(t)+B\sin(t)$$

$$\frac{d}{dt}y^{(p)}(t)=-A\sin(t)+B\cos(t)$$

$$\frac{d^2}{dt^2}y^{(p)}(t)=-A\cos(t)-B\sin(t)$$

$$\frac{d^2}{dt^2}y(t)+2\frac{d}{dt}y(t)+y(t)=\frac{d}{dt}x(t)$$

$$-A\cos(t)-B\sin(t)-2A\sin(t)+2B\cos(t)+A\cos(t)+B\sin(t)=2\cos(t)$$

$$(-A+2B+A)\cos(t)+(-B-2A+B)\sin(t)=2\cos(t)$$

$$\begin{cases}-A+2B+A=2\\-B-2A+B=0\end{cases}\Rightarrow\begin{cases}A=0\\B=1\end{cases}\Rightarrow y^{(p)}(t)=\sin(t)$$

$$y(t)=c_1e^{-t}+c_2te^{-t}+\sin(t)$$

$$y(0^-)=c_1=-1$$

$$\frac{d}{dt}y(t)=-c_1e^{-t}+c_2e^{-t}-c_2te^{-t}+\cos(t)$$

$$=e^{-t}+c_2e^{-t}-c_2te^{-t}+\cos(t)$$

$$\left.\frac{d}{dt}y(t)\right|_{t=0^-}=1+c_2+1=1\Rightarrow c_2=-1$$

$$\therefore y(t)=-e^{-t}-te^{-t}+\sin(t)$$

2. $\frac{d^2}{dt^2}y(t)+y(t)=3\frac{d}{dt}x(t),\ y(0^-)=-1,\ \left.\frac{d}{dt}y(t)\right|_{t=0^-}=1,\ x(t)=2te^{-t}u(t)$

$$r^2+1=0\Rightarrow r=\pm j\Rightarrow y^{(h)}(t)=A\cos(t)+B\sin(t)$$

Natural response:

$$y^{(n)}(t)=c_1\cos(t)+c_2\sin(t)$$

$$y(0^-)=c_1=-1$$

$$\left.\frac{d}{dt}y(t)\right|_{t=0^-}=-c_1\sin(0^-)+c_2\cos(0^-)=c_2=1$$

$$\therefore y^{(n)}(t)=-\cos(t)+\sin(t)$$

Forced response:

$$y^{(p)}(t)=kte^{-t}u(t)$$

$$\frac{d}{dt}y^{(p)}(t)=ke^{-t}-kte^{-t};\ \frac{d^2}{dt^2}y^{(p)}(t)=-ke^{-t}-ke^{-t}+kte^{-t}$$

$$\frac{d}{dt}x(t)=2e^{-t}-2te^{-t}$$

$$-2ke^{-t}+kte^{-t}+kte^{-t}=3\left(2e^{-t}-2te^{-t}\right)\Rightarrow k=-3$$

$$\therefore y^{(p)}(t)=-3te^{-t}u(t)$$

$$y^{(f)}(t)=-3te^{-t}+c_1\cos(t)+c_2\sin(t),\ t>0$$

$$y(0)=c_1=0$$

$$\left.\frac{d}{dt}y(t)\right|_{t=0^-}=-3e^{-t}+3te^{-t}-c_1\sin(t)+\left.c_2\cos(t)\right|_{t=0^-}=-3+c_2=0\Rightarrow c_2=3$$

$$y^{(f)}(t)=-3te^{-t}+3\sin(t),\ t>0$$

3. $y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]+2x[n-1],\quad y[-1]=1,\ y[-2]=0$

$$r^2+r+\frac{1}{4}=0\Rightarrow r=-\frac{1}{2},-\frac{1}{2}\Rightarrow\therefore\ y^{(h)}[n]=c_1\left(-\frac{1}{2}\right)^n+c_2n\left(-\frac{1}{2}\right)^n$$

(1) $x[n]=u[n]$

$$y^{(p)}[n]=ku[n]$$
$$y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]+2x[n-1]$$
$$\Rightarrow k+k+\frac{1}{4}k=1+2\Rightarrow k=\frac{4}{3}\Rightarrow y^{(p)}[n]=\frac{4}{3}u[n]$$
$$y[n]=c_1\left(-\frac{1}{2}\right)^n+c_2n\left(-\frac{1}{2}\right)^n+\frac{4}{3}u[n]$$

Translate initial condition

$$y[n]=-y[n-1]-\frac{1}{4}y[n-2]+x[n]+2x[n-1]$$
$$y[0]=-y[-1]-\frac{1}{4}y[-2]+x[0]+2x[-1]=-1+1=0$$
$$y[1]=-y[0]-\frac{1}{4}y[-1]+x[1]+2x[0]=-\frac{1}{4}+1+2=\frac{11}{4}$$
$$y[0]=c_1+\frac{4}{3}=0\Rightarrow c_1=-\frac{4}{3}$$
$$y[1]=c_1\left(-\frac{1}{2}\right)+c_2\left(-\frac{1}{2}\right)+\frac{4}{3}=\frac{2}{3}+c_2\left(-\frac{1}{2}\right)+\frac{4}{3}=\frac{11}{4}\Rightarrow c_2=-\frac{3}{2}$$
$$\therefore\ y[n]=-\frac{4}{3}\left(-\frac{1}{2}\right)^n-\frac{3}{2}n\left(-\frac{1}{2}\right)^n+\frac{4}{3}u[n]$$

