Homework No. 2 Solution

1. The signal $x[n]$ is as shown in Fig. 1. $x[n]$ can be obtained by flipping $u[n]$ and then shifting the flipped signal by 3 to the right. Therefore, $x[n] = u[-n + 3]$. This implies that $M = -1$ and $n_0 = -3$.

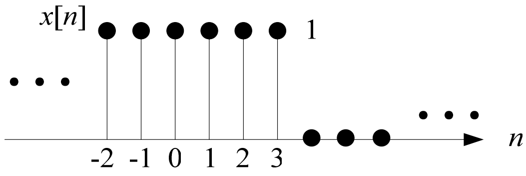

Figure 1

2. Let us name the output of H_1 as $w[n]$ and the output of H_2 as $z[n]$ (see Fig. 2). Then,

$$y[n] = z[2n] = w[2n] + \frac{1}{2}w[2n-1] + \frac{1}{4}w[2n-2]$$
$$= x[n] + \frac{1}{2}x[n-1] + \frac{1}{4}x[n-2]$$

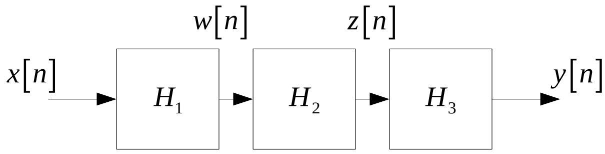

Figure 2

Linearity:

$$y_1[n] = x_1[n] + \frac{1}{2}x_1[n-1] + \frac{1}{4}x_1[n-2]$$
$$y_2[n] = x_2[n] + \frac{1}{2}x_2[n-1] + \frac{1}{4}x_2[n-2]$$
$$x_3[n] = \alpha x_1[n] + \beta x_2[n]$$
$$y_3[n] = \alpha x_1[n] + \beta x_2[n] + \frac{1}{2}\left(\alpha x_1[n-1] + \beta x_2[n-1]\right)$$
$$+\frac{1}{4}\left(\alpha x_1[n-2] + \beta x_2[n-2]\right)$$
$$= \alpha y_1[n] + \beta y_2[n]$$

Time invariant:

$$y_4[n] = x_4[n] + \frac{1}{2}x_4[n-1] + \frac{1}{4}x_4[n-2]$$
$$x_5[n] = x_4[n-n_0]$$
$$y_5[n] = x_5[n] + \frac{1}{2}x_5[n-1] + \frac{1}{4}x_5[n-1]$$
$$= x_4[n-n_0] + \frac{1}{2}x_4[n-n_0-1] + \frac{1}{4}x_4[n-n_0-2]$$

$$\therefore y_4[n-n_0] = x_4[n-n_0] + \frac{1}{2}x_4[n-n_0-1] + \frac{1}{4}x_4[n-n_0-2] = y_5[n]$$

3.

	Memory-less	Stable	Causal	Linear	Time Invariant
$y[n] = \cos(2\pi x[n+1]) + x[n]$	×	○	×	×	○
$y(t) = \frac{d}{dt}\{e^{-t}x(t)\}$	×	×	○	○	×
$y[n] = \log_{10}(\lvert x[n]\rvert)$	○	×	○	×	○
$y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$	×	×	×	○	×
$y(t) = x(t/2)$	×	○	×	○	×
$y[n] = 2x[2^n]$	×	○	×	○	×

(1) $y[n] = \cos(2\pi x[n+1]) + x[n]$

- $y(t)$ depends on future time, so it is memory and noncausal.
- $$\begin{aligned} &y_1[n] = \cos(2\pi x_1[n+1]) + x_1[n];\ y_2[n] = \cos(2\pi x_2[n+1]) + x_2[n] \\ &y_3[n] = \cos(2\pi\{\alpha x_1[n+1] + \beta x_2[n+1]\}) + \alpha x_1[n] + \beta x_2[n] \\ &\quad = \cos(2\pi\alpha x_1[n+1])\cos(2\pi\beta x_2[n+1]) \\ &\quad\ -\sin(2\pi\alpha x_1[n+1])\sin(2\pi\beta x_2[n+1]) + \alpha x_1[n] + \beta x_2[n] \\ &\quad \neq \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

