Chapter 6 The Laplace Transform

6-1 Definition of the Laplace Transform

1. For the linear time-invariant system with impulse response $h(t)$, the output $y(t)$ corresponding to the input of the form e^{st} is

$$y(t) = \underbrace{H(s)}_{\text{eigenvalue}} \cdot \underbrace{e^{st}}_{\text{eigenfunction}} \tag{6.1}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt \tag{6.2}$$

is referred to as the Laplace transform of $h(t)$.

$s = j\omega \Rightarrow H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$ is the Fourier transform of $h(t)$.

2. The Laplace transform of a general signal $x(t)$:

$$\begin{cases} X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt \equiv \mathcal{L}\{x(t)\} \\ x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \end{cases} \tag{6.3}$$

$$X(s)\Big|_{s=j\omega} = X(j\omega) = \mathcal{F}\{x(t)\} \tag{6.4}$$

3. The Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)$ after multiplication by a real exponential.

$$s = \sigma + j\omega$$

$$\begin{aligned} X(s) = X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \\ &= \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma t}\right]e^{-j\omega t}dt = \mathcal{F}\left\{x(t)e^{-\sigma t}\right\} \end{aligned} \tag{6.5}$$

***Example* 6.1**: $x(t) = e^{-at}u(t)$, $X(j\omega)$ converges for $a > 0$ $\left(\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty\right)$.

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = \frac{1}{j\omega + a},\ a > 0$$

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} e^{-(s+a)t}dt$$

With $s = \sigma + j\omega$, we have

$$X(\sigma + j\omega) = \int_0^{\infty} e^{-(\sigma+a)t} \cdot e^{-j\omega t} dt = \mathcal{F}\left\{e^{-(\sigma+a)t} u(t)\right\}$$
$$= \frac{1}{j\omega + (\sigma + a)},\ \sigma + a > 0, \text{ i.e., } \sigma > -a \text{ or } \operatorname{Re}\{s\} > -a$$
$$= \frac{1}{s+a},\ \operatorname{Re}\{s\} > -a$$

$$\Rightarrow X(s) = \frac{1}{s+a},\ \operatorname{Re}\{s\} > -a$$

Note:

- The Laplace transform converges for some values of $\operatorname{Re}\{s\}$, and not for the others.
- The existence of the Laplace transform does not imply the existence of the Fourier transform, e.g., $x(t) = e^{-at}u(t),\ a < 0$.

■

***Example* 6.2**: $x(t) = -e^{-at}u(-t)$

$$X(s) = -\int_{-\infty}^{\infty} e^{-at} e^{-st} u(-t)\, dt = -\int_{-\infty}^{0} e^{-(s+a)t} dt = \frac{1}{s+a}$$
$$\left(\begin{array}{l} = -\int_{-\infty}^{\infty} e^{-(\sigma+a)t} e^{-j\omega t} u(-t)\, dt = -\mathcal{F}\left\{e^{-(\sigma+a)t} u(-t)\right\} \\ (\because t < 0,\ \therefore \sigma + a < 0 \Rightarrow \sigma < -a) \end{array} \right)$$

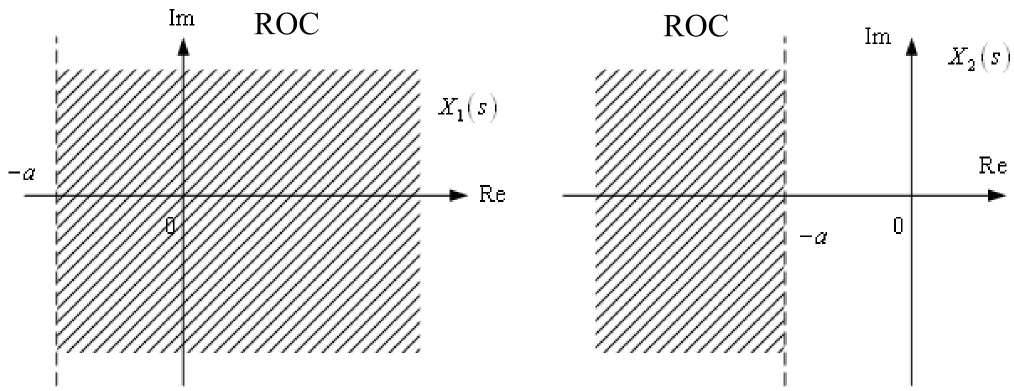
$\Rightarrow X(s)$ exists only for $\operatorname{Re}\{s\} < -a$.

