Chapter 7 The z-Transform

7-1 Definition of the z-Transform

The z-transform of a sequence $x[n]$ is defined as

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad \begin{cases} x[n] \overset{z}{\longleftrightarrow} X(z) \\ Z\{x[n]\} = X(z) \end{cases} \tag{7.1}$$

where z is a complex variable.

$x[n] = z^n$ is an eigenfunction of discrete-time LTI systems

$$\begin{aligned} y[n] &= h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} \\ &= z^n H(z) = H(z) z^n \end{aligned} \tag{7.2}$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k} \;\cdots\cdots \text{eigenvalue} \tag{7.3}$$

1. $z = e^{j\Omega} \Rightarrow X(z)\big|_{z=e^{j\Omega}}$ is the discrete-time Fourier transform of $x[n]$.

   Note:

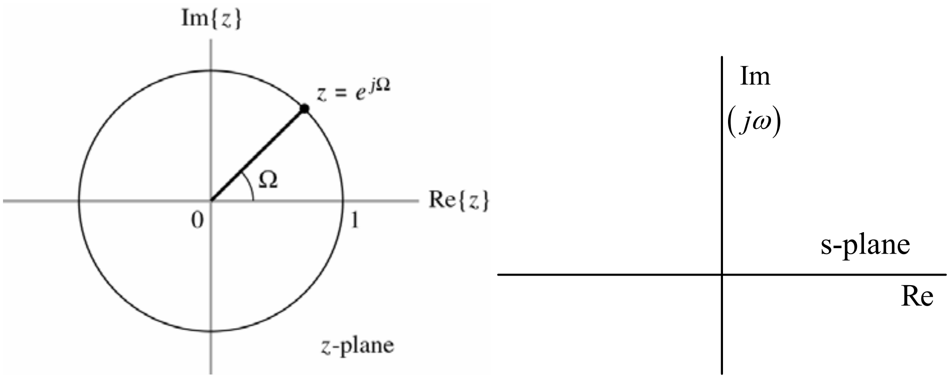
   - Bilateral z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ (7.4)
   - Unilateral z-transform: $X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$ (7.5)

2. The z-transform of $x[n]$ can be interpreted as the Fourier transform of $x[n]$ after multiplication by a real exponential r^{-n}. $\left(z = re^{j\Omega}\right)$

$$\begin{aligned} X(z) = X\left(re^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\Omega}\right)^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left\{x[n] r^{-n}\right\} e^{-j\Omega n} \\ &= \mathcal{F}\left\{x[n] r^{-n}\right\} \end{aligned} \tag{7.6}$$

Note:

- The z-transform reduces to the Fourier transform when the magnitude of transform variable z is unity (i.e., for $z = e^{j\Omega}$).
- The Laplace-transform reduces to the Fourier transform when the real part of the transform variable is zero (i.e., for $s = j\omega$).



(a) (b)

■**Figure 7.1** The z-plane and s-plane. (a) z-plane. (b) s-plane.

Note:

- The unit circle of z-plane and the $j\omega$-axis of s-plane play a similar role.
- For convergence of the z-transform, we require that the Fourier transform of $x[n]r^{-n}$ converges.
- The range of values for which the z-transform exists is referred to as the region of convergence (ROC) of the z-transform.
- If the ROC includes the unit circle, then the Fourier transform also converges.

***Example* 7.1**: The z-transform and ROC

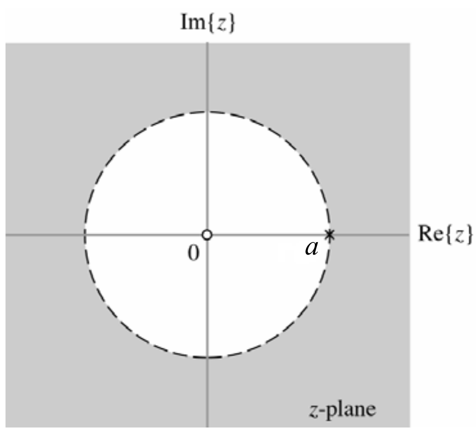
$$x[n] = a^n u[n],\ a > 0$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

For convergence of $X(z)$, we have $\left|az^{-1}\right| < 1 \quad \left(|z| > |a|\right)$

$$\Rightarrow X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

pole-zero plot and ROC:



The Fourier transform of $x[n]$ converges only if $|a| < 1$. ■

***Example* 7.2**: The z-transform and ROC

$$x[n] = -a^n u[-n-1],\ a > 0$$

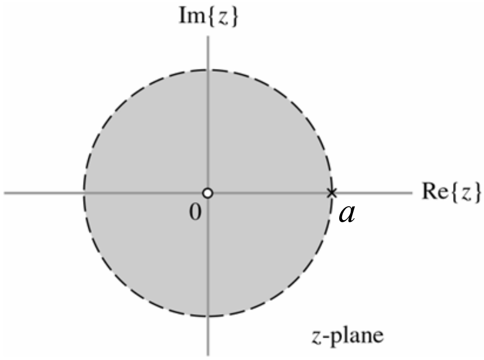
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n$$

$X(z)$ converges if $\left|a^{-1}z\right| < 1 \quad \left(|z| < a\right)$

$$\Rightarrow X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < a$$

pole-zero plot and ROC:



■

***Example* 7.3**: The z-transform and ROC

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n] \right\} z^{-n}$$

$$= \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n}_{} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^n}_{}$$

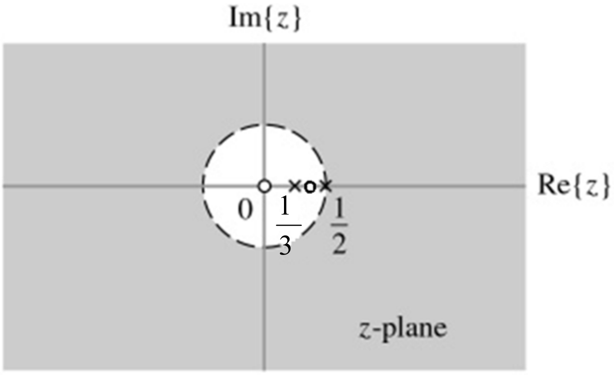
converges if $\left|\frac{1}{2} z^{-1}\right| < 1$ $\left(\text{i.e. } |z| > \frac{1}{2}\right)$; converges if $\left|\frac{1}{3} z^{-1}\right| < 1$ $\left(\text{i.e. } |z| > \frac{1}{3}\right)$

$$\Rightarrow X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{2 - \frac{5}{6} z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$= \frac{z\left(2z - \frac{5}{6}\right)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}, \qquad |z| > \frac{1}{2}$$

pole-zero plot and ROC:



■

***Example* 7.4**: The z-transform

$$h[n] = \frac{r^n \sin\left[(n+1)\Omega\right]}{\sin\Omega} u[n], \quad 0 < \mathrm{r} < 1$$

$$H(z) = \sum_{n=0}^{\infty} \frac{r^n \sin\left[(n+1)\Omega\right]}{\sin\Omega} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{r^n z^{-n}}{2j\sin\Omega}\left(e^{j(n+1)\Omega} - e^{-j(n+1)\Omega}\right)$$

$$= \sum_{n=0}^{\infty}\left[\left(rz^{-1}e^{j\Omega}\right)^n \frac{e^{j\Omega}}{2j\sin\Omega}\right] - \sum_{n=0}^{\infty}\left[\left(rz^{-1}e^{-j\Omega}\right)^n \frac{e^{-j\Omega}}{2j\sin\Omega}\right]$$

Both series converge if the ROC is given by

$$\left|rz^{-1}e^{j\Omega}\right| = \left|rz^{-1}e^{-j\Omega}\right| < 1$$

$$\left(\text{i.e. } |z| > r\right)$$

$$\Rightarrow H(z) = \frac{1}{2j\sin\Omega}\left(\frac{e^{j\Omega}}{1 - rz^{-1}e^{j\Omega}} - \frac{e^{-j\Omega}}{1 - rz^{-1}e^{-j\Omega}}\right)$$

$$= \frac{1}{\left(1 - rz^{-1}e^{j\Omega}\right)\left(1 - rz^{-1}e^{-j\Omega}\right)}$$

$$= \frac{z^2}{z^2 - 2r(\cos\Omega)z + r^2}$$

poles: $z = re^{j\Omega}$ zeros: $z = 0$

$z = re^{-j\Omega}$ $z = 0$

■

7-2 The Region of Convergence for the z-transform

Properties of the ROC for the z-transform:

1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

$$X\left(re^{j\Omega}\right) = \mathcal{F}\left\{x[n]r^{-n}\right\} \tag{7.7}$$

The convergence of $X(z)$ is dependent only on $r = |z|$ but not on Ω.

2. The ROC does not contain any poles.
3. If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$.

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n} \tag{7.8}$$

$\Rightarrow$ The z-transform is the sum of a finite number of terms.

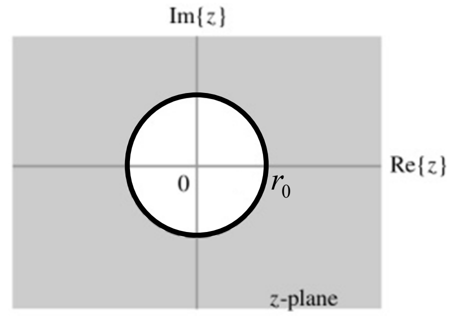
$\Rightarrow X(z)$ will converge for z not equal to zero or infinity.

(1) $N_1 \geq 0$ (only negative powers of z) $\Rightarrow z = \infty$ is included in the ROC, and $z = 0$ is not included in the ROC.

(2) $N_1 < 0$ and $N_2 > 0 \Rightarrow z = 0$ and $z = \infty$ are not included in the ROC.

(3) $N_2 \leq 0$ (only positive powers of z) $\Rightarrow z = 0$ is included in the ROC, and $z = \infty$ is not included in the ROC.

4. If $x[n]$ is a right-sided sequence and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| \geq r_0$ will also be in the ROC.



■**Figure 7.2** The ROC of right-sided sequence

$$X(z) = \sum_{n=N_1}^{\infty} x[n]z^{-n} \tag{7.9}$$

When $N_1 < 0$, the summation includes terms with positive powers of z which become unbounded as $|z| \to \infty$. Consequently, for right-sided sequences, in general, the ROC will not include infinity.

Suppose the z-transform of $x[n]$ converges for some value of $r = r_0$, i.e., $|z| = r_0$ is in the ROC. Then

$$\sum_{n=-\infty}^{\infty}|x[n]|r_0^{-n}<\infty \Rightarrow \sum_{n=N_1}^{\infty}|x[n]|r_0^{-n}<\infty \tag{7.10}$$

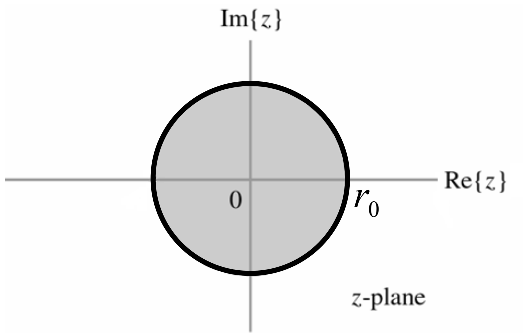
For $r_1 \geq r_0$

$$\Rightarrow \sum_{n=N_1}^{\infty}|x[n]|r_1^{-n}=\sum_{n=N_1}^{\infty}|x[n]|r_0^{-n}\left(\frac{r_1}{r_0}\right)^{-n} \leq \left(\frac{r_1}{r_0}\right)^{-N_1}\cdot\sum_{n=N_1}^{\infty}|x[n]|r_0^{-n}<\infty \tag{7.11}$$

$\left(\because \text{ The maximum value of } \left(\frac{r_1}{r_0}\right)^{-n} \text{ in the summation is } \left(\frac{r_1}{r_0}\right)^{-N_1}\right)$

$\Rightarrow$ The z-plane for $|z| \geq r_0$ is in the ROC.

5. If $x[n]$ is a left-sided sequence and if the circle $|z|=r_0$ is in the ROC, then all values of z for which $0<|z|\leq r_0$ will also be in the ROC.



■**Figure 7.3** The ROC of left-sided sequence

$$X(z)=\sum_{n=-\infty}^{N_2}x[n]z^{-n} \tag{7.12}$$

When $N_2>0$, the summation includes terms with negative powers of z which become unbound when $|z|\to 0$. Consequently, for left-sided sequences, in general, the ROC will not include $z=0$.

Note: When $N_2<0$, ROC will include $z=0$.