(2) $x[n]=\left(-\frac{1}{4}\right)^n u[n]$

$$y^{(p)}[n]=k\left(-\frac{1}{4}\right)^n u[n]$$
$$y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]+2x[n-1]$$
$$k\left(-\frac{1}{4}\right)^n+k\left(-\frac{1}{4}\right)^{n-1}+\frac{1}{4}k\left(-\frac{1}{4}\right)^{n-2}=\left(-\frac{1}{4}\right)^n+2\left(-\frac{1}{4}\right)^{n-1}$$
$$k+k\left(-\frac{1}{4}\right)^{-1}+\frac{1}{4}k\left(-\frac{1}{4}\right)^{-2}=1+2\left(-\frac{1}{4}\right)^{-1}\Rightarrow k=-7$$

$$y[n] = c_1\left(-\frac{1}{2}\right)^n + c_2 n\left(-\frac{1}{2}\right)^n - 7\left(-\frac{1}{4}\right)^n u[n]$$

Translate initial condition

$$x[n] = \left(-\frac{1}{4}\right)^n u[n]$$

$$y[n] = -y[n-1] - \frac{1}{4}y[n-2] + x[n] + 2x[n-1]$$

$$y[0] = -y[-1] - \frac{1}{4}y[-2] + x[0] + 2x[-1] = -1 + 1 = 0$$

$$y[1] = -y[0] - \frac{1}{4}y[-1] + x[1] + 2x[0] = -\frac{1}{4} - \frac{1}{4} + 2 = \frac{3}{2}$$

$$y[0] = c_1 + \frac{4}{3} = 0 \Rightarrow c_1 = -\frac{4}{3}$$

$$y[1] = c_1\left(-\frac{1}{2}\right) + c_2\left(-\frac{1}{2}\right) - 7\left(-\frac{1}{4}\right) = \frac{2}{3} + c_2\left(-\frac{1}{2}\right) + \frac{7}{4} = \frac{3}{2} \Rightarrow c_2 = -\frac{11}{6}$$

$$\therefore y[n] = -\frac{4}{3}\left(-\frac{1}{2}\right)^n - \frac{11}{6}n\left(-\frac{1}{2}\right)^n + \frac{4}{3}u[n]$$

4. $y[n] - \frac{1}{2}y[n-1] = 2x[n],\ y[-1] = 3,\ x[n] = \left(\frac{-1}{2}\right)^n u[n]$

Natural response:

$$r - \frac{1}{2} = 0 \Rightarrow r = \frac{1}{2} \Rightarrow y^{(h)}[n] = c\left(\frac{1}{2}\right)^n$$

$$y[-1] = 3 = c\left(\frac{1}{2}\right)^{-1} \Rightarrow c = \frac{3}{2} \Rightarrow y^{(n)}[n] = \frac{3}{2}\left(\frac{1}{2}\right)^n$$

Forced response:

$$y^{(p)}[n] = k\left(\frac{-1}{2}\right)^n u[n]$$

$$k\left(\frac{-1}{2}\right)^n - k\frac{1}{2}\left(\frac{-1}{2}\right)^{n-1} = 2\left(\frac{-1}{2}\right)^n \Rightarrow \left(\frac{-1}{2}\right)k - k\frac{1}{2} = 2\left(\frac{-1}{2}\right) \Rightarrow k = 1$$

$$\therefore y^{(p)}[n] = \left(\frac{-1}{2}\right)^n u[n]$$

$$y^{(f)}[n] = c\left(\frac{1}{2}\right)^n + \left(\frac{-1}{2}\right)^n,\ n \ge 0$$

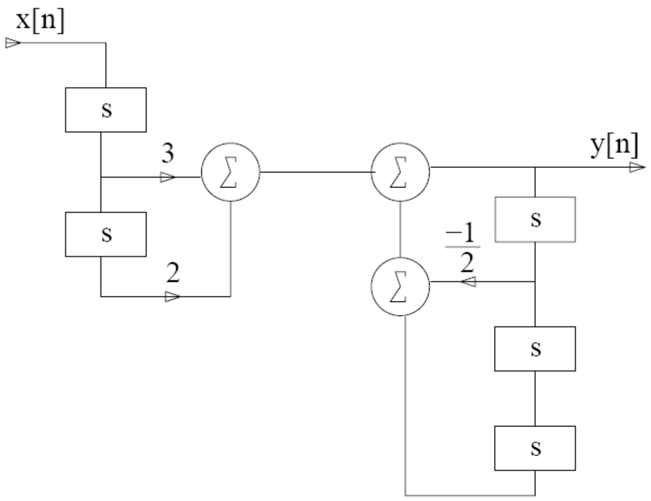
Translate initial condition

$$y[n]=\frac{1}{2}y[n-1]+2x[n]$$
$$y[0]=\frac{1}{2}y[-1]+2x[0]=\frac{1}{2}0+2=2$$
$$y[0]=2=c+1\Rightarrow c=1$$
$$\therefore y^{(f)}[n]=\left(\frac{1}{2}\right)^n+\left(\frac{-1}{2}\right)^n,\ n\geq 0$$