Note:

- In specifying the Laplace transform of a signal, both the algebraic expression and the range of values for which this expression is valid are required.
- The range of values for which the Laplace transform exists is referred to as the region of convergence (ROC) of the Laplace transform.

■

***Example* 6.3**: Region of convergence (ROC) of $X_1(s) = \mathcal{L}\left\{e^{-at}u(t)\right\} = \frac{1}{s+a}$ and $X_2(s) = \mathcal{L}\left\{-e^{-at}u(-t)\right\} = \frac{1}{s+a}$.



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***Example* 6.4**: $x(t) = e^{-t}u(t) + e^{-2t}u(t)$

$$X(s) = \int_{-\infty}^{\infty} \left[e^{-t}u(t) + e^{-2t}u(t) \right] e^{-st} dt$$

$$= \int_{-\infty}^{\infty} e^{-t}e^{-st}u(t)dt + \int_{-\infty}^{\infty} e^{-2t}e^{-st}u(t)dt$$

$$= \frac{1}{s+1} + \frac{1}{s+2}, \ \mathrm{Re}\{s\} > -1$$

$$\left(\begin{aligned} &= \underbrace{\mathcal{L}\{e^{-t}u(t)\}}_{\frac{1}{s+1},\ \mathrm{Re}\{s\}>-1} + \underbrace{\mathcal{L}\{e^{-2t}u(t)\}}_{\frac{1}{s+2},\ \mathrm{Re}\{s\}>-2} \\ &\Rightarrow \mathcal{L}\{e^{-t}u(t) + e^{-2t}u(t)\} = \frac{1}{s+1} + \frac{1}{s+2}, \ \mathrm{Re}\{s\} > -1 \\ &\qquad = \frac{2s+3}{s^2+3s+2}, \ \mathrm{Re}\{s\} > -1 \\ &\qquad = \frac{N(s)}{D(s)} \begin{array}{l} \rightarrow \text{numerator polynomial} \\ \rightarrow \text{denominator polynomial} \end{array} \end{aligned} \right)$$

Note:

- Whenever $x(t)$ is a linear combination of real or complex exponentials, $X(s)$ can be expressed by $X(s) = N(s)/D(s)$, i.e., $X(s)$ is rational.
- The roots of the numerator polynomial (denominator polynomial) are referred to as the zeros (poles) of $X(s)$ since for those values of *s*, $X(s) = 0$ ($X(s) \rightarrow \infty$).

- $X(s) = N(s)/D(s)$

  [The order of $N(s)$]<[The order of $D(s)$]

  ⇒exist zeros at infinity $(s \to \infty,\ X(s) \to 0)$

  [The order of $N(s)$]>[The order of $D(s)$]

  ⇒exist poles at infinity $(s \to \infty,\ X(s) \to \infty)$

- The representation of $X(s) = N(s)/D(s)$ through its poles and zeros in the *s*-plane is referred to as the pole-zero diagram or the pole-zero plot.

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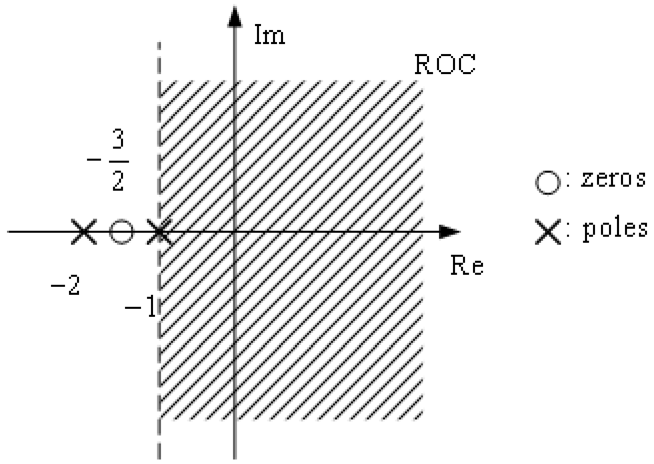
***Example* 6.5**: $X(s) = \dfrac{2s+3}{s^2+3s+2}$