Suppose $|z|=r_0$ is in the ROC, then

$$\sum_{n=-\infty}^{N_2}|x[n]|r_0^{-n}<\infty \tag{7.13}$$

For $r_1 \leq r_0$

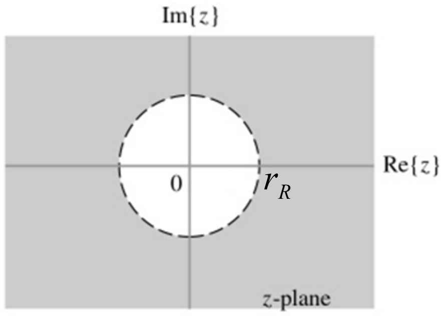
$$\sum_{n=-\infty}^{N_2}|x[n]|r_1^{-n}=\sum_{n=-\infty}^{N_2}|x[n]|r_0^{-n}\left(\frac{r_1}{r_0}\right)^{-n} \leq \left(\frac{r_1}{r_0}\right)^{-N_2}\sum_{n=-\infty}^{N_2}|x[n]|r_0^{-n}<\infty \tag{7.14}$$

$\left(\because \text{ The maximum value of } \left(\frac{r_1}{r_0}\right)^{-n} \text{ in the summation is } \left(\frac{r_1}{r_0}\right)^{-N_2}\right)$

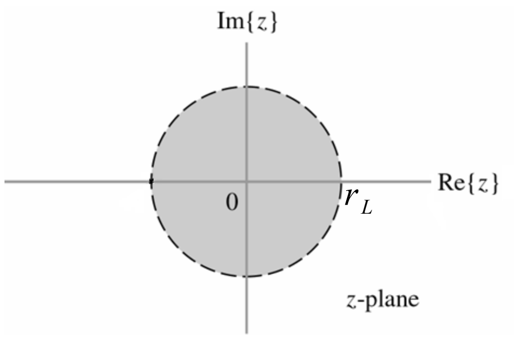
6. If $x[n]$ is a two-sided and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane which includes the circle $|z| = r_0$.

Note:

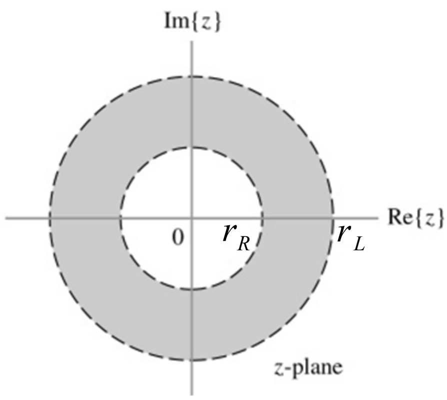
- A two-sided sequence can be expressed as a sum of a right-sided sequence and a left-sided sequence.



(ROC for the right-sided sequence)



(ROC for the left-sided sequence)

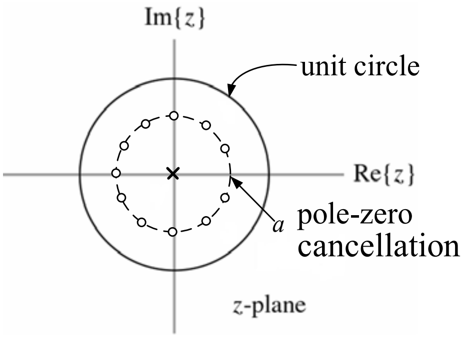


■**Figure 7.4** The ROC of two-sided sequence. (r_L must be greater than r_R; otherwise $X(z)$ does not converge.)

***Example* 7.5**: The z-transform

$$x[n]=\begin{cases} a^n \ , & 0\le n\le N-1 \ , \ a>0 \\ 0 \ , & \text{otherwise} \end{cases}$$

$$X(z)=\sum_{n=0}^{N-1}a^n z^{-n}=\sum_{n=0}^{N-1}\left(az^{-1}\right)^n$$
$$=\frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}=\frac{1}{z^{N-1}}\cdot\frac{z^N-a^N}{z-a}$$



The ROC includes the entire z-plane except the origin.

pole: $z=0$ $\quad$ $z=a$
zero: $z^N-a^N=0$ $\quad$ $z=a$ $\Big\}$ pole-zero cancellation

$z=ae^{j(2\pi k/N)}$, $\quad k=1,2,\cdots,N-1$

■

***Example* 7.6**: The z-transform

$$x[n]=b^{|n|} \ , \quad b>0$$
$$\Rightarrow x[n]=b^n u[n]+b^{-n}u[-n-1]$$

$$b^n u[n]\overset{z}{\longleftrightarrow}\frac{1}{1-bz^{-1}} \ , \quad |z|>b \tag{7.15}$$

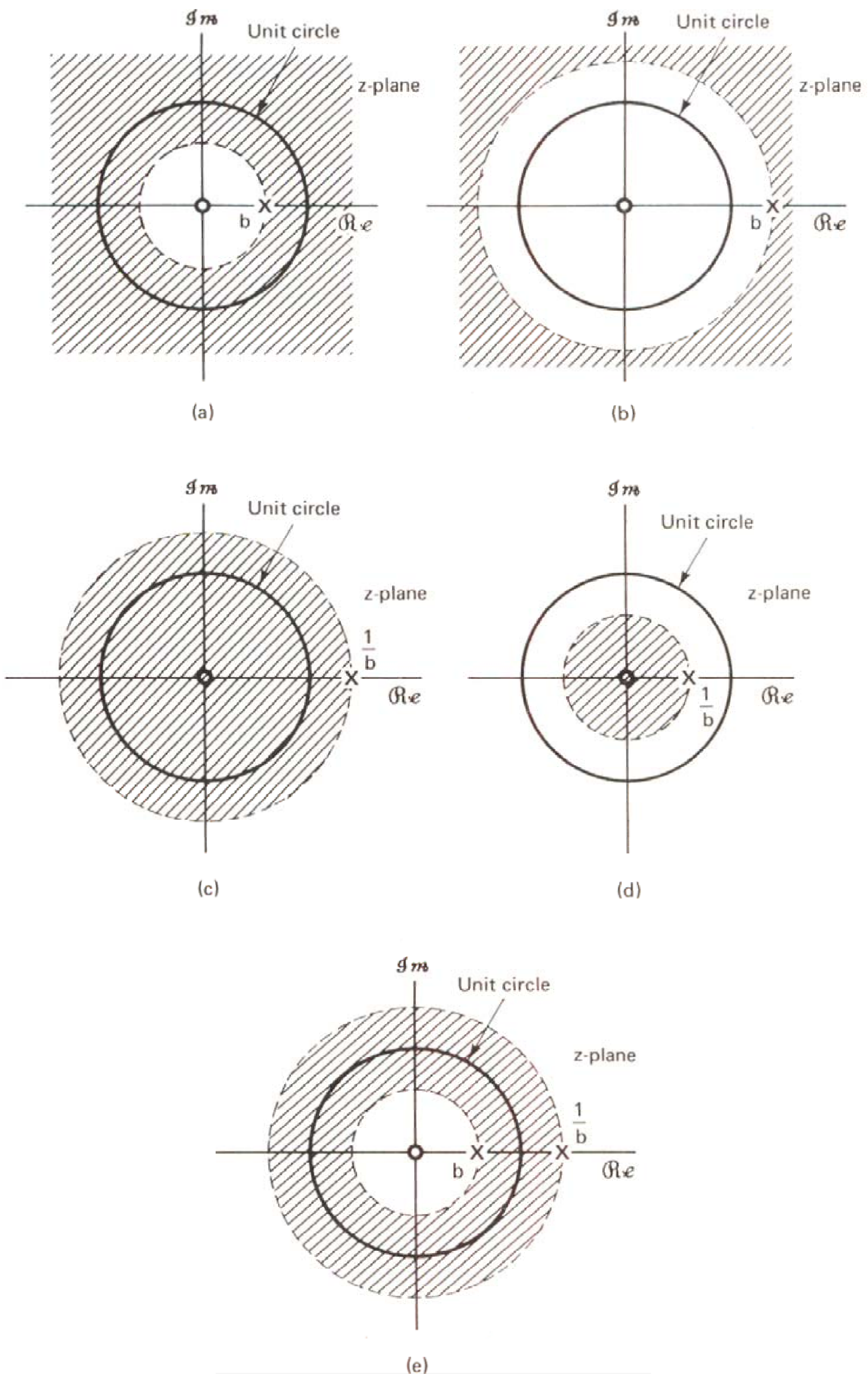
$$b^{-n}u[-n-1]\overset{z}{\longleftrightarrow}\frac{-1}{1-b^{-1}z^{-1}} \ , \quad |z|<\frac{1}{b} \tag{7.16}$$

For $b<1$:

$$\Rightarrow X(z)=\frac{1}{1-bz^{-1}}+\frac{-1}{1-b^{-1}z^{-1}} \ , \quad b<|z|<\frac{1}{b} \tag{7.17}$$

$$=\frac{b^2-1}{b}\cdot\frac{z}{(z-b)\left(z-b^{-1}\right)} \tag{7.18}$$

For $b > 1$: There is no common ROC and thus the sequence $x[n]$ has no z-transform.



■**Figure 7.5** Pole-zero plots and ROCs for Example 7.6. (a) Eq. (7.15) for $|b| < 1$. (b) Eq. (7.15) for $|b| > 1$. (c) Eq. (7.16) for $|b| < 1$. (d) Eq. (7.16) for $|b| > 1$. (e) pole-zero plot and ROC for Eq. (7.18) with $|b| < 1$.

■

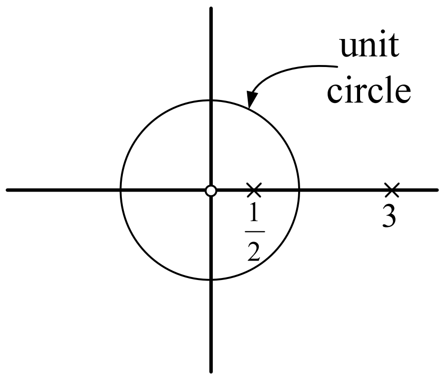
Note: For any rational z-transform

- The ROC will be bounded by poles or will extend to infinity.
- For a right-sided sequence, the ROC is bounded on the inside by the pole with the largest magnitude and on the outside by infinity.
- For a left-sided sequence, the ROC is bounded on the outside by the pole with the smallest magnitude and on the inside by zero.

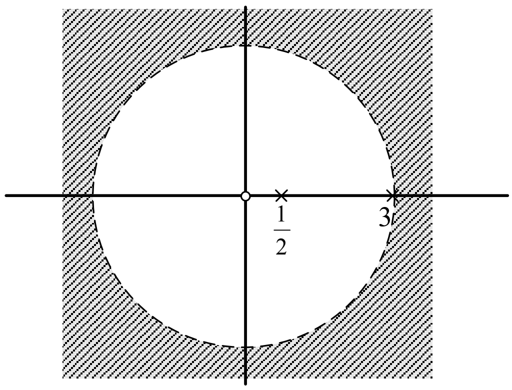
***Example* 7.7**: Characteristics of ROC

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right)}$$

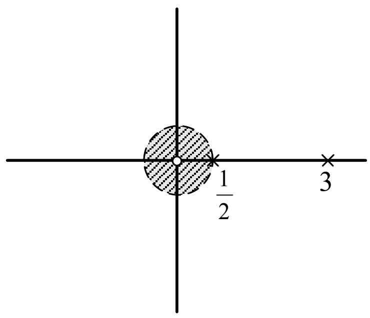
pole-zero plot:



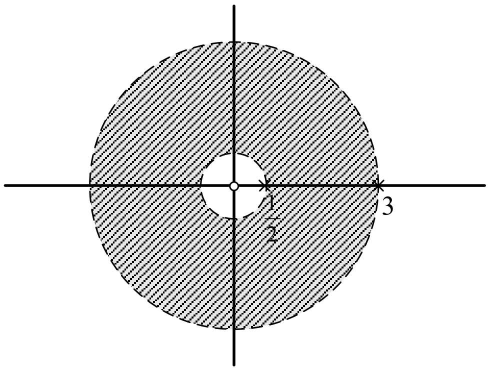
pole-zero plot and ROC if $x[n]$ is right-sided:



pole-zero plot and ROC if $x[n]$ is left-sided:



pole-zero plot and ROC if $x[n]$ is two-sided:



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7-3 The Inverse z-transform

Very often, we will be able to analyze or design discrete-time signals and systems using their z transforms without having to convert the transforms back to the corresponding sequences. However, such conversion is sometimes desired or necessary and is accomplished via the ***inverse z transform***. The formal definition of the inverse z transform is simple in concept, but somewhat cumbersome to use; and for rational transforms, in particular, we will obtain simpler methods to invert the z transform.

$$x[n] = Z^{-1}\{X(z)\}$$

1.

$$X\left(re^{j\Omega}\right) = \mathcal{F}\left\{x[n]r^{-n}\right\}$$

$$\Rightarrow x[n]r^{-n} = \mathcal{F}^{-1}\left\{X\left(re^{j\Omega}\right)\right\}$$

$$\Rightarrow x[n] = r^{n}\cdot\mathcal{F}^{-1}\left\{X\left(re^{j\Omega}\right)\right\}$$

$$\Rightarrow x[n] = r^{n}\cdot\frac{1}{2\pi}\int_{2\pi}X\left(re^{j\Omega}\right)e^{j\Omega n}d\Omega$$

$$=\frac{1}{2\pi}\int_{2\pi}X\left(re^{j\Omega}\right)\left(re^{j\Omega}\right)^{n}d\Omega$$

$$=\frac{1}{2\pi j}\oint_{2\pi}X(z)z^{n-1}dz \qquad (7.19)$$

$$z = re^{j\Omega} \text{ and } r \text{ fixed}$$

$$\Rightarrow dz = jre^{j\Omega}d\Omega = jzd\Omega$$

$$\Rightarrow d\Omega = \frac{1}{j}z^{-1}dz$$

$\oint$: denotes a counterclockwise closed circular contour centered at the origin and with radius r.

The value of r can be chosen as any value for which $X(z)$ converges.