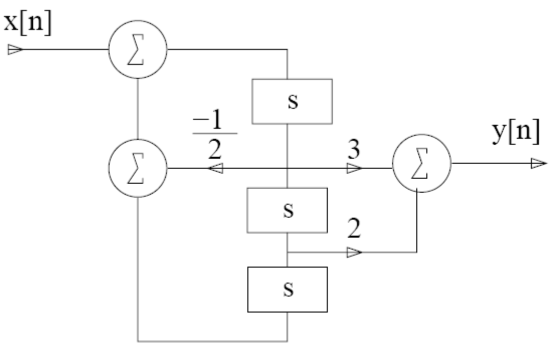
5.

(1) (a) $y[n]+\frac{1}{2}y[n-1]-y[n-3]=3x[n-1]+2x[n-2]$

Direct form I



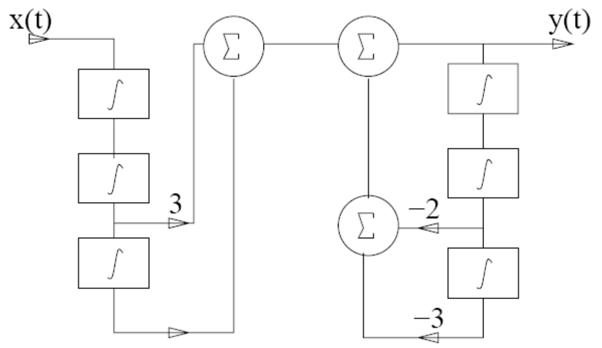
Direct form II



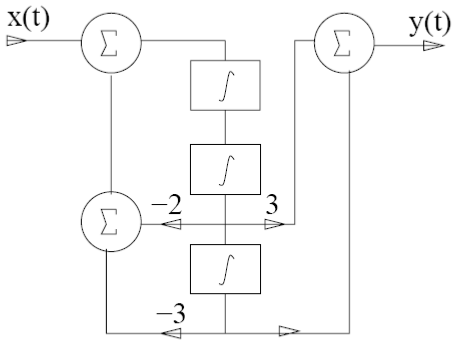
(b) $\frac{d^3}{dt^3}y(t)+2\frac{d}{dt}y(t)+3y(t)=x(t)+3\frac{d}{dt}x(t)$

$$y(t)+2y^{(2)}(t)+3y^{(3)}(t)=x^{(3)}(t)+3x^{(2)}(t)$$
$$y(t)=x^{(3)}(t)+3x^{(2)}(t)-2y^{(2)}(t)-3y^{(3)}(t)$$

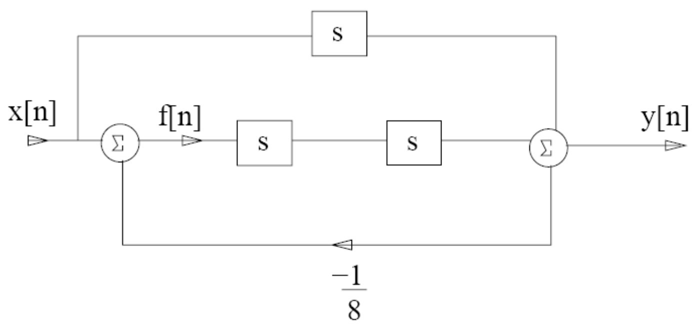
Direct form I



Direct form II



(2) (a)

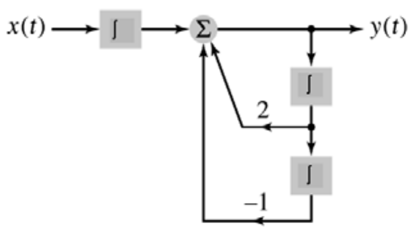


$$f[n] = x[n] - \frac{1}{8} y[n]$$

$$y[n] = x[n-1] + f[n-2] = x[n-1] + x[n-2] - \frac{1}{8} y[n-2]$$

$$\Rightarrow y[n] + \frac{1}{8} y[n-2] = x[n-1] + x[n-2]$$

(b)



$$y(t) = x^{(1)}(t) + 2y^{(1)}(t) - y^{(2)}(t)$$

$$\frac{d^2}{dt^2}y(t) = \frac{d}{dt}x(t) + 2\frac{d}{dt}y(t) - y(t)$$

$$\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t)$$