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***Example* 6.6**: $x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1,\ \text{ROC: entire s plane}$$

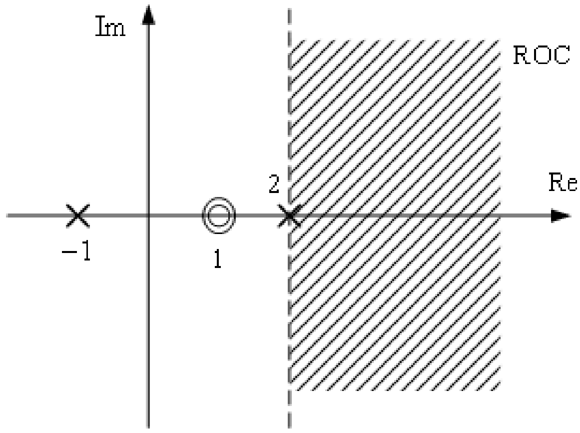
$$\frac{4}{3}e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{4}{3}\cdot\frac{1}{s+1},\ \ \text{Re}\{s\} > -1$$

$$\frac{1}{3}e^{2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{3}\cdot\frac{1}{s-2},\ \ \text{Re}\{s\} > 2$$

$$\left(\int_{-\infty}^{\infty} e^{2t}e^{-st}u(t)\,dt = \int_{-\infty}^{\infty} e^{(2-\sigma)t}e^{-j\omega t}u(t)\,dt,\ 2-\sigma < 0 \Rightarrow \sigma > 2 \Rightarrow \text{Re}\{s\} > 2\right)$$

$$X(s) = 1 - \frac{4}{3}\cdot\frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s-2},\ \ \mathrm{Re}\{s\} > 2$$
$$= \frac{(s-1)^2}{(s+1)(s-2)},\ \ \mathrm{Re}\{s\} > 2$$

We will refer to the order of pole or zero as the number of times it is repeated at a given location.



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6-2 The Region of Convergence for Laplace Transforms

Properties of ROC for Laplace transforms:

1. The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the *s*-plane.

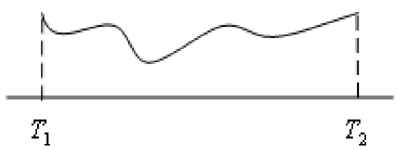
   ***Example* 6.7**: $X(s)$ converges only for $\mathrm{Re}\{s\} > a$ (or $\mathrm{Re}\{s\} < a$). The ROC depends only on the real part of *s*.

■

2. For rational Laplace transforms, the ROC does not contain any poles.

$$s = \text{pole} \ \Rightarrow X(s) \to \infty$$

3. If $x(t)$ is of finite duration and if there is at least one value of *s* for which the Laplace transform converges, then the ROC is the entire *s*-plane.



■**Figure 6.1** Finite-duration signal.

Proof:

Let $x(t)$ be zero outside the interval between T_1 and T_2. Then

$$X(s)=\int_{T_1}^{T_2} x(t)e^{-st}dt \tag{6.6}$$

Assume the line $\mathrm{Re}\{s\}=\sigma_0$ is in the ROC. Then

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma_0 t}dt<\infty \tag{6.7}$$

(1) For $\sigma_1>\sigma_0$,

$$\begin{aligned}\int_{T_1}^{T_2} |x(t)|e^{-\sigma_1 t}dt &= \int_{T_1}^{T_2} |x(t)|e^{-\sigma_0 t}e^{-(\sigma_1-\sigma_0)t}dt \\ &< e^{-(\sigma_1-\sigma_0)T_1}\int_{T_1}^{T_2} |x(t)|e^{-\sigma_0 t}dt<\infty\end{aligned} \tag{6.8}$$

($\because$ The maximum value of $e^{-(\sigma_1-\sigma_0)t}$ in the interval of integration is $e^{-(\sigma_1-\sigma_0)T_1}$.)