***Example* 7.8**: The inverse z-transform

$$X(z)=\frac{1}{1-az^{-1}} \quad , \quad |z|>a$$

$$\Rightarrow X(z)=1+az^{-1}+a^2z^{-2}+\cdots\cdots=\sum_{n=0}^{\infty}a^nz^{-n}=\sum_{n=-\infty}^{\infty}a^nu[n]z^{-n}$$

$$\Rightarrow x[n]=a^nu[n]$$

■

***Example* 7.9**: The inverse z-transform

$$X(z)=\log\left(1+az^{-1}\right) \quad , \quad |z|>|a| \text{ or } \left|az^{-1}\right|<1$$

Using the Taylor's series expansion for $\log(1+\omega)$, $|\omega|<1$, we have

$$\log(1+\omega)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\omega^n}{n} \quad , \quad |\omega|<1$$

$$\Rightarrow X(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}a^nz^{-n}}{n}$$

$$=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

$$\Rightarrow x[n]=\begin{cases}\dfrac{(-1)^{n+1}a^n}{n} & , \quad n\geq 1\\ 0 & , \quad n\leq 0\end{cases}$$

$$\Rightarrow x[n]=\frac{-(-a)^n}{n}u[n-1]$$

■

***Example* 7.10**: Consider a right-sided sequence with z-transform

$$X(z)=\frac{1}{\left(1-az^{-1}\right)\left(1-bz^{-1}\right)} \quad , \quad a\neq b$$

$$\Rightarrow X(z)=\frac{z^2}{(z-a)(z-b)}=\frac{a^{-1}b^{-1}}{\left(z^{-1}-a^{-1}\right)\left(z^{-1}-b^{-1}\right)}$$

$$=\frac{1}{b-a}\cdot\frac{1}{z^{-1}-a^{-1}}+\frac{1}{a-b}\cdot\frac{1}{z^{-1}-b^{-1}}$$

$$=\frac{a}{a-b}\cdot\frac{1}{1-az^{-1}}+\frac{b}{b-a}\cdot\frac{1}{1-bz^{-1}}$$

$$\Rightarrow x[n]=\frac{a}{a-b}a^nu[n]+\frac{b}{b-a}b^nu[n]$$

■

2.

The basis of the inverse z transform is the Cauchy Integral Theorem from the theory of complex variables, which states that

$$\frac{1}{2\pi j}\oint_{\Gamma} z^{k-1}dz = \begin{cases} 1, & k=0 \\ 0, & k\neq 0 \end{cases}, \tag{7.20}$$

where Γ is a counterclockwise contour of integration enclosing the origin. Therefore, to find $x[n]$ from $X(z)$, we multiply both sides of Eq. (7.1) by $\frac{z^{k-1}}{2\pi j}$, and integrate along a suitable Γ in R to obtain

$$\begin{aligned}\frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{k-1}dz &= \frac{1}{2\pi j}\oint_{\Gamma}\sum_{n=-\infty}^{\infty} x[n]z^{-n+k-1}dz \\ &= \sum_{n=-\infty}^{\infty} x[n]\frac{1}{2\pi j}\oint_{\Gamma} z^{-n+k-1}dz = x[k].\end{aligned}$$

Thus, the inverse z transform is given by

$$x[n] = \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz, \tag{7.21}$$

where Γ is a counterclockwise contour in the region of convergence of $X(z)$ enclosing the origin. We know that a suitable Γ enclosing the origin can always be found since R is an annular ring centered on the origin.

3.

Computation of the inverse z-transform:

(1) Cauchy integration

In the usual case where $X(z)$ is a rational function of z, the Cauchy Residue Theorem states that Eq. (7.21) can be evaluated by

$$x[n] = \sum_i \rho_i\ , \tag{7.22}$$

where the ρ_i are the residues of $X(z)z^{n-1}$ at the poles inside Γ. To show the k poles at $z = p_i$ explicitly, we write

$$X(z)z^{n-1} = \frac{\Phi_i(z)}{(z-p_i)^k} \tag{7.23}$$

and the residue at p_i is then given by

$$\rho_i = \frac{1}{(k-1)!}\left.\frac{d^{k-1}\Phi_i(z)}{dz^{k-1}}\right|_{z=p_i}. \tag{7.24}$$

Very often, $k=1$, in which case Eq. (7.24) becomes simply

$$\rho_i = \Phi_i(p_i). \tag{7.25}$$

***Example* 7.11**: Consider the z transform

$$X(z) = \frac{z}{z-a} \quad , \qquad |z| > |a|.$$

The function $X(z)z^{n-1} = \dfrac{z^n}{z-a}$ has poles at $z = a$ and, for $n < 0$, at $z = 0$. Any Γ in the region of convergence $|z| > |a|$ will enclose all of these poles. Thus, for $n \geq 0$, we have only the residue $\rho_1 = z^n\big|_{z=a} = a^n$, $n \geq 0$.

For $n = -1$, there are residues at both $z = a$ and $z = 0$ given by

$$\rho_1 = z^{-1}\Big|_{z=a} = a^{-1} \quad \text{and} \quad \rho_2 = \frac{1}{z-a}\bigg|_{z=0} = -a^{-1}$$

and, therefore $x[-1] = \rho_1 + \rho_2 = 0$.

For all $n < -1$, we must use the general form of (7.24) to obtain the residues, and the reader can verify that $x[n] = 0$, $n \leq -1$. Thus, we have determined that

$$x[n] = a^n u[n],$$

which checks with our previous derivation of the particular $X(z)$.

■

(2) Long division:

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Starting with the lowest powers of z^{-1}, we divided $N(z)$ by $D(z)$ to expand $X(z)$ in power series form.

***Example* 7.12**:

$$X(z) = \frac{1 + 2z^{-1}}{1 + 2z^{-1} + 4z^{-2}}$$

$$
\begin{array}{r|l}
 & 1-4z^{-2}+8z^{-3}-32z^{-5} \\
1+2z^{-1}+4z^{-2} & 1+2z^{-1} \\
 & \underline{1+2z^{-1}+4z^{-2}} \\
 & \quad -4z^{-2} \\
 & \underline{\quad -4z^{-2}-8z^{-3}-16z^{-4}} \\
 & \qquad 8z^{-3}+16z^{-4} \\
 & \underline{\qquad 8z^{-3}+16z^{-4}+32z^{-5}} \\
 & \qquad\qquad -32z^{-5} \\
 & \qquad\qquad \vdots
\end{array}
$$

$$\Rightarrow x[0]=1,\ x[1]=0,\ x[2]=-4,\ x[3]=8,\ x[4]=0,\ x[5]=-32,\cdots$$

■

(3) The Cauchy product and a recurrence relation:

$$X(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_mz^{-m}}{a_0+a_1z^{-1}+a_2z^{-2}+\cdots+a_mz^{-m}}=\sum_{n=0}^{\infty}x[n]z^{-n}$$

Note:

- The numerator and denominator have the same degree. If this is not the case, we merely have some coefficients that are zero.

$$\left(\sum_{n=0}^{\infty}b_nz^{-n}\right)=\left(\sum_{n=0}^{\infty}a_nz^{-n}\right)\left(\sum_{n=0}^{\infty}x[n]z^{-n}\right)$$

where $a_n=b_n=0$ for $n>m$

Applying the Cauchy product to the right-hand side results in

$$\sum_{n=0}^{\infty}b_nz^{-n}=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}x[k]a_{n-k}\right]z^{-n}$$

$$\Rightarrow b_n=\sum_{k=0}^{n}x[k]a_{n-k}=x[n]a_0+\sum_{k=0}^{n-1}x[k]a_{n-k}$$

Assume $a_0\neq 0$

$$\Rightarrow\begin{cases}x[0]=\dfrac{b_0}{a_0}, & n=0\\[2ex] x[n]=\dfrac{1}{a_0}\left[b_n-\sum_{k=0}^{n-1}x[k]a_{n-k}\right], & n>0\end{cases}$$

***Example* 7.13**:

$$X(z)=\frac{1+2z^{-1}}{1+2z^{-1}+4z^{-2}}=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{a_0+a_1z^{-1}+a_2z^{-2}}$$

$$\begin{cases} a_0 = 1,\ a_1 = 2,\ a_2 = 4,\ a_n = 0,\ \text{for } n \geq 3 \\ b_0 = 1,\ b_1 = 2,\ b_2 = 0,\ b_n = 0,\ \text{for } n \geq 3 \end{cases}$$

$$\Rightarrow \begin{cases} x[0] = \dfrac{b_0}{a_0} = 1 \\ x[1] = b_1 - x[0]a_1 = 2 - 1 \times 2 = 0 \\ x[2] = b_2 - x[0]a_2 - x[1]a_1 = 0 - 1 \times 4 - 0 \times 2 = -4 \\ x[3] = b_3 - x[0]a_3 - x[1]a_2 - x[2]a_1 = -(-4) \times 2 = 8 \\ \qquad\vdots \\ \qquad\vdots \end{cases}$$

■

(4) Partial-fraction expansion:

$$X(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

(a) If $M < N$ and $X(z)$ has no multiple poles, it may be expanded in a partial-fraction of the form

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}, \qquad |z| > r, \quad \text{(causal)} \tag{7.26}$$

with the p_k being poles of $X(z)$. But each term in Eq. (7.26) is just the z transform of an exponential sequence, and thus the inverse z transform for $X(z)$ is given by

$$x[n] = \sum_{k=1}^{N} A_k p_k^n u[n]. \tag{7.27}$$

(b) If $M \geq N$, we divide $N(z)$ and $D(z)$ starting with the highest powers of z^{-1} to produce

$$a_N z^{-N} + \cdots + a_0 \overline{\smash{\big)}\ b_M z^{-M} + \cdots + b_1 z^{-1} + b_0}^{\ C_{M-N} z^{-M+N} + \cdots + C_1 z^{-1} + C_0} \quad + \frac{R(z)}{D(z)} \tag{7.28}$$

where the remainder polynomial $R(z)$ is of order $M' = N - 1$, or less. Then, $R(z)/D(z)$ can be expanded in a partial-fraction expansion as before and $x[n]$ is given by

$$x[n] = \sum_{i=0}^{M-N} C_i \delta[n-i] + \sum_{k=1}^{N} A'_k p_k^n u[n]. \tag{7.29}$$

(c) For the case of multiple poles, e.g., K multiple poles of p_1, $X(z)$ should be expanded as

$$X(z) = \frac{A_{11}}{1-p_1z^{-1}} + \frac{A_{12}}{\left(1-p_1z^{-1}\right)^2} + \cdots + \frac{A_{1K}}{\left(1-p_1z^{-1}\right)^K} + \frac{A_{21}}{1-p_2z^{-1}} + \cdots + \frac{A_{N1}}{1-p_Nz^{-1}} \tag{7.30}$$

***Example* 7.14**: Assume that

$$X(z) = \frac{z^{-2}+2z^{-1}+2}{z^{-1}+1}, \quad |z|>1 .$$

By long division, we obtain

$$X(z) = 2 + z^{-2} - z^{-3} + z^{-4} - z^{-5} + \cdots$$

and thus

$$x[n] = \begin{cases} 0, & n<0 \\ 2, & n=0 \\ 0, & n=1 \\ (-1)^n, & n\geq 2. \end{cases}$$

By the partial-fraction expansion method

$$X(z) = z^{-1} + 1 + \frac{1}{z^{-1}+1}, \quad |z|>1$$

and thus

$$x[n] = \delta[n-1] + \delta[n] + (-1)^n u[n],$$

which checks with our previous result.

The above techniques can also be employed even if $x[n]$ is not causal, with suitable modification. Common z transform pairs are given in table 7.1.

■

***Example* 7.15**:

$$X(z) = \frac{z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \frac{z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = \frac{4}{1-\frac{1}{2}z^{-1}} - \frac{4}{1-\frac{1}{4}z^{-1}}$$

ROC: $|z| > \frac{1}{2} \Rightarrow x[n] = 4\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]$

ROC: $\frac{1}{4} < |z| < \frac{1}{2} \Rightarrow x[n] = -4\left(\frac{1}{2}\right)^n u[-n-1] - 4\left(\frac{1}{4}\right)^n u[n]$

ROC: $|z| < \frac{1}{4} \Rightarrow x[n] = 4\left[-\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right]u[-n-1]$

■

***Example* 7.16**:

$$X(z) = \frac{3 + \frac{11}{2}z^{-1} + 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1} + 4z^{-2}\right)}$$
$$= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B + Cz^{-1}}{1 + 2z^{-1} + 4z^{-2}}$$

$$A = \left(1 - \frac{1}{2}z^{-1}\right)X(z)\Big|_{z^{-1}=2} = \frac{3 + \frac{1}{2}z^{-1} + 7z^{-2}}{1 + 2z^{-1} + 4z^{-2}}\Bigg|_{z^{-1}=2} = 2$$

$$\Rightarrow X(z) - \frac{2}{1 - \frac{1}{2}z^{-1}} = \frac{B + Cz^{-1}}{1 + 2z^{-1} + 4z^{-2}}$$

$$\frac{1 + \frac{3}{2}z^{-1} - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1} + 4z^{-2}\right)} = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1} + 4z^{-2}\right)}$$
$$= \frac{1 + 2z^{-1}}{1 + 2z^{-1} + 4z^{-2}} = \frac{B + Cz^{-1}}{1 + 2z^{-1} + 4z^{-2}}$$

$$\Rightarrow X(z) = \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{1 + 2z^{-1}}{1 + 2z^{-1} + 4z^{-2}}$$

$$\Rightarrow x[n] = \left[2\left(\frac{1}{2}\right)^n + 2^n\left(\cos\left(\frac{2\pi}{3}n\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi}{3}n\right)\right)\right]u[n]$$

