Homework No. 4 Solution

1. (20%)

(1) Fundamental period of $x(t) = T = 4 \Rightarrow \omega_0 = 2\pi/4 = \pi/2$

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{4}\int_{-1}^{1} x(t)dt = 0.5$$

$$a_k = \frac{1}{4}\int_{-1}^{1} x(t)e^{-jk\omega_0 t}dt = \frac{1}{-j4k\omega_0}e^{-jk\omega_0 t}\Big|_{-1}^{1} = \frac{1}{jk2\pi}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right) = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}$$

(2) Fundamental period of $y(t) = T = 2 \Rightarrow \omega_0 = 2\pi/2 = \pi$

$$a_0 = \frac{1}{T}\int_T y(t)dt = 0;\ y(t) = \sin \pi t = \frac{e^{j\pi t} - e^{-j\pi t}}{2j} \Rightarrow a_k = \begin{cases} \frac{-j}{2}, & k = 1 \\ \frac{j}{2}, & k = -1 \\ 0, & o.w. \end{cases}$$

2. (20%)

(1) $x(t) = \sum_{m=-\infty}^{\infty} a_k e^{j2\pi kt} = je^{j2\pi t} - je^{-j2\pi t} + e^{j(3)2\pi t} + e^{j(-3)2\pi t} = -2\sin(2\pi t) + 2\cos(6\pi t)$

(2) $x(t) = \sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k e^{jk\pi t} + \sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^{-k} e^{-jk\pi t} = \frac{1}{1+\frac{1}{3}e^{j\pi t}} + \frac{-\frac{1}{3}e^{-j\pi t}}{1+\frac{1}{3}e^{-j\pi t}}$

$$= \frac{4}{5+3\cos(\pi t)}$$

3. (30%)

(1) $X(j\omega) = \frac{1}{1+j\omega}$, and $Y(\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega} = \frac{5+2j\omega}{(2+j\omega)(3+j\omega)}$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{5+7j\omega-2\omega^2}{(2+j\omega)(3+j\omega)} = 2 - \frac{1}{2+j\omega} - \frac{2}{3+j\omega}$$

$$\therefore h(t) = 2\delta(t) - (e^{-2t} + 2e^{-3t})u(t)$$

(2) $X(j\omega) = \frac{1}{2+j\omega}$, and $Y(j\omega) = \frac{2}{(2+j\omega)^2}e^{-j2\omega}$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{2+j\omega}e^{-j2\omega} \quad \therefore h(t) = 2e^{-2(t-2)}u(t-2)$$

4. (30%)

(1)

$$\begin{aligned} x(t) &= \sin(2\pi t)e^{-t}u(t) \\ &= \frac{1}{2j}e^{j2\pi t}e^{-t}u(t) - \frac{1}{2j}e^{-j2\pi t}e^{-t}u(t) \end{aligned}$$

$$\begin{aligned} e^{-t}u(t) &\stackrel{FT}{\longleftrightarrow} \frac{1}{1+j\omega} \\ e^{j2\pi t}s(t) &\stackrel{FT}{\longleftrightarrow} S(j(\omega - 2\pi)) \\ X(j\omega) &= \frac{1}{2j}\left[\frac{1}{1+j(\omega-2\pi)} - \frac{1}{1+j(\omega+2\pi)}\right] \end{aligned}$$

(2)

$$\begin{aligned} \frac{\sin(Wt)}{\pi t} &\stackrel{FT}{\longleftrightarrow} \begin{cases} 1 & \omega \le W \\ 0, & \text{otherwise} \end{cases} \\ s_1(t)s_2(t) &\stackrel{FT}{\longleftrightarrow} \frac{1}{2\pi}S_1(j\omega) * S_2(j\omega) \end{aligned}$$

$$X(j\omega) = \begin{cases} 5 - \frac{|\omega|}{\pi} & \pi < |\omega| \le 5\pi \\ 4 & |\omega| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

(3)

Since $\frac{1}{(1+j\omega)^2} \stackrel{F.T.}{\longleftrightarrow} te^{-t}u(t)$ and $j\omega S(\omega) \stackrel{F.T.}{\longleftrightarrow} \frac{d}{dt}s(t)$

$$\therefore x(t) = \frac{d}{dt}\left[te^{-t}u(t)\right] = (1-t)e^{-t}u(t)$$