This implies the *s*-plane for $\mathrm{Re}\{s\}>\sigma_0$ is in the ROC.

(2) For $\sigma_2<\sigma_0$,

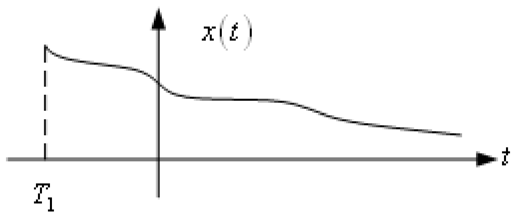
$$\begin{aligned}\int_{T_1}^{T_2} |x(t)|e^{-\sigma_2 t}dt &= \int_{T_1}^{T_2} |x(t)|e^{-\sigma_0 t}e^{-(\sigma_2-\sigma_0)t}dt \\ &< e^{-(\sigma_2-\sigma_0)T_2}\int_{T_1}^{T_2} |x(t)|e^{-\sigma_0 t}dt<\infty\end{aligned} \tag{6.9}$$

This implies the *s*-plane for $\mathrm{Re}\{s\}<\sigma_0$ is in the ROC.

From Eq. (6.1) and Eq. (6.2), we can see that the ROC of a finite-duration signal includes the entire *s*-plane.

■

4. If $x(t)$ is right-sided and if the line $\mathrm{Re}\{s\}=\sigma_0$ is in the ROC, then all values of *s* for which $\mathrm{Re}\{s\}>\sigma_0$ will also be in the ROC.



■ **Figure 6.2** Right-sided signal.

$x(t)=e^{t^2}u(t)$: there is no value of *s* for which the Laplace transform will converge.

Suppose the Laplace transform of $x(t)$ converges for some value of σ, denoted by σ_0. Then

$$\int_{-\infty}^{\infty} |x(t)| e^{-\sigma_0 t} dt < \infty \tag{6.10}$$

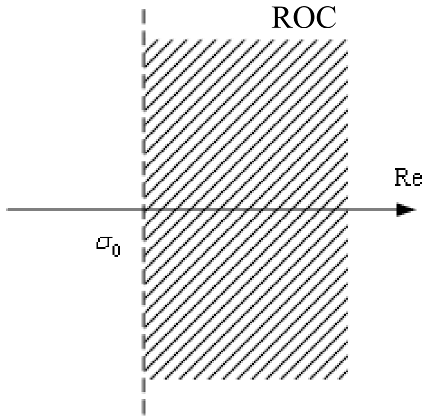
$$\Rightarrow \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} dt < \infty \tag{6.11}$$

For $\sigma_1 > \sigma_0$,

$$\Rightarrow \int_{T_1}^{\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1-\sigma_0)t} dt < e^{-(\sigma_1-\sigma_0)T_1} \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} dt < \infty \tag{6.12}$$

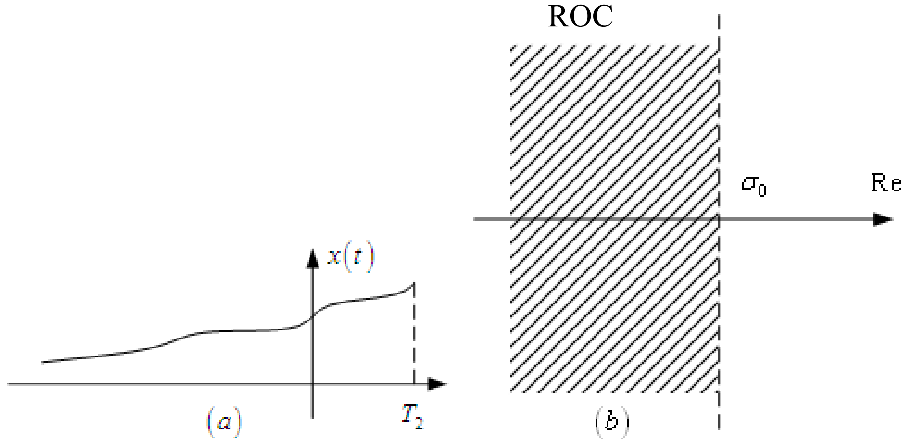
($\because$ The maximum value of $e^{-(\sigma_1-\sigma_0)t}$ in the interval of integration is $e^{-(\sigma_1-\sigma_0)T_1}$.)