Alternate solution:

$$X(z)=\frac{A}{1-\frac{1}{2}z^{-1}}+\frac{B}{1-pz^{-1}}+\frac{C}{1-p^{*}z^{-1}},\quad \begin{matrix} p=-1+\sqrt{3}i \\ p^{*}=-1-\sqrt{3}i \end{matrix}$$

■

***Example* 7.17**:

$$X(z)=\frac{4-8z^{-1}+6z^{-2}}{\left(1-2z^{-1}\right)^{2}\left(1+z^{-1}\right)}$$

$$=\frac{A}{1-2z^{-1}}+\frac{B}{\left(1-2z^{-1}\right)^{2}}+\frac{C}{1+z^{-1}}$$

$$B=\left(1-2z^{-1}\right)^{2}X(z)\Big|_{z^{-1}=\frac{1}{2}}=1$$

$$C=\left(1+z^{-1}\right)X(z)\Big|_{z^{-1}=-1}=2$$

$$X(z)\left(1-2z^{-1}\right)^{2}=\frac{4-8z^{-1}+6z^{-2}}{1+z^{-1}}=A\left(1-2z^{-1}\right)+B+\frac{C}{1+z^{-1}}\left(1-2z^{-1}\right)^{2}$$

Applying differentiation to the both sides

$$\frac{d\left[X(z)\left(1-2z^{-1}\right)^{2}\right]}{dz^{-1}}=-2A+C\frac{d}{dz^{-1}}\left[\frac{\left(1-2z^{-1}\right)^{2}}{1+z^{-1}}\right]$$

$$\Rightarrow A=-\frac{1}{2}\left\{\frac{d\left[\frac{4-8z^{-1}+6z^{-2}}{1+z^{-1}}\right]}{dz^{-1}}\right\}\Bigg|_{z^{-1}=\frac{1}{2}}-\frac{1}{2}C\frac{d}{dz^{-1}}\left[\frac{\left(1-2z^{-1}\right)^{2}}{1+z^{-1}}\right]\Bigg|_{z^{-1}=\frac{1}{2}}$$

$$=-\frac{1}{2}\left\{\frac{\left(-8+12z^{-1}\right)\left(1+z^{-1}\right)-\left(4-8z^{-1}+6z^{-2}\right)\cdot 1}{\left(1+z^{-1}\right)^{2}}\right\}\Bigg|_{z^{-1}=\frac{1}{2}}-0$$

$$=-\frac{1}{2}\frac{\left(-3\cdot\frac{3}{2}\right)-\left(4-4+\frac{3}{2}\right)}{\frac{9}{4}}=1$$

$$\Rightarrow X(z)=\frac{1}{1-2z^{-1}}+\frac{1}{\left(1-2z^{-1}\right)^{2}}+\frac{2}{1+z^{-1}}$$

or

$$X(z)=\frac{1}{1-2z^{-1}}+\frac{\left(1-2z^{-1}\right)+2z^{-1}}{\left(1-2z^{-1}\right)^2}+\frac{2}{1+z^{-1}}$$

$$=\frac{2}{1-2z^{-1}}+\frac{2z^{-1}}{\left(1-2z^{-1}\right)^2}+\frac{2}{1+z^{-1}}$$

$$\Rightarrow x[n]=\left[2\left(2^n\right)+n\left(2^n\right)+2(-1)^n\right]u[n]$$

(5) Power series:
If $X(z)$ is not a rational function of z, its inverse z-transform $x[n]$ may still be obtained from the power series expansion of $X(z)$.

***Example* 7.18**: Assume a z transform of the form

$$X(z)=e^{a/z},\qquad |z|>0.$$

Since R contains $z=\infty$, the sequence $x[n]$ must be causal. The power (Maclaurin) series for $X(z)$ is given by

$$X(z)=\sum_{n=0}^{\infty}\frac{a^n}{n!}z^{-n}$$

from which we have immediately that

$$x[n]=\frac{a^n}{n!}u[n].$$

■

***Example* 7.19**: Let

$$X(z)=\log\left(1-az^{-1}\right)\ ,\qquad |z|>|a|.$$

The power series expansion for $\log(1-y)$ is of the form

$$\log(1-y)=\sum_{n=1}^{\infty}\frac{-1}{n}y^n$$

from which

$$X(z)=\sum_{n=1}^{\infty}\frac{-1}{n}a^n z^{-n}\ .$$

Hence,

$$x[n]=-\frac{a^n}{n}u[n-1].$$

7-4 Properties of the z-transform

1. Linearity

$$x_1[n]\overset{z}{\longleftrightarrow}X_1(z)\ ,\qquad \text{ROC}=R_1$$

$$x_2[n]\overset{z}{\longleftrightarrow}X_2(z)\ ,\qquad \text{ROC}=R_2$$

$\Rightarrow a_1x_1[n]+a_2x_2[n]\overset{z}{\longleftrightarrow}a_1X_1(z)+a_2X_2(z)$, ROC containing $R_1\cap R_2$ (7.31)
If pole-zero cancellation occurs, the ROC may be larger than $R_1\cap R_2$.

■

2. Time shifting

$$x[n]\overset{z}{\longleftrightarrow}X(z)\quad,\quad \text{ROC}=R_x$$

$\Rightarrow x[n-n_0]\overset{z}{\longleftrightarrow}z^{-n_0}X(z)$, $\text{ROC}=R_x$ except for the possible addition or deletion of the origin or infinity. (7.32)

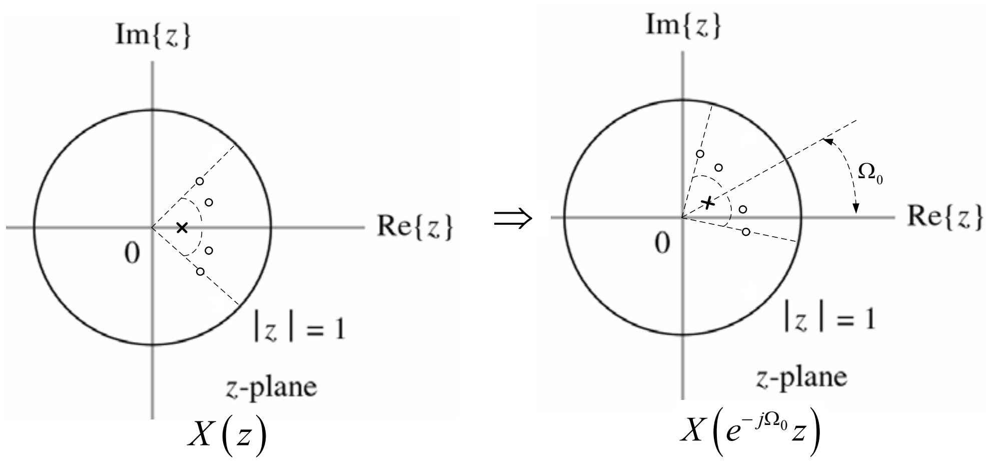
$$n_0>0\Rightarrow z^{-n_0}\to\infty \text{ as } z=0$$
$$\therefore \text{introducing a pole } z=0$$
$$\text{a zero } z=\infty$$
$$n_0<0\Rightarrow z^{-n_0}\to\infty \text{ as } z=\infty$$
$$\therefore \text{introducing a pole } z=\infty$$
$$\text{a zero } z=0$$

■

3. Frequency shifting

$$x[n]\overset{z}{\longleftrightarrow}X(z)\quad,\quad \text{ROC}=R_x$$

$$\Rightarrow e^{j\Omega_0 n}x[n]\overset{z}{\longleftrightarrow}X\left(e^{-j\Omega_0}z\right)\quad,\quad \text{ROC}=R_x \tag{7.33}$$



$X(z)$ $\qquad$ $X\left(e^{-j\Omega_0}z\right)$

$$\Rightarrow z_0^n x[n]\overset{z}{\longleftrightarrow}X\left(\frac{z}{z_0}\right)\quad,\quad \text{ROC}=z_0R_x \tag{7.34}$$

$|z_0|=1\Rightarrow z_0=e^{j\Omega_0}$, reduce to the above

$z_0=re^{j\Omega_0}\Rightarrow$ the pole and zero locations are rotated in the z-plane by an angle of Ω_0 and scaled in position radially by a factor of r.

4. Time Reversal

$$x[n] \overset{z}{\longleftrightarrow} X(z) \quad , \quad \text{ROC} = R_x$$

$$\Rightarrow x[-n] \overset{z}{\longleftrightarrow} X\left(\frac{1}{z}\right) \quad , \quad \text{ROC} = \frac{1}{R_x} \tag{7.35}$$

$$\frac{1}{z} = z_p \Rightarrow z = \frac{1}{z_p}$$

■

5. Convolution Property

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \quad \text{ROC} = R_1$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad \text{ROC} = R_2$$

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z), \quad \text{ROC contains } R_1 \cap R_2 \tag{7.36}$$

Proof:

$$y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

$$\therefore Y(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]\left\{\sum_{k=-\infty}^{\infty} x_2[n-k]z^{-n}\right\}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \cdot X_2(z)$$

$$= X_1(z)X_2(z)$$

The ROC may be larger than $R_1 \cap R_2$ if pole-zero cancellation occurs in $X_1(z)X_2(z)$.

■

Note:

- When two polynomials or power series $X_1(z)$ and $X_2(z)$ are multiplied, the coefficients in the polynomial representing the product are the convolution of the coefficients in the polynomials $X_1(z)$ and $X_2(z)$.

  $X_1(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N} \rightarrow (N+1)$ points

  $X_2(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N} \rightarrow (N+1)$ points

  if $X_3(z) = X_1(z)X_2(z)$ then

  $X_3(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_{2N} z^{-2N} \rightarrow (2N+1)$ points

  $c_n = \sum_{k=0}^{N} a_k b_{n-k} = a_n * b_n \quad , \quad n = 0, 1, 2, \cdots, 2N$

6. Differentiation in the z-Domain

$$x[n] \overset{z}{\longleftrightarrow} X(z) \quad , \quad \text{ROC} = R_x$$

$$\Rightarrow nx[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z) \quad , \quad \text{ROC} = R_x \quad (\text{poles are the same}) \quad (7.37)$$

If there is a pole at $z = 0$ originally, then an extra pole at $z = 0$ will occur after differentiating and that will be cancelled with the new zero $z = 0$.

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} -nx[n] z^{-n+1}$$

$$\Rightarrow -z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} nx[n] z^{-n}$$

■

***Example* 7.20**:

$$X(z) = \ln\left(1 + az^{-1}\right) \quad , \quad |z| > a$$

Find $x[n] = ?$

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z) = \frac{az^{-1}}{1+az^{-1}} \quad , \quad |z| > |a|$$

$$\because (-a)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1+az^{-1}} \quad , \quad |z| > |a|$$

$$\therefore a(-a)^n u[n] \overset{z}{\longleftrightarrow} \frac{a}{1+az^{-1}} \quad , \quad |z| > |a|$$

$$\Rightarrow a(-a)^{n-1} u[n-1] \overset{z}{\longleftrightarrow} \frac{az^{-1}}{1+az^{-1}} \quad , \quad |z| > |a|$$

$$\Rightarrow nx[n] = -(-a)^n u[n-1]$$

$$\Rightarrow x[n] = \frac{-(-a)^n}{n} u[n-1]$$

■

***Example* 7.21**:

$$X(z) = \frac{az^{-1}}{\left(1 - az^{-1}\right)^2} \quad , \quad |z| > |a|$$

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}} \quad , \quad |z| > |a|$$

$$\Rightarrow na^n u[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad , \quad |z| > |a|$$

■

7. The Initial Value Theorem

If $x[n]=0$ for $n<0$, then

$$x[0]=\lim_{z\to\infty}X(z) \tag{7.38}$$

Proof:

$$\lim_{z\to\infty}X(z)=\lim_{z\to\infty}\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$
$$=\lim_{z\to\infty}\sum_{n=0}^{\infty}x[n]z^{-n}$$
$$=\lim_{z\to\infty}\left[x[0]z^{0}+x[1]z^{-1}+x[2]z^{-2}+\cdots\right]$$
$$=x[0]$$

8. The Final Value Theorem

If $x[n]$ is causal and stable with z-transform $X(z)$, then

$\lim_{n\to\infty}x[n]=\lim_{z\to 1}\left(1-z^{-1}\right)X(z)$ ($X(z)$ has poles inside the unit circle.) (7.39)

Proof:

First we know $Z\{x[k]-x[k-1]\}=X(z)-z^{-1}X(z)=\left(1-z^{-1}\right)X(z)$

Then take limit as $z\to 1$ on both sides:

$$\lim_{z\to 1}\left(1-z^{-1}\right)X(z)$$
$$=\lim_{z\to 1}\left\{\sum_{k=0}^{\infty}\left(x[k]-x[k-1]\right)z^{-k}\right\}=\sum_{k=0}^{\infty}\left(x[k]-x[k-1]\right)$$
$$=x[0]+\left(x[1]-x[0]\right)+\left(x[2]-x[1]\right)+\ldots x[\infty]$$
$$=x[\infty]=\lim_{n\to\infty}x[n]$$

■

Note:

- If the system is stable, then the impulse response $h[n]$ is absolutely summable.
  $\Rightarrow$ the Fourier transform of the impulse response $h[n]$ converges
  $\Rightarrow$ the ROC of $H(z)$ must include the unite circle
- If the system is both causal and stable, the ROC of the z-transform of the impulse response must include the unit circle and be outside the outermost pole.
- For a causal and stable system, all the poles of the system function must be inside the unit circle.

***Example* 7.22**:

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Assume the system is causal and stable.

