Homework No. 3 Solution

1.

<Sol.>

Natural response:

$$r-\frac{1}{2}=0 \;\Rightarrow\; r=\frac{1}{2} \;\Rightarrow\; y^{(h)}[n]=c\left(\frac{1}{2}\right)^{n}$$

$$y[-1]=3=c\left(\frac{1}{2}\right)^{-1} \;\Rightarrow\; c=\frac{3}{2} \;\Rightarrow\; y^{(h)}[n]=\frac{3}{2}\left(\frac{1}{2}\right)^{n}$$

Forced response:

$$y^{(p)}[n]=k\left(\frac{-1}{2}\right)^{n}u[n]$$

$$k\left(\frac{-1}{2}\right)^{n}-k\frac{1}{2}\left(\frac{-1}{2}\right)^{n-1}=2\left(\frac{-1}{2}\right)^{n} \;\Rightarrow\; \left(\frac{-1}{2}\right)k-k\frac{1}{2}=2\left(\frac{-1}{2}\right)\Rightarrow\; k=1.$$

$$\therefore y^{(p)}[n]=\left(\frac{-1}{2}\right)^{n}u[n]$$

$$y^{(f)}[n]=c\left(\frac{1}{2}\right)^{n}+\left(\frac{-1}{2}\right)^{n},n\geq 0.$$

Translate initial condition

$$y[n]=\frac{1}{2}y[n-1]+2x[n]$$

$$y[0]=\frac{1}{2}y[-1]+2x[0]=\frac{1}{2}\cdot 0+2=2$$

$$y[0]=2=c+1\Rightarrow\; c=1$$

$$\therefore y^{(f)}[n]=\left(\frac{1}{2}\right)^{n}+\left(\frac{-1}{2}\right)^{n},\text{n}\geq 0.$$

2.

<Sol.>

1. Homogeneous solution

$$r^{2}-\frac{1}{4}r-\frac{1}{8}=0\Rightarrow r=\frac{1}{2},-\frac{1}{4}$$

$$y^{h}[n]=c_{1}(\frac{1}{2})^{n}+c_{2}(-\frac{1}{4})^{n}$$

2. Particular solution

(a) $x[n]=nu[n]$

$$y^p[n] = (p_1 n + p_2)u[n]$$

$$p_1 n + p_2 - \frac{1}{4}(p_1(n-1) + p_2) - \frac{1}{8}(p_1(n-2) + p_2) = n + n - 1 \Rightarrow p_1 = \frac{16}{5}, p_2 = -\frac{104}{25}$$

$$y[n] = c_1(\frac{1}{2})^n + c_2(-\frac{1}{4})^n + (\frac{16}{5}n - \frac{104}{25})u[n]$$

From $y[-1] = 1,\ y[-2] = 0$

$$\Rightarrow \begin{cases} \text{y}[0] - \frac{1}{4}y[-1] - \frac{1}{8}y[-2] = 0 \ \Rightarrow y[0] = \frac{1}{4} \\ y[1] - \frac{1}{4}y[0] - \frac{1}{8}y[-1] = 1 \ \Rightarrow \ y[1] = \frac{19}{16} \end{cases}$$

$$\Rightarrow \begin{cases} y[0] = \frac{1}{4} = c_1 + c_2 - \frac{104}{25} \\ y[1] = \frac{19}{16} = \frac{1}{2}c_1 - \frac{1}{4}c_2 + \frac{16}{5} - \frac{104}{25} \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = \frac{441}{100} \\ \frac{1}{2}c_1 - \frac{1}{4}c_2 = \frac{859}{400} \end{cases} \Rightarrow \begin{cases} c_1 = \frac{13}{3} \\ c_2 = \frac{23}{300} \end{cases}$$

$$\therefore y[n] = \frac{13}{3}(\frac{1}{2})^n + \frac{23}{300}(-\frac{1}{4})^n + (\frac{16}{5}n - \frac{104}{25})u[n]$$