  Nonlinear

(2) $y(t) = \frac{d}{dt}\{e^{-t}x(t)\} = \lim_{\Delta\to 0}\frac{e^{-(t+\Delta)}x(t+\Delta) - e^{-t}x(t)}{\Delta}$

- The value of $x(t)$ does not specify its slope at the same time instant. $\Rightarrow$ memory
- Let $x(t) = \sin(t^2)$. Then $y(t) = -e^{-t}\sin(t^2) + 2te^{-t}\cos(t^2)$. $y(t)$ is unbounded while $x(t)$ is bounded by 1.$\Rightarrow$ unstable
- For continuously differentiable x, the left limit is the same as the right

limit, so the slope can be determined with past information only.

$\Rightarrow$ causal

- $$y_1(t)=\frac{d}{dt}\left\{e^{-t}x_1(t)\right\}=\lim_{\Delta\to 0}\frac{e^{-(t+\Delta)}x_1(t+\Delta)-e^{-t}x_1(\Delta)}{\Delta}$$
  $$y_2(t)=\frac{d}{dt}\left\{e^{-t}x_1(t-t_0)\right\}=\lim_{\Delta\to 0}\frac{e^{-(t+\Delta)}x_1(t-t_0+\Delta)-e^{-t}x_1(t-t_0)}{\Delta}$$
  $$\begin{aligned}y_1(t-t_0)&=\frac{d}{dt}\left\{e^{-(t-t_0)}x_1(t-t_0)\right\}\\&=\lim_{\Delta\to 0}\frac{e^{-(t-t_0+\Delta)}x_1(t-t_0+\Delta)-e^{-t-t_0}x_1(t-t_0)}{\Delta}\\&\neq y_2(t)\end{aligned}$$

  Time-varying

(3)

- $x[n]=0,\ \left|y[n]\right|=\left|\log_{10}(0)\right|=\infty$, unstable
- $$\begin{aligned}&y_1[n]=\log_{10}\left(\left|\alpha x_1[n]\right|\right);\ y_2[n]=\log_{10}\left(\left|\beta x_2[n]\right|\right)\\&y_3[n]=\log_{10}\left(\left|\alpha x_1[n]+\beta x_2[n]\right|\right)\neq y_1[n]+y_2[n]\end{aligned}$$, nonlinear

(4)

- Since the integrated range starts from negative infinite, the system has memory.
- $$\begin{aligned}\left|y(t)\right|&=\left|\int_{-\infty}^{2t}x(\tau)d\tau\right|\leq\int_{-\infty}^{2t}\left|x(\tau)\right|d\tau\\&=M_x\int_{-\infty}^{2t}1d\tau=M_x(2t+\infty)=\infty\end{aligned}$$, unstable
- $y(1)$ requires knowledge of $x(2)$ (future value). Noncausal.
- $$\begin{aligned}&y_1(t-t_0)=\int_{-\infty}^{2t-2t_0}x_1(\tau)d\tau\\&x_2(t)=x_1(t-t_0)\\&y_2(t)=\int_{-\infty}^{2t}x_2(\tau)d\tau=\int_{-\infty}^{2t}x_1(\tau-t_0)d\tau\\&\overset{\tau'=\tau-t_0}{=}\int_{-\infty}^{2t-t_0}x_1(\tau')d\tau'\neq y_1(t-t_0)\end{aligned}$$, time-varying

(5)

- $y(-1)=x(-1/2)$, noncausal and memory

- $y_1(t) = x_1(t/2);\ x_2(t) = x_1\left(\frac{t-t_0}{2}\right)$
  $\Rightarrow y_2(t) = x_2(t/2) = x_1\left(\frac{t/2-t_0}{2}\right) \neq y_1(t-t_0) = x_1\left(\frac{t-t_0}{2}\right)$