Then the ROC is $|z|>0.5$.

$$\Rightarrow h[n]=z^{-1}\{H(z)\}$$

$$=\left(\frac{1}{2}\right)^{n}u[n]+\frac{1}{3}\left(\frac{1}{2}\right)^{n-1}u[n-1]$$

■

7-5 The System Function for LTI Systems

Invertible Systems:

If an LTI system *h*[*n*] is invertible, there must exist an inverse system with impulse response $h_I[n]$ such that

$$h[n]*h_I[n]=\delta[n]. \qquad (7.40)$$

Expressing this relationship in terms of z-transforms, we thus have

$$H(z)H_I(z)=1 \quad \text{or} \quad H_I(z)=\frac{1}{H(z)}. \qquad (7.41)$$

If *H*(*z*) is the rational fraction *B*(*z*)/*A*(*z*), then $H_I(z)$ is the rational faction *A*(*z*)/*B*(*z*), and the poles of *H*(*z*) are the zeros of $H_I(z)$, and vice versa. In general, the inverse system $H_I(z)$ for a given *H*(*z*) is not unique because multiple ROCs can be defined for a rational fraction *A*(*z*)/*B*(*z*) having at least poles at other than $z=0$ or $z=\infty$. However, if we set the requirements of stability and/or causality on $H_I(z)$, it will be unique.

***Example* 7.23**: Given the accumulator system function

$$H(z)=\frac{1}{1-z^{-1}}, \quad |z|>1,$$

the associated inverse system is

$$H_I(z)=1-z^{-1}, \quad |z|>0,$$

corresponding to the impulse response

$$h_I[n]=\delta[n]-\delta[n-1].$$

This system is known as a *first-difference operator* and is unique because $H_I(z)$ has only a pole at $z = 0$. Checking that Eq. (7.30) is indeed satisfied by $h_I[n]$, we have

$$\begin{aligned} h[n] * h_I[n] &= u[n] * \{\delta[n] - \delta[n-1]\} \\ &= u[n] - u[n-1] \\ &= \delta[n]. \end{aligned}$$

■

***Example* 7.24**: Given the stable and causal system

$$H(z) = \frac{1+0.8z^{-1}}{1-0.5z^{-1}}, \quad |z| > 0.5,$$

we can identify two corresponding inverse systems, as follows:

$$H_{I1}(z) = \frac{1-0.5z^{-1}}{1+0.8z^{-1}}, \quad |z| > 0.8,$$

and

$$H_{I2}(z) = \frac{1-0.5z^{-1}}{1+0.8z^{-1}}, \quad |z| < 0.8.$$

In most practical applications, however, only $H_{I1}(z)$ is useful because it is both stable and causal.

On the other hand, for the stable and causal system

$$H(z) = \frac{1-2z^{-1}}{1-0.5z^{-1}}, \quad |z| > 0.5,$$

the two possible inverse system are

$$H_{I3}(z) = \frac{1-0.5z^{-1}}{1-2z^{-1}}, \quad |z| > 2,$$

and

$$H_{I4}(z) = \frac{1-0.5z^{-1}}{1-2z^{-1}}, \quad |z| < 2.$$

Hence, in this case, we must choose between stability and causality for the inverse system because $H_{I3}(z)$ is causal but not stable, while $H_{I4}(z)$ is stable but not causal.

■

Difference Equations:

Given a finite-order linear difference equation with constant coefficients

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \tag{7.42}$$

we can obtain its z-transform

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

and the corresponding system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \qquad (7.43)$$

***Example* 7.25**: In the first-order linear difference equation

$$y[n] - ay[n-1] = x[n],$$

we have

$$(1 - az^{-1})Y(z) = X(z).$$

Hence, the actual system function can be either

$$H_1(z) = \frac{1}{1 - az^{-1}}, \quad |z| > a,$$

or

$$H_2(z) = \frac{1}{1 - az^{-1}}, \quad |z| < a,$$

corresponding to the causal ad anticausal impulse response

$$h_1[n] = a^n u[n]$$

and

$$h_2[n] = -a^n u[-n-1],$$

respectively. Since $h_1[n]$ and $h_2[n]$ are both nonzero for an infinite time duration, they are classified as *infinite-impulse-response* (*IIR*) filters. Clearly, any filter with at least one nonzero, finite pole (i.e., a pole at other than $z = 0$ or $z = \infty$) that is not canceled by a zero, will be IIR because such poles imply exponential components in $h[n]$. ■

***Example* 7.26**: The first-difference operator was defined in Example 7.23 by the system function

$$H(z) = 1 - z^{-1}, \quad |z| > 0.$$

Recognizing that $H(z)$ is a first-order rational fraction of the form in Eq. (7.43), with $b_0 = 1$, $b_1 = -1$, and $a_0 = 1$ (and thus $M = 1$ and $N = 0$), we can write the corresponding difference equation from Eq. (7.42) as simply

$$y[n] = x[n] - x[n-1].$$

Since the associated impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

is nonzero for only a finite time duration, this filter is classified as a *finite-impulse response* (*FIR*) filter. Note, in particular, that in contrast with the IIR case, this filter has only a pole at $z = 0$.

■

7-6 Geometric Evaluation of the Fourier Transform from the Pole-Zero Plot

1. As the continuous-time systems we introduced in Section 6-4. First, factoring the numerator and denominator polynomials of the rational fraction into products of first-order factors of the form

$$H(z) = \frac{C\prod_{k=1}^{M}\left(1 - z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - p_k z^{-1}\right)}, \quad (7.44)$$

where z_k and p_k are the zeros and poles, respectively, of $H(z)$ and $C = b_0/a_0$, we may write $H(z)$ in the equivalent form

$$H(z) = \frac{Cz^{N-M}\prod_{k=1}^{M}\left(z - z_k\right)}{\prod_{k=1}^{N}\left(z - p_k\right)}. \quad (7.45)$$

The corresponding frequency response $H(e^{j\Omega})$ is then simply

$$H(e^{j\Omega}) = \frac{Ce^{j\Omega(N-M)}\prod_{k=1}^{M}\left(e^{j\Omega} - z_k\right)}{\prod_{k=1}^{N}\left(e^{j\Omega} - p_k\right)}. \quad (7.46)$$

Therefore, for a given frequency Ω, each complex-valued numerator term $(e^{j\Omega} - z_k)$ can be thought of as a vector in the complex (z) plane from the zeros z_k to the point $e^{j\Omega}$ on the unit circle; and likewise, each denominator term $(e^{j\Omega} - p_k)$ is effectively a vector from the pole p_k to the point $e^{j\Omega}$. Also, the $N - M$ zeros (or $M - N$ poles if $M > N$) at $z = 0$ produce an additional factor $e^{j\Omega(N-M)}$ in the frequency response.

Utilizing Eq. (7.46) to write the magnitude response $|H(e^{j\Omega})|$, we thus have

$$\left|H(e^{j\Omega})\right| = \frac{|C|\prod_{k=1}^{M}\left|e^{j\Omega} - z_k\right|}{\prod_{k=1}^{N}\left|e^{j\Omega} - p_k\right|}. \quad (7.47)$$

That is, the magnitude response as the frequency Ω equals the scaled product of the lengths of all vectors $(e^{j\Omega} - z_k)$ from the zeros to the point

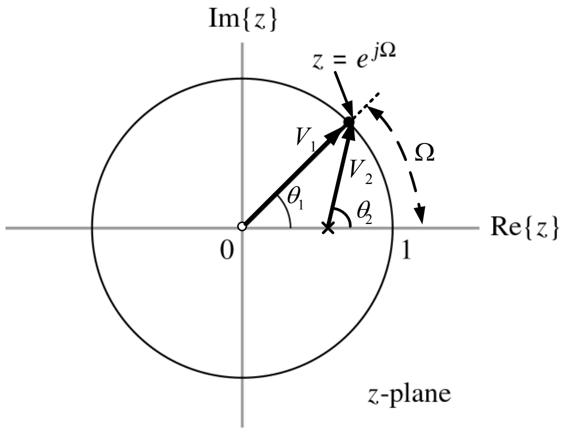
$e^{j\Omega}$ divided by the product of the lengths of all vectors $(e^{j\Omega} - p_k)$ from the poles to the point $e^{j\Omega}$, with the scaling constant being $|C| = |b_0/a_0|$. Similarly, the phase response $\angle H(e^{j\Omega})$ can be written from Eq. (7.46) as

$$\angle H(e^{j\Omega}) = \sum_{k=1}^{M} \angle\left(e^{j\Omega} - z_k\right) - \sum_{k=1}^{N} \angle\left(e^{j\Omega} - p_k\right) + \left(N - M\right)\Omega + \angle C \, ; \qquad (7.48)$$

and thus $\angle H(e^{j\Omega})$ is simply the sum of the angles of all numerator vectors $(e^{j\Omega} - z_k)$ minus the sum of the angles of all denominator vectors $(e^{j\Omega} - p_k)$ plus a linear-phase term $(N - M)\Omega + \angle C$.

2. The z-transform reduces to the Fourier transform for $|z| = 1$.

$$X\left(e^{j\Omega}\right) = X\left(z\right)\Big|_{z=e^{j\Omega}} \qquad (7.49)$$



***Example* 7.27**:

$$H\left(z\right) = \frac{1}{1 - az^{-1}} \qquad |z| > |a| \qquad h\left[n\right] = a^n u\left[n\right]$$

$$= \frac{z}{z - a} \qquad \text{zero: } z = 0 \qquad \text{pole: } z = a$$

$$H\left(e^{j\Omega}\right) = \left|H\left(e^{j\Omega}\right)\right| \cdot e^{\angle H\left(e^{j\Omega}\right)}$$

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{|\vec{V}_1|}{|\vec{V}_2|} = \frac{1}{|\vec{V}_2|} \qquad \vec{V}_1: \text{ zero vector} \quad \vec{V}_2: \text{ pole vector}$$

$$\angle H\left(e^{j\Omega}\right) = \theta_1 - \theta_2 = \Omega - \theta_2$$

■

Note:

- The magnitude of the frequency response will be the maximum at

$\Omega = 0$ and will decrease monotonically as Ω increases from 0 to π.

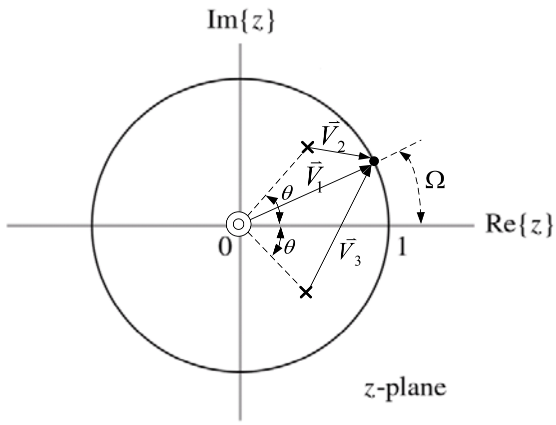
- The angle of the pole vector begins at zero and increases monotonically but not linearly as Ω increases from 0 to π.
- As $|a|$ decreases, the impulse response decays more sharply and the step response settles more quickly. (a is similar to time constant.)

***Example* 7.28**:

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}}$$

$$= \frac{z^2}{z^2-(2r\cos\theta)z+r^2} = \frac{z^2}{(z-re^{j\theta})(z-re^{-j\theta})}$$

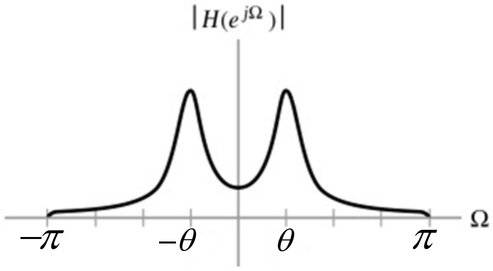
$$h[n] = r^n\frac{\sin(n+1)\theta}{\sin\theta}u[n]$$



$\vec{V}_1$: zero vector

$\vec{V}_2$, $\vec{V}_3$: pole vector

$$\left|H(e^{j\Omega})\right| = \frac{|\vec{V}_1|^2}{|\vec{V}_2||\vec{V}_3|} = \frac{1}{|\vec{V}_2||\vec{V}_3|}$$



The length of $\vec{V}_2$ has a minimum length when $\Omega = \theta$.

$\Rightarrow$The magnitude of the frequency response peaks for Ω near θ.

Note:

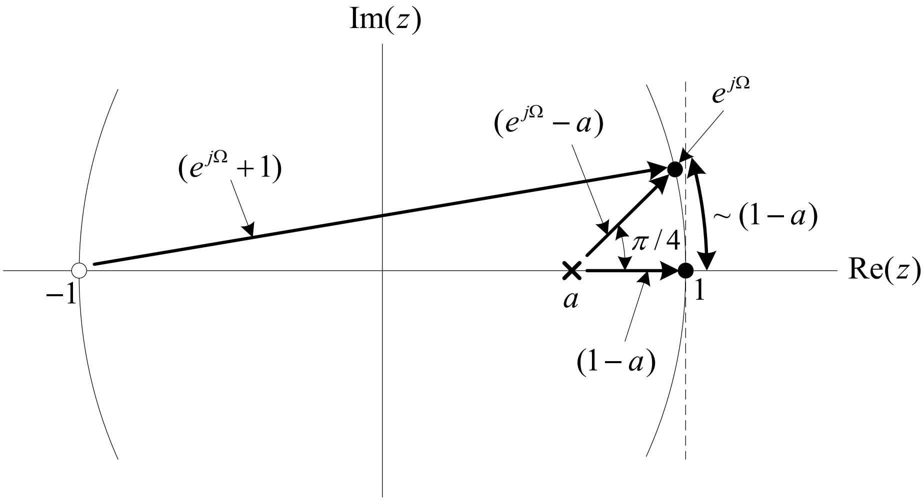
- $r\downarrow\Rightarrow$ the impulse response decays more sharply and the step response settles more quickly.