(b) $x[n] = (\frac{1}{8})^n u[n]$

$$y_p[n] = p(\frac{1}{8})^n u[n]$$

$$p(\frac{1}{8})^n - \frac{1}{4}p(\frac{1}{8})^{n-1} - \frac{1}{8}p(\frac{1}{8})^{n-2} = (\frac{1}{8})^n + (\frac{1}{8})^{n-1} \Rightarrow p = -1$$

$$\therefore y_p[n] = -(\frac{1}{8})^n u[n]$$

$$y[n] = c_1(\frac{1}{2})^n + c_2(-\frac{1}{4})^n - (\frac{1}{8})^n u[n]$$

From $y[-1] = 1,\ y[-2] = 0$

$$\Rightarrow \text{y}[0] = \frac{5}{4},\ \text{y}[1] = \frac{25}{16}$$

$$\Rightarrow \begin{cases} y[0] = c_1 + c_2 - 1 = \frac{5}{4} \\ y[1] = \frac{1}{2}c_1 - \frac{1}{4}c_2 - \frac{1}{8} = \frac{25}{16} \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = \frac{9}{4} \\ \frac{1}{2}c_1 - \frac{1}{4}c_2 = \frac{27}{16} \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -\frac{3}{4} \end{cases}$$

$$y[n] = 3(\frac{1}{2})^n - \frac{3}{4}(-\frac{1}{4})^n - (\frac{1}{8})^n u[n]$$

There exists another solution approach which can be calculated as follows:

<Another Solution>:

$$y_p[n] = p(\frac{1}{8})^n, n \geq 1$$

$$p(\frac{1}{8})^n - \frac{1}{4}p(\frac{1}{8})^{n-1} - \frac{1}{8}p(\frac{1}{8})^{n-2} = (\frac{1}{8})^n + (\frac{1}{8})^{n-1} \Rightarrow p = -1$$

$$\therefore y_p[n] = -(\frac{1}{8})^n u[n-1]$$

$$y[n] = c_1(\frac{1}{2})^n + c_2(-\frac{1}{4})^n - (\frac{1}{8})^n u[n-1]$$

From $y[-1] = 1,\ y[-2] = 0, \mathrm{y}[0] = \frac{5}{4}$

$$\Rightarrow \mathrm{y}[1] = \frac{25}{16},\ \mathrm{y}[2] = \frac{11}{16}$$

$$\Rightarrow \begin{cases} y[1] = \frac{1}{2}c_1 - \frac{1}{4}c_2 - \frac{1}{8} = \frac{25}{16} \\ y[2] = \frac{1}{4}c_1 + \frac{1}{16}c_2 - \frac{1}{64} = \frac{11}{16} \end{cases} \Rightarrow \begin{cases} 8c_1 - 4c_2 = 27 \\ 16c_1 + 4c_2 = 45 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -\frac{3}{4} \end{cases}$$

p.s. subproblems (a) and (c) can be also solved by this approach .

(c) $x[n] = e^{j\frac{\pi}{4}n} u[n]$

$$y^p[n] = pe^{j\frac{\pi}{4}n} u[n]$$

$$pe^{j\frac{\pi}{4}n} - \frac{1}{4}pe^{j\frac{\pi}{4}(n-1)} - \frac{1}{8}pe^{j\frac{\pi}{4}(n-2)} = e^{j\frac{\pi}{4}n} + e^{j\frac{\pi}{4}(n-1)}$$

$$p = \frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}}$$

$$y^p[n] = -\frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}} e^{j\frac{\pi}{4}n} u[n]$$

$$y[n]=c_1(\frac{1}{2})^n+c_2(-\frac{1}{4})^n-\frac{1+e^{-j\frac{\pi}{4}}}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}}e^{j\frac{\pi}{4}n}u[n]$$

From $y[-1]=1,\ y[-2]=0$

$$\Rightarrow y[0]=\frac{5}{4},\ y[1]=\frac{23}{16}+e^{j\frac{\pi}{4}},\ \text{we set K}=1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}$$

$$\Rightarrow\begin{cases} y[0]=\frac{5}{4}=c_1+c_2-\left(1+e^{-j\frac{\pi}{4}}\right)\text{K}^{-1} \\ y[1]=\frac{23}{16}+e^{j\frac{\pi}{4}}=\frac{1}{2}c_1-\frac{1}{4}c_2-\left(1+e^{j\frac{\pi}{4}}\right)\text{K}^{-1}\end{cases}$$

$$\Rightarrow\begin{cases} c_1=\frac{7}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}+\left(\frac{5}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}+\frac{1}{3}e^{-j\frac{\pi}{4}}\right)\text{K}^{-1} \\ c_2=-\frac{13}{12}-\frac{4}{3}e^{j\frac{\pi}{4}}-\left(\frac{2}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}-\frac{2}{3}e^{-j\frac{\pi}{4}}\right)\text{K}^{-1}\end{cases}$$

3.