  Time-varying

(6) $y[n] = 2x\left[2^n\right]$

- $y[1] = 2x[2]$, memory and noncausal
- $y_1[n] = 2x_1\left[2^n\right];\ x_2[n] = x_1[n-n_0]$
  $y_2[n] = 2x_2\left[2^n\right] = 2x_1\left[2^n - n_0\right] \neq y_1[n-n_0] = 2x_1\left[2^{n-n_0}\right]$,

  time-varying

4.

(1) Using the given input-output relation:

$$y[n] = a_0x[n] + a_1x[n-1] + a_2x[n-2] + a_3x[n-3]$$

We may write

$$\begin{aligned} |y[n]| &= |a_0x[n] + a_1x[n-1] + a_2x[n-2] + a_3x[n-3]| \\ &\leq |a_0x[n]| + |a_1x[n-1]| + |a_2x[n-2]| + |a_3x[n-3]| \\ &\leq |a_0|M_x + |a_1|M_x + |a_2|M_x + |a_3|M_x \\ &= (|a_0| + |a_1| + |a_2| + |a_3|)M_x \end{aligned}$$

where $|x[n]| \leq M_x$. Hence, provide that M_x is finite, the absolute value of output will always be finite. This assumes that the coefficients a_0, a_1, a_2, and a_3 have finite values of their own. The system is BIBO stable.

(2) The memory of the discrete-time system extends 3 time units into the past.

5.

(1) $y(t) = x(2t)$

(a) If $x(t)$ is periodic, then $y(t)$ is periodic. **True**.

$$y(t) = x(2t + T_x) = x(2(t + T_x/2)) = x(2(t + T_y)) = y(t + T_y)$$
$$\therefore T_y = T_x/2$$

(b) If $y(t)$ is periodic, then $x(t)$ is periodic. **True**.

$$y(t) = y(t + T_y) = x(2(t + T_y)) = x(2t + 2T_y) = x(2t + T_x)$$
$$\therefore T_x = 2T_y$$

(2) $y[n] = x[2n]$.

(a) If $x[n]$ is periodic, then $y[n]$ is periodic. **True**.

$$y[n] = x[2n] = x[2n + N_x]$$

If N_x is even, then

$$y[n] = x[2n] = x[2n + N_x] = x\left[2\left(n + N_x/2\right)\right] = y\left[n + N_y\right]$$
$$\therefore N_y = N_x/2$$

There is also a special case, if $x[n]$ is also periodic when those odd points are excluded, i.e., $x[2n] = x[2n + N_x/2] \Rightarrow N_y = N_x/4$.

$$\therefore N_y = \begin{cases} N_x/2, & \text{even part is not periodic.} \\ N_x/4, & \text{even part is periodic.} \end{cases}$$

If N_x is odd, then

$$y[n] = x[2n] = x[2n + N_x] \stackrel{?}{=} y\left[n + N_y\right]$$
$$y\left[n + N_y\right] = x\left[2n + 2N_y + N_x\right]$$
$$\therefore 2N_y + N_x = mN_x,\ m \text{ is a nonnegative integer.}$$
$$N_y = \frac{m-1}{2} N_x \Rightarrow m = 3\left(\because N_x \text{ is odd and } N_y \text{ is a positive integer.}\right)$$
$$\Rightarrow N_y = N_x$$

(b) If $y[n]$ is periodic, then $x[n]$ is periodic. **False**.

Let $x[n] = g[n] + h[n]$ where

$$g[n] = \begin{cases} 1, & n \text{ is even.} \\ 0, & n \text{ is odd.} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} 0, & n \text{ is even.} \\ (1/2)^n, & n \text{ is odd.} \end{cases}$$

Then $y[n]$ is periodic, but $x[n]$ is clearly not periodic.