- $r\uparrow\Rightarrow$ the peak is larger.

***Example* 7.29**: The 3-dB bandwidth of these filters is easily approximated in narrowband cases using geometric analysis, as follows:

$$H(e^{j\Omega}) = C\frac{(1+e^{-j\Omega})}{(1-ae^{-j\Omega})} = C\frac{(e^{j\Omega}+1)}{(e^{j\Omega}-a)}, \qquad (7.50)$$

where $C=(1-a)/2$ for unity gain at dc. The vectors $(e^{j\Omega}+1)$, $(e^{j\Omega}-a)$, and also $(1-a)$ are depicted in Fig. 7.6 for $0 \ll a < 1$.



■**Figure 7.6** Geometric approximation of 3-dB bandwidth for first-order LPF.

Since the dc gain of the LPF is unity, the 3-dB point occurs at the value Ω_b for which $H(e^{j\Omega_b}) = 1/\sqrt{2}$.

Approximating the unit circle in the vicinity of $z=1$ by the dotted vertical line shown in the Fig. 7.6, we note that the vectors $(e^{j\Omega}-a)$ and $(1-a)$ and the dotted line approximate an isosceles triangle for $0 \ll a < 1$ when the angle of $(e^{j\Omega}-a)$ is $\pi/4$, as illustrated. Hence, since the vector $(e^{j\Omega}-a)$ forms the approximate hypotenuse of this triangle, its length can be estimated as $\sqrt{2}(1-a)$ at this angle, while, on the other hand, the length of the numerator vector $(e^{j\Omega}+1)$, which equals 2 for $\Omega=0$, is only slightly less in this case.

We thus find that $\Omega \approx \Omega_b$ for this geometric situation

$$\left|H(e^{j\Omega})\right| \approx |C| \frac{2}{\sqrt{2}(1-a)} = \frac{1}{\sqrt{2}}.$$

Finally, to estimate the value of Ω_b, we note that the two sides of an isosceles triangle have equal lengths and that the length of an arc on the unit circle equals the associated angle (in radians). Therefore the vertical side of the triangle has length $1-a$, and for $0 \ll a < 1$, the associated angle (bandwidth) Ω_b is also approximately

$$\Omega_b \approx 1-a, \tag{7.51}$$

as depicted in the Fig. 7.6.

A similar geometric derivation can be employed to estimate the bandwidth of a first-order HPF in the narrowband case.

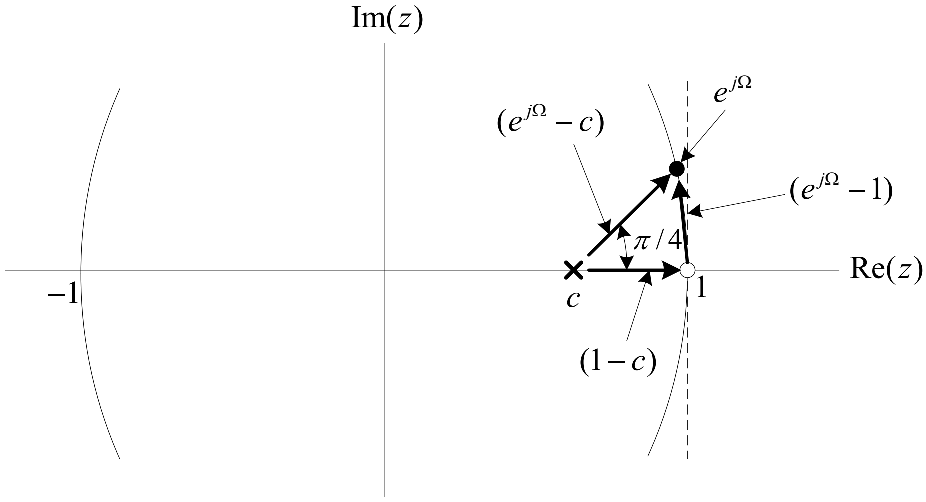
$$\begin{aligned} H(e^{j\Omega}) &= C\frac{(1-e^{-j\Omega})}{(1-ce^{-j\Omega})} \\ &= C\frac{(e^{j\Omega}-1)}{(e^{j\Omega}-c)}, \end{aligned} \tag{7.52}$$

where $C=(1+c)/2$, the vectors $(e^{j\Omega}-1)$, $(e^{j\Omega}-c)$ and also $(1-c)$ are shown in Fig. 7.7 for $0 \ll c < 1$. Again, since the maximum gain of the HPF is unity (at $\Omega=\pi$), the 3-dB point occurs at the value Ω_b for which $H(e^{\Omega_b}) = 1/\sqrt{2}$. Note that the vector $(e^{j\Omega}-1)$ is almost vertical and thus forms an approximate isosceles triangle with the other two vectors when the angle of $(e^{j\Omega}-c)$ is $\pi/4$, as illustrated. Therefore the length of $(e^{j\Omega}-1)$ is approximately $1-c$, while the length of the hypotenuse $(e^{j\Omega}-c)$ is approximately $\sqrt{2}(1-c)$. Note also that $C \approx 1$ for $0 \ll c < 1$. Hence, the magnitude response in this situation is approximated by

$$\left|H(e^{j\Omega})\right| \approx \frac{1-c}{\sqrt{2}(1-c)} = \frac{1}{\sqrt{2}},$$

and thus $\Omega \approx \Omega_b$. As before, the associated value of the angle Ω_b (bandwidth of the stopband) is then simply

$$\Omega_b \approx 1-c. \tag{7.53}$$



■**Figure 7.7** Geometric approximation of 3-dB bandwidth for first-order HPF.

***Example* 7.30**: Second-Order IIR Filters

The second-order underdamped system function

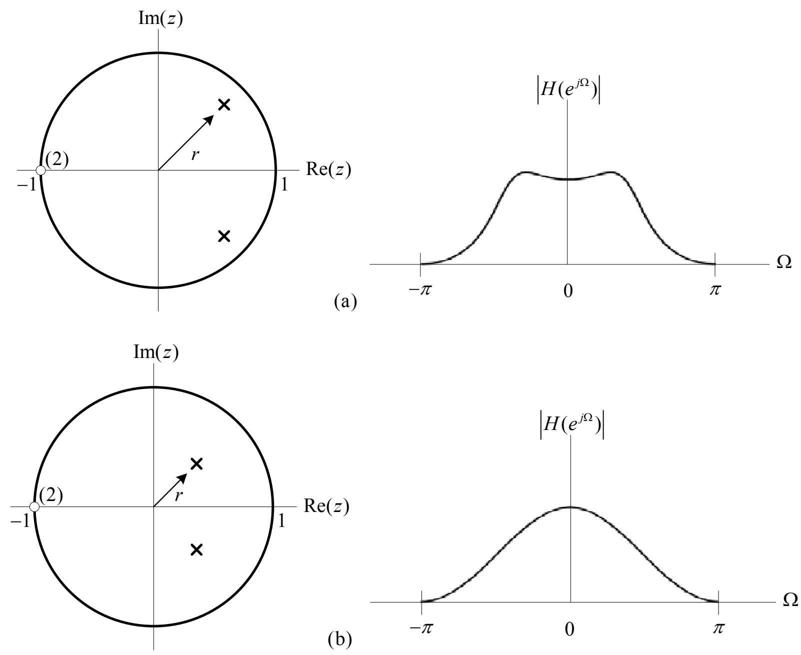
$$H(z) = \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 - 2r(\cos\Omega_0)z^{-1} + r^2 z^{-2}}, \quad |z| > r, \tag{7.54}$$

can provide an LPF, HPF, BPF, or BSF response, depending upon the values of the numerator coefficients b_1 and b_2 , as illustrated by the following cases：

LPF Case: For $b_1 = 2$ and $b_2 = 1$, Eq. (7.54) becomes

$$\begin{aligned} H(z) &= \frac{1 + 2z^{-1} + z^{-2}}{1 - 2r(\cos\Omega_0)z^{-1} + r^2 z^{-2}} \\ &= \frac{(1 + z^{-1})^2}{1 - 2r(\cos\Omega_0)z^{-1} + r^2 z^{-2}}, \quad |z| > r, \end{aligned} \tag{7.55}$$

and hence there is a double zero at $z = -1$. Therefore $H(e^{j\pi}) = 0$, implying an LPF response. Sketching the corresponding pole/zero plot and magnitude response, we can actually identify two possible cases, as illustrated in Fig. 7.8. In particular, if the poles are close enough to the unit circle to produce discernible peaks in $|H(e^{j\Omega})|$, the response is nonmonotonic in the passband, as shown in Fig. 7.8(a). By analogy with the corresponding continuous-time case, such filters are called *highly underdamped*. On the other hand, if the radius (r) of the poles is sufficiently small, distinct peaks due to the poles are not discernible in $|H(e^{j\Omega})|$, and the response decreases monotonically, as depicted in Fig. 7.8(b).

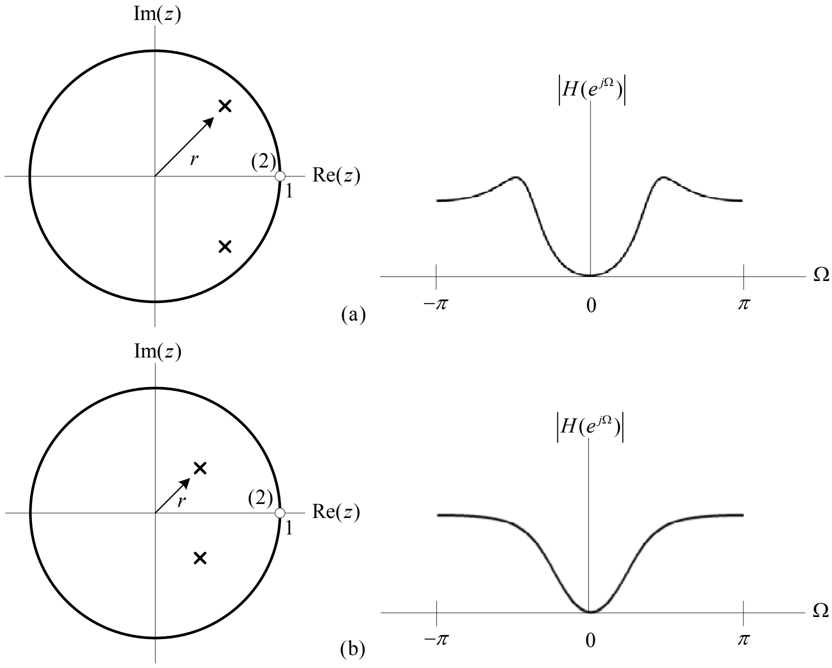


■**Figure 7.8** Pole/zero plots and magnitude responses for second-order LPF.

HPF Case: For $b_1 = -2$ and $b_2 = 1$, we have instead

$$\begin{aligned} H(z) &= \frac{1-2z^{-1}+z^{-2}}{1-2r(\cos\Omega_0)z^{-1}+r^2z^{-2}} \\ &= \frac{(1-z^{-1})^2}{1-2r(\cos\Omega_0)z^{-1}+r^2z^{-2}}, \quad |z|>r. \end{aligned} \tag{7.56}$$

Thus there is double zero at $z=+1$, implying an HPF response with $H(e^{j0})=0$. As in the LPF case, we have again depicted two possible forms for the associated magnitude response in Fig. 7.9. That is, if the poles are close enough to the unit circle to produce discernible peaks in $|H(e^{j\Omega})|$, the response is nonmonotonic in the passband, as shown in Fig. 7.9(a), and the filter is said to be *highly underdamped*. However, if the radius (r) of the poles is sufficiently small, distinct peaks due to the poles are not evident in $|H(e^{j\Omega})|$, and the response increases monotonically, as illustrated in Fig. 7.9(b).

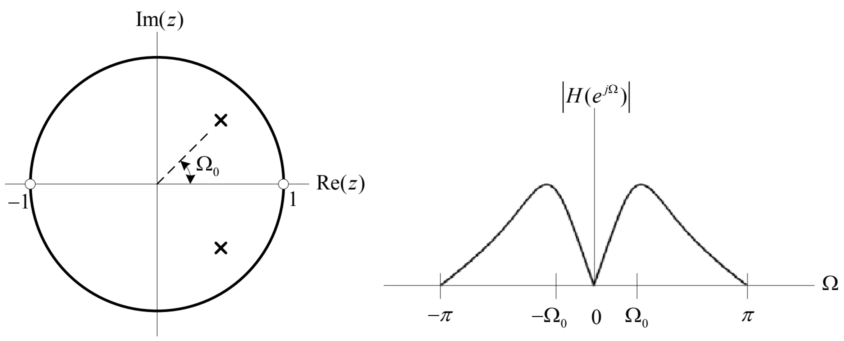


■**Figure 7.9** Pole/zero plots and magnitude responses for second-order HPF.