<Sol.>

1. Homogeneous solution

$$r^2+4=0\Rightarrow r=\pm j2$$
$$y^h(t)=c1e^{j2t}+c2e^{-j2t}$$

2. Particular solution

(a) (5%) $x(t)=t$

$$y^p(t)=p_1t+p_2$$

$$4p_1t+4p_2=3\Rightarrow p_1=0,p_2=\frac{3}{4}$$

$$\therefore y^p(t)=\frac{3}{4}$$

$$\therefore y(t)=y^h(t)+y^p(t)=c_1e^{j2t}+c_2e^{-j2t}+\frac{3}{4}$$

$$\therefore y(t)=b_1\sin(2t)+b_2\cos(2t)+\frac{3}{4}$$

From $y(0^-)=-1,\frac{d}{dt}y(t)|_{t=0^-}=1$

$$\text{We get } \Rightarrow b_1 = \frac{1}{2}, b_2 = -\frac{7}{4}$$

$$y(t) = -\frac{7}{4}\cos(2t) + \frac{1}{2}\sin(2t) + \frac{3}{4}$$

(b) (5%) $x(t) = e^{-t}$

$$y^p(t) = pe^{-t}$$

$$pe^{-t} + 4pe^{-t} = -3e^{-t} \Rightarrow p = -\frac{3}{5}$$

$$\therefore y^p(t) = -\frac{3}{5}e^{-t}$$

$$y(t) = y^h(t) + y^p(t) = c_1 e^{j2t} + c_2 e^{-j2t} - \frac{3}{5}e^{-t}$$

$$\therefore y(t) = b_1 \sin(2t) + b_2 \cos(2t) - \frac{3}{5}e^{-t}$$

$$\text{From } y(0^-) = -1, \frac{d}{dt}y(t)|_{t=0^-} = 1 \Rightarrow b_1 = \frac{1}{5}, b_2 = -\frac{2}{5}$$

$$\therefore y(t) = -\frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t) - \frac{3}{5}e^{-t}$$

(c) (10%) $x(t) = \sin(t) + \cos(t)$

$$y^p(t) = p_1 \cos(t) + p_2 \sin(t)$$

$$y^{'p}(t) = -p_1 \sin(t) + p_2 \cos(t)\ , y^{''p}(t) = -p_1 \cos(t) - p_2 \sin(t)$$

$$\text{We get } p_1 = 1, p_2 = -1$$

$$\therefore y^p(t) = \cos(t) - \sin(t)$$

$$\therefore y(t) = b1\cos(2t) + b2\sin(2t) + \cos(t) - \sin(t)$$

$$\text{From } y(0^-) = -1, \frac{d}{dt}y(t)|_{t=0^-} = 1 \Rightarrow b_1 = -2, b_2 = 1$$

$$\therefore y(t) = -2\cos(2t) + \sin(2t) + \cos(t) - \sin(t)$$

4.

<Sol.>

$$x[n] = u[n] \Rightarrow y[n] = s[n]$$

(a) Homogeneous solution: $r^2 - 5r + 6 = 0 \Rightarrow r = 2,\ 3.$

Hence, $y^{(h)}(n) = c_1(2)^n + c_2(3)^n$.