BPF Case: For $b_1 = 0$ and $b_2 = -1$, Eq. (7.54) becomes

$$\begin{aligned} H(z) &= \frac{1 - z^{-2}}{1 - 2r(\cos\Omega_0)z^{-1} + r^2 z^{-2}} \\ &= \frac{(1 - z^{-1})(1 + z^{-1})}{1 - 2r(\cos\Omega_0)z^{-1} + r^2 z^{-2}}, \qquad |z| > r, \end{aligned} \tag{7.57}$$

implying single zeros at both $z = 1$ and $z = -1$, and thus $H(e^{j0}) = H(e^{j\pi}) = 0$. Figure 7.10 depicts the corresponding pole/zero plot and magnitude response. Note that the center frequency for the BPF response is approximately Ω_0 since the denominator vector from the pole at $re^{j\Omega_0}$ to the point $e^{j\Omega}$ on the unit circle is shortest when $\Omega = \Omega_0$. The associated 3-dB bandwidth is readily shown to be about $2(1-r)$ radians for narrowband filters, that is, $0 \ll r < 1$.

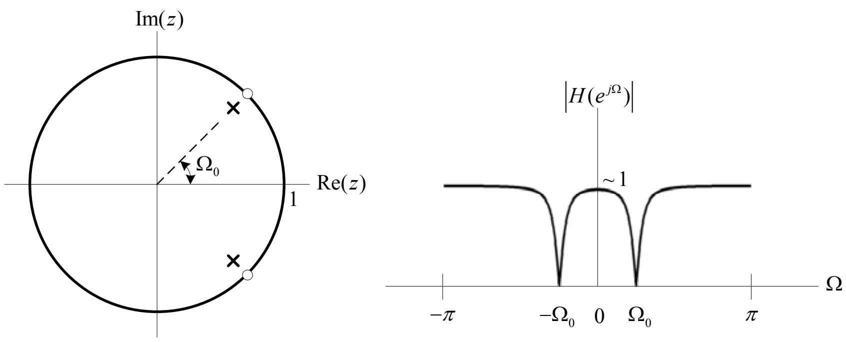


■**Figure 7.10** Pole/zero plot and magnitude response for second-order BPF.

BSF Case: For $b_1 = -2(\cos\Omega_0)$ and $b_2 = 1$, Eq. (7.54) takes the form

$$H(z) = \frac{1-2(\cos\Omega_0)z^{-1}+z^{-2}}{1-2r(\cos\Omega_0)z^{-1}+r^2z^{-2}}, \qquad |z| > r, \tag{7.58}$$

implying complex-conjugate zeros on the unit circle at angles of $\pm\Omega_0$, as shown in Fig. 7.11. That is, $H(e^{j\Omega_0}) = H(e^{-j\Omega_0}) = 0$. Note then that the pole angles and the zeros angles are the same. Using the geometric method to sketch the resulting *notch-filter* response in Fig.7.11, we produce $|H(e^{j\Omega})|$ as shown. Note that at $\Omega = 0$, and also at $\Omega = \pi$, the numerator and denominator vectors all have about the same length, and hence $H(e^{j0}) \approx H(e^{j\pi}) \approx 1$. As in BPF case, the associated 3-dB bandwidth (of the stopband) is readily shown to be about $2(1-r)$ radians for BSF responses with narrow stopbands, that is, $0 \ll r < 1$.



■**Figure 7.11** Pole/zero plot and magnitude response for second-order BSF.

***Example* 7.31**: *Linear-Phase FIR Filters*

Letting $a_0 = 1$ and $a_k = 0$ for all $k > 0$ in the general difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{7.59}$$

we produce the *nonrecursive* difference equation

$$y[n] = \sum_{k=0}^{M} b_k x[n-k], \tag{7.60}$$

and thus, setting $x[n] = \delta[n]$, we find that the corresponding impulse response is simply

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k].$$

That is,

$$h[n] = \begin{cases} bn, & n = 0,\ 1, \dots, M \\ 0, & \text{otherwise.} \end{cases} \tag{7.61}$$

Therefore any discrete-time system satisfying a finite-order nonrecursive difference equation is FIR. (As might then be expected, a *recursive* difference equation having $a_k \neq 0$ for some $k > 0$ usually implies an IIR system, but pole/zero cancelations can still cause such systems to be FIR.) Because of their special properties, discrete-time FIR filters find wide application in digital signal processing and communications.

The most important class of FIR filters in practice are those having piecewise linear-phase responses. Assuming $h[n]$ to be real, such linear-phase filters have either even or odd symmetry about the midpoint of $h[n]$, that is,

$$b_n = b_{M-n} \tag{7.62}$$

or

$$b_n = -b_{M-n}. \tag{7.63}$$

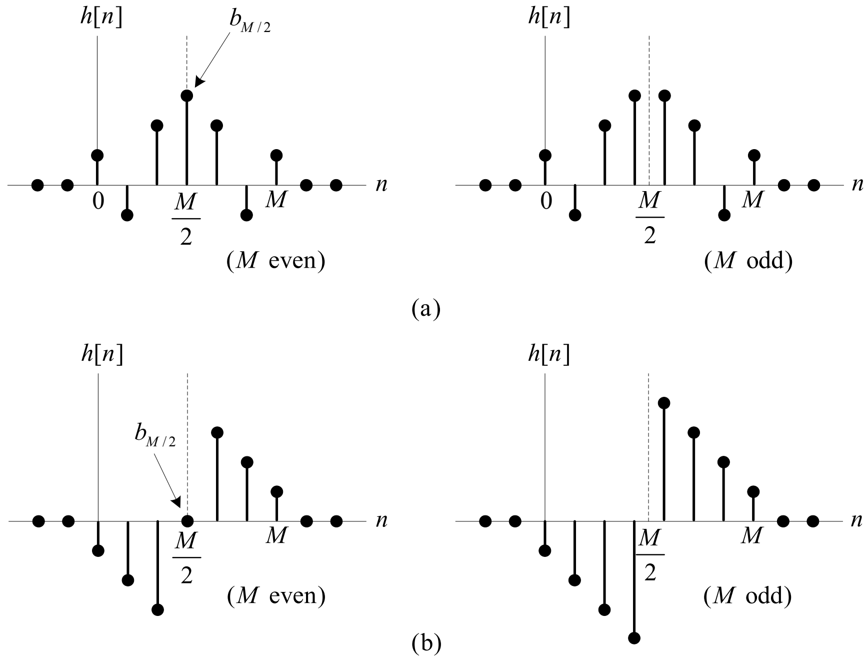
Examples of even- and odd-symmetric impulse responses are shown in Fig. 7.12 for even and odd values of M. Note that the center of symmetry (shown by a dotted line) occurs at the coefficient $b_{M/2}$ for M even, but between two coefficients for M odd. Note also that $b_{M/2}$ must equal zero for odd symmetry and M even.

To show the linear-phase property of such filters, we first express the FIR system function $H(z)$ as

$$\begin{aligned} H(z) &= \sum_{n=0}^{M} b_n z^{-n} \\ &= b_c z^{-M/2} + \sum_{n=0}^{L} (b_n z^{-n} + b_{M-n} z^{-(M-n)}) \end{aligned} \tag{7.64}$$

where L is the integer part of $(M-1)/2$ and b_c is the central coefficient (if there is one), that is,

$$b_c = \begin{cases} b_{M/2}, & M \text{ even} \\ 0, & M \text{ odd.} \end{cases}$$



■Figure 7.12 Four cases of symmetry for linear-phase FIR filters: (a) even symmetry and (b) odd symmetry.

In the even-symmetry case $(b_n = b_{M-n})$, we then have

$$\begin{aligned} H(e^{j\Omega}) &= b_c e^{-j\Omega M/2} + \sum_{n=0}^{L} b_n (e^{-j\Omega n} + e^{-j\Omega(M-n)}) \\ &= e^{-j\Omega M/2} \{ b_c + \sum_{n=0}^{L} b_n (e^{j\Omega(M/2-n)} + e^{-j\Omega(M/2-n)}) \} \\ &= e^{-j\Omega M/2} \{ b_c + \sum_{n=0}^{L} 2b_n \cos[\Omega(\frac{M}{2} - n)] \} \\ &= e^{-j\Omega M/2} R(\Omega), \end{aligned} \tag{7.65}$$

where $R(\Omega)$ is a purely real function of Ω. Therefore the associated magnitude and phase responses are simply

$$|H(e^{j\Omega})| = |R(\Omega)|$$

and

$$\angle H(e^{j\Omega}) = \frac{-\Omega M}{2} + \angle R(\Omega), \tag{7.66}$$

where $\angle R(\Omega) = 0$ if $R(\Omega) > 0$, and $\angle R(\Omega) = \pm\pi$ if $R(\Omega) < 0$. Hence the phase response is a piecewise linear function having a discontinuity of π radians at each zero crossing of $R(\Omega)$.

A similar derivation for odd symmetry ($b_n = -b_{M-n}$) leads to the result

$$\begin{aligned} H(e^{j\Omega}) &= je^{-j\Omega M/2}\sum_{n=0}^{L} 2b_n \sin\left[\Omega(\frac{M}{2}-n)\right] \\ &= je^{-j\Omega M/2}R(\Omega) \\ &= e^{j(\pi/2-\Omega M/2)}R(\Omega) \end{aligned} \tag{7.67}$$

for real $R(\Omega)$. Therefore the associated magnitude response is again simply $|H(e^{j\Omega})| = |R(\Omega)|$,
but the phase response has an additional component of $\pi/2$ (90^o), that is,

$$\angle H(e^{j\Omega}) = \frac{\pi}{2} - \frac{\Omega M}{2} + \angle R(\Omega). \tag{7.68}$$

A simple example of this case is the first-difference operation $H(z) = 1 - z^{-1}$, which has the frequency response

$$H(e^{j\Omega}) = 2je^{-j\Omega/2}\sin\frac{\Omega}{2}. \tag{7.69}$$

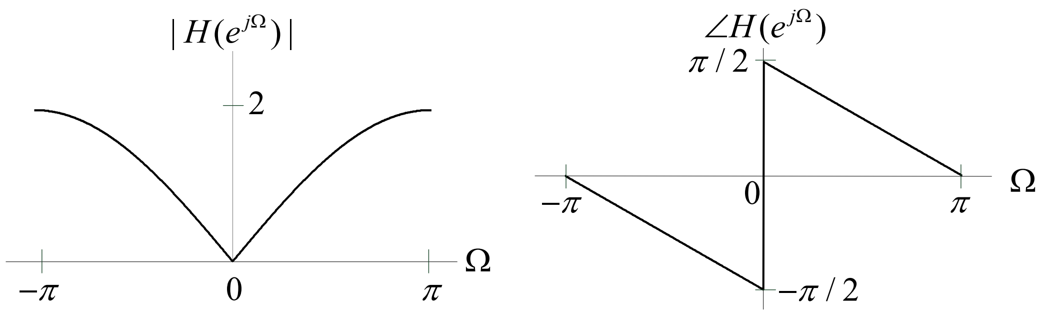
The corresponding magnitude and phase responses are then

$$|H(e^{j\Omega})| = 2|\sin\frac{\Omega}{2}| \tag{7.70}$$

and

$$\angle H(e^{j\Omega}) = \begin{cases} \pi/2 - \Omega/2, & 0 < \Omega \le \pi, \\ -\pi/2 - \Omega/2, & -\pi \le \Omega < 0, \end{cases} \tag{7.71}$$

As shown in Fig.7.13. Note, in particular, the phase discontinuity of $-\pi$ radians at $\Omega = 0$ due to the real factor $R(\Omega) = 2\sin\Omega/2$, which changes sign at $\Omega = 0$.



■**Figure 7.13** Magnitude and phase responses for $H(z) = 1 - z^{-1}$.

***Example* 7.32**:

A crude but common technique for smoothing a noisy data sequence is to average the sequence over *M+1* adjacent data samples, that is,

$$y[n] = \frac{1}{M+1}\sum_{k=0}^{M} x[n-k]. \tag{7.72}$$

Hence, $y[n]$ is computed as the average of $x[n]$ and M preceding samples $x[n-1]$, $x[n-2]$, …, $x[n-M]$. The corresponding impulse response is thus

$$h[n] = \frac{1}{M+1}\sum_{k=0}^{M} \delta[n-k] = \frac{1}{M+1}(u[n]-u[n-M-1]), \tag{7.73}$$

Implying the system function

$$H(z) = \frac{1}{M+1}\sum_{k=0}^{M} z^{-k} = \frac{1-z^{-(M+1)}}{(M+1)(1-z^{-1})}. \tag{7.74}$$

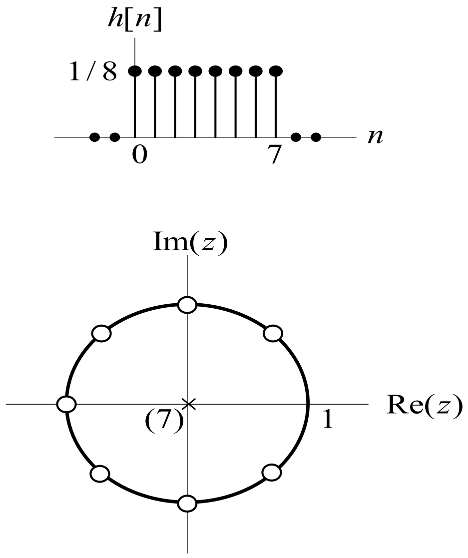
The zeros of $H(z)$ occur at values of z satisfying

$$z^{-(M+1)} = 1 \tag{7.75}$$

And thus equation the (M+1)st roots of unity, that is,

$$z_k = e^{j2\pi k/(M+1)}, \quad k = 1,\ 2,\ ...,\ M. \tag{7.76}$$

The zeros at z = 1 for k = 0 is not included in Eq. (7.76) because, as seen from Eq. (7.74), this zero is canceled by a pole at z = 1.