Particular solution: Set $y^{(p)}[n] = Au[n]$.

$$\because\ y^{(p)}[n] - 5y^{(p)}[n-1] + 6y^{(p)}[n-2] = u[n] + u[n-1]$$

$$A - 5A + 6A = 1 + 1 = 2 \Rightarrow A = 1 \ \Rightarrow\ \therefore \mathrm{y}^{(p)}[n] = u[n].$$

Complete solution:

$$y[n] = y^{(h)}[n] + y^{(p)}[n] = c_1(2)^n + c_2(3)^n + u[n].$$

$$y[-1] = y[2] = 0 \Rightarrow\ \mathrm{y}[0] = 1,\ \mathrm{y}[1] = 7.$$

$$\begin{cases} c_1 + c_2 = 0 \\ 2c_1 + 3c_2 = 6 \end{cases} \Rightarrow\ \mathrm{c}_1 = 6,\ \mathrm{c}_2 = -6.$$

$$\therefore y[n] = y^{(h)}[n] + y^{(p)}[n] = -6(2)^n + 6(3)^n + u[n].$$

(b) Natural response:

$$y^{(n)}[n] = c_3(2)^n + c_4(3)^n,\ \mathrm{y}[-1] = y[-2] = 0.$$

$$c_3 = 0,\ c_4 = 0 \Rightarrow y^{(n)}[n] = 0.$$

(c) Forced response:

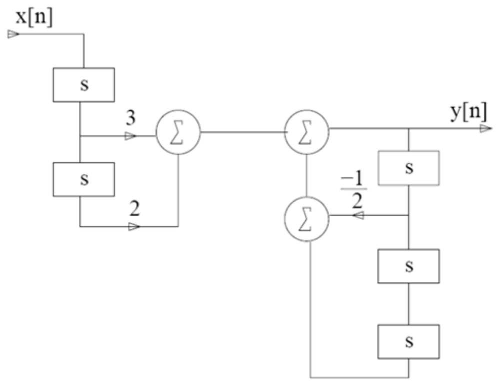
$$y^{(f)}[n] = c_1(2)^n + c_2(3)^n + u[n],\ \mathrm{y}[-1] = y[-2] = 0.$$

$$\therefore\ y^{(f)}[n] = -6(2)^n + 6(3)^n + u[n].$$

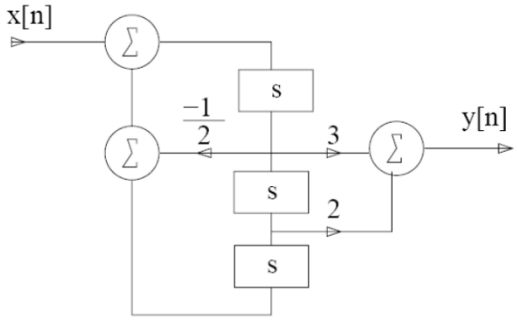
5.

<Sol.>

(a)

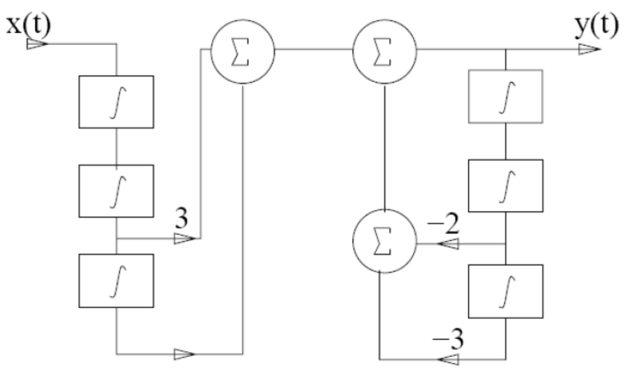


Direct form I



Direct form II

(b)



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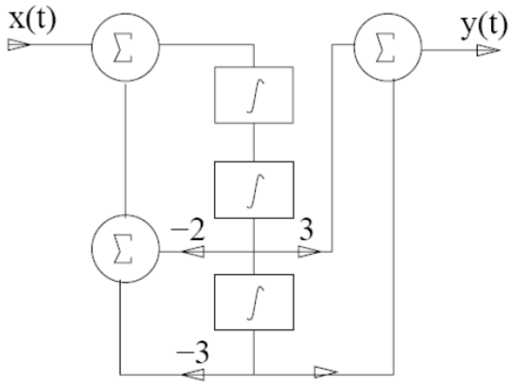

Direct form II