■**Figure 7.14** Impulse response and pole/zero plot for simple-averaging filter.

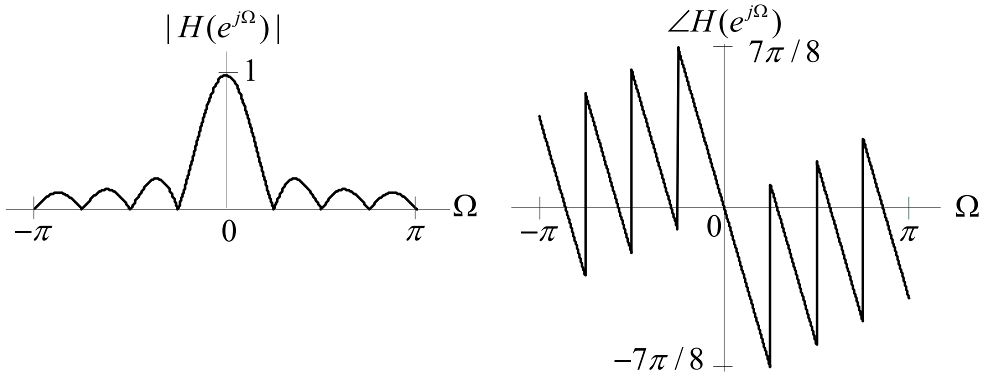
Therefore $H(z)$ has M zeros on the unit circle spaced by $2\pi/(M+1)$ radians and M poles at z = 0, as illustrated in Fig. 7.14 for M = 7. Note that M zeros are expected since this is an Mth-order FIR filter, and M poles at z = 0 result from the fact that the filter is causal. Also, since $h[n]$ has even symmetry about its midpoint, this is a linear-phase FIR filter.

To determine the magnitude and phase responses, we set $z = e^{j\Omega}$ in Eq.

(7.74) to produce

$$
\begin{aligned}
H(e^{j\Omega}) &= \frac{1-e^{-j\Omega(M+1)}}{(M+1)(1-e^{-j\Omega})} \\
&= \frac{e^{-j\Omega(M+1)/2}(e^{j\Omega(M+1)/2}-e^{-j\Omega(M+1)/2})}{(M+1)e^{-j\Omega/2}(e^{j\Omega/2}-e^{-j\Omega/2})} \\
&= \frac{e^{-j\Omega M/2}\sin[e^{-j\Omega(M+1)/2}(M+1)/2]}{(M+1)\sin(\Omega/2)} \\
&= e^{-j\Omega M/2}R(\Omega),
\end{aligned}
\tag{7.77}
$$

which is consistent with Eq. (7.65). Fig. 7.15 shows $|H(e^{j\Omega})|$ and $\angle H(e^{j\Omega})$ for M = 7. Hence the simple-average filter defined by Eq. (7.72) has a lowpass response and a bandwidth of about $\pi/(M+1)$. Note that the zeros on the unit circle in the z plane produce zeros of transmission in $|H(e^{j\Omega})|$ at $\Omega = \pm 2\pi k/(M+1)$, k = 1, 2, …, M, and that phase discontinuities of π radians occur in $\angle H(e^{j\Omega})$ at the same frequencies.



■**Figure 7.15** Magnitude and phase responses for simple-averaging filter.

7-7 The Unilateral *z* Transform

The z transform defined in Eq. (7.4) is sometimes referred to as the two-sided or *bilateral z transform* (*BZT*) to distinguish it from the one-sided or *unilateral z transform* (*UZT*) defined by

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{7.78}$$

The UZT is useful for calculating the response of a causal system to a causal input when the system is described by a linear difference equation with constant coefficients but nonzero initial conditions. That is, the system need not be at initial rest. Specifically, the zero-input response $y_{zi}[n]$, as well as the zero-state response $y_{zs}[n]$, is readily determined using the UZT. Note that such analysis is not anticipated by the relationship $Y(z) = H(z)X(z)$ using the BZT because this assumes that the system is LTI, not merely incrementally linear. (On the other hand, if the nonzero initial conditions are replaced by an equivalent nonzero input $x[n]$ for $-\infty < n < 0$, then the BZT can be employed.)

The basic properties of the UZT that are useful in this application relate to the transforms of the delayed signals $x[n-k]$ and are listed in Table 7.2. These properties may be derived as follows: Computing the UZT of the unit delay $x[n-1]$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} x[n-1]z^{-n} &= x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n} \\ &= x[-1] + z^{-1}\sum_{m=0}^{\infty} x[m]z^{-m} \\ &= x[-1] + z^{-1}X(z),\end{aligned} \tag{7.79}$$

as indicated in Table 7.2. Likewise,

$$\begin{aligned}\sum_{n=0}^{\infty} x[n-2]z^{-n} &= x[-2] + \sum_{n=1}^{\infty} x[n-2]z^{-n} \\ &= x[-2] + z^{-1}\sum_{m=0}^{\infty} x[m-1]z^{-m} \\ &= x[-2] + z^{-1}x[-1] + z^{-2}X(z),\end{aligned} \tag{7.80}$$

and so forth.

***Example* 7.33:**

A discrete-time system described by the linear difference equation

$$y[n] - ay[n-1] = x[n] = b^n u[n],$$

with $y[-1] = Y_I$. Applying the UZT to both sides of this equation, we obtain

$$Y(z) - az^{-1}Y(z) - ay[-1] = \frac{1}{1-bz^{-1}}$$

or

$$(1-az^{-1})Y(z) - aY_I = \frac{1}{1-bz^{-1}},$$

and thus

$$Y(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} + \frac{aY_I}{1-az^{-1}},$$

from which

$$y[n] = \frac{b^{n+1} - a^{n+1}}{b-a} + Y_I a^{n+1}, \qquad n \geq 0 .$$

In particular, note that

$$y_{zs}[n] = \frac{b^{n+1} - a^{n+1}}{b-a},$$

while

$$y_{zi}[n] = Y_I a^{n+1}.$$

■

***Example* 7.34:**

The response of an all-pole discrete-time system with zero input for $n \geq 0$, but nonzero initial conditions, can be modeled as the impulse response of a pole/zero system at initial rest. To see this, consider the second-order difference equation

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = x[n], \tag{7.81}$$

with $x[n] = 0$, $n \geq 0$, and initial conditions $y[-1] = Y_{I1}$ and $y[-2] = Y_{I2}$, corresponding to the LTI system

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}},$$

with an unknown input for $n < 0$. Taking the UZT of both sides of Eq. (7.81), we obtain

$$Y(z) + a_1\{z^{-1}Y(z) + y[-1]\} + a_2\{z^{-2}Y(z) + z^{-1}y[-1] + y[-2]\} = 0$$

or

$$Y(z)[1 + a_1 z^{-1} + a_2 z^{-2}] = -[a_1 Y_{I1} + a_2 Y_{I2}] - a_2 Y_{I1} z^{-1},$$

from which

$$Y(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}},$$

where $b_0 = -[a_1 Y_{I1} + a_2 Y_{I2}]$ and $b_1 = -a_2 Y_{I1}$. Hence $Y(z)$ has the form of a system function with the same poles as $H(z)$, plus a zero at $z = -b_1 / b_0$, and $y[n]$ can be thought of as the corresponding impulse response.

■

7-8 Structures for Discrete-Time Filters

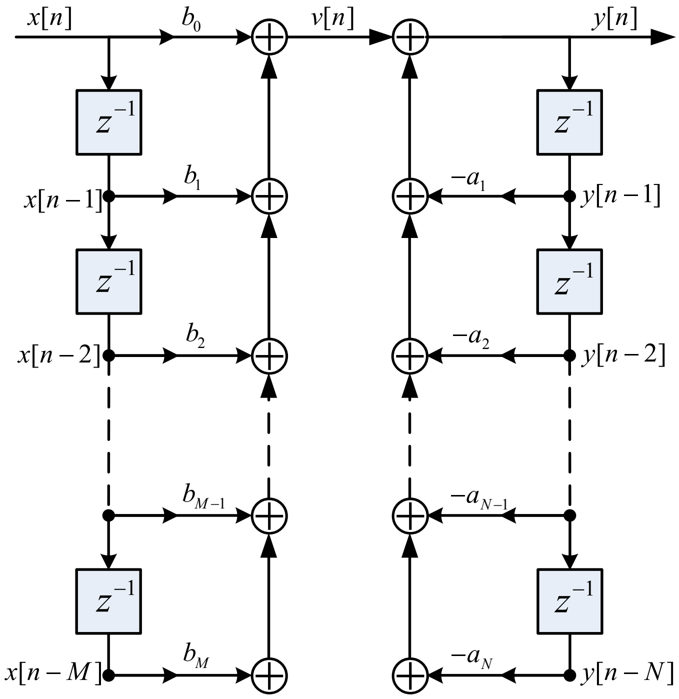
The structure corresponding directly to the general difference equation in Eq. (7.42) was called the *direct form* and is shown in Fig. 7.16, with z^{-1} denoting each unit delay.

Note that the structure in Fig. 7.16 consists effectively of the cascade of two subsystems. The first subsystem corresponds to the nonrecursive difference equation

$$v[n] = \sum_{k=0}^{M} b_k x[n-k] \tag{7.82}$$

and is thus FIR because its system function is

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k}. \tag{7.83}$$



■Figure 7.16 General discrete-time direct form structure.

In general, the second subsystem is IIR because it implements the recursive difference equation

$$y[n] = v[n] - \sum_{k=1}^{N} y[n-k] \tag{7.84}$$

and has the system function

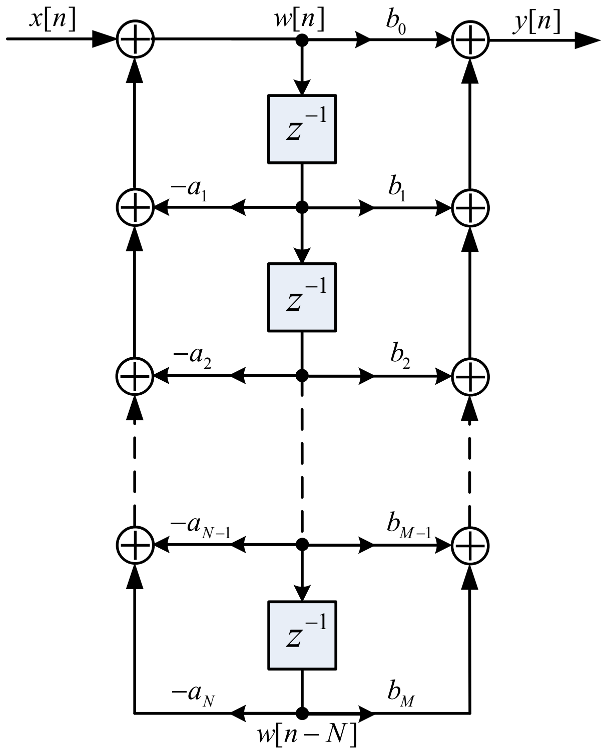
$$H_2(z) = \frac{1}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{7.85}$$

Hence, the system function of the total system can be represented by

$$H(z) = H_1(z)H_2(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{7.86}$$

as Eq. (7.43).

By reversing the order of $H_1(z)$ and $H_2(z)$ and eliminating the redundant delays, we produce the canonical *direct form II*, shown in Fig. 7.17.



■**Figure 7.17** General discrete-time direct form II structure.

***Table* 7.1** Common z-transform pairs:

Sequence	z-transform	ROC
$\delta[n]$	1	all z
$\delta[n-m],\quad m>0$	z^{-m}	$\lvert z\rvert > 0$
$\delta[n+m],\quad m>0$	z^{m}	$\lvert z\rvert < \infty$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{\left(1-az^{-1}\right)^2}$	$\lvert z\rvert > \lvert a\rvert$
$\cos[n\theta]u[n]$	$\frac{1-z^{-1}\cos\theta}{1-2z^{-1}\cos\theta+z^{-2}}$	$\lvert z\rvert > 1$
$\sin[n\theta]u[n]$	$\frac{1-z^{-1}\sin\theta}{1-2z^{-1}\cos\theta+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos[n\theta]u[n]$	$\frac{1-az^{-1}\cos\theta}{1-2az^{-1}\cos\theta+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n\sin[n\theta]u[n]$	$\frac{1-az^{-1}\sin\theta}{1-2az^{-1}\cos\theta+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

***Table* 7.2** *Unilateral z Transforms*

Delayed signal	Transform
$x[n-1]$	$z^{-1}X(z)+x[-1]$
$x[n-2]$	$z^{-2}X(z)+x[-1]z^{-1}+x[-2]$
$x[n-3]$	$z^{-3}X(z)+x[-1]z^{-2}+x[-2]z^{-1}+x[-3]$
$x[n-k]$	$z^{-k}X(z)+x[-1]z^{-(k-1)}+\cdots+x[-(k-1)]z^{-1}+x[-k]$

References:

[1] V. Oppenheim, A. S. Willsky, and I. T. Young, *Signals and Systems*, Englewood Cliffs, NJ: Prentice-Hall, 1983.

[2] S. Haykin and B. Van Veen, *Signals and Systems*, 2nd Ed., Hoboken, NJ: John Wiley & Sons, 2003.

[3] Leland B. Jackson, *Signals, Systems, and Transforms*, NJ: Addison-Wesley, 1991.