ROC of a right-sided signal:



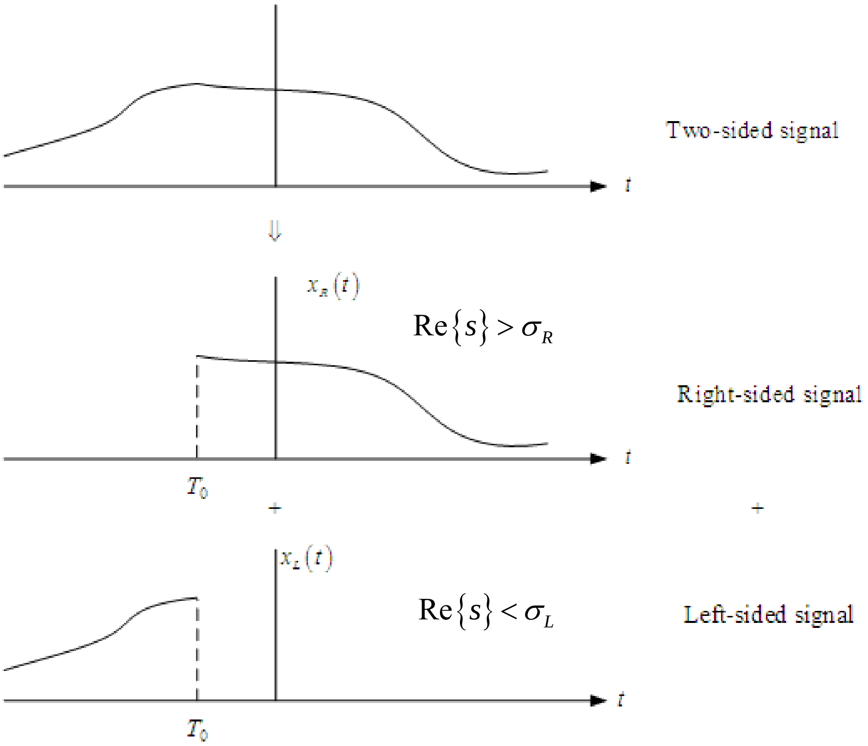
■ **Figure 6.3** ROC of a right-sided signal.

5. If $x(t)$ is left-sided and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathrm{Re}\{s\} < \sigma_0$ will also be in the ROC.
ROC of a left-sided signal:

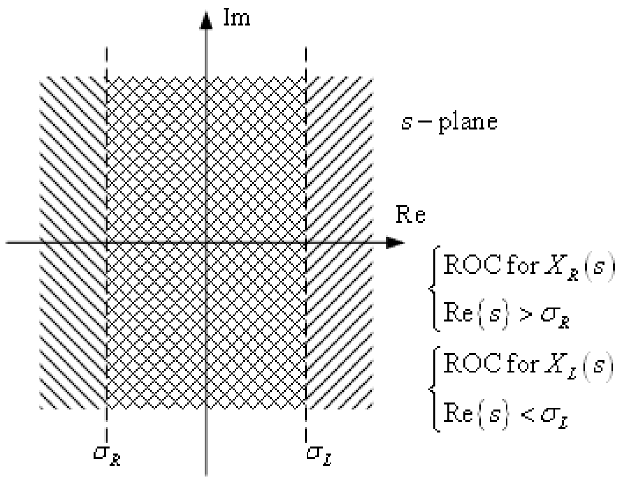


■ **Figure 6.4** (a) Left-sided signal; (b) ROC of a left-sided signal.

6. If $x(t)$ is two-sided and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane which includes the line $\mathrm{Re}\{s\} = \sigma_0$.



■ **Figure 6.5** Two-sided signal divided into the sum of a right-sided and left-sided signal.



■ **Figure 6.6** ROCs for $x_R(t)$ and for $x_L(t)$ assuming that they overlap. The overlap of the two ROCs is the ROC for $x(t) = x_R(t) + x_L(t)$.

Note:

- σ_L must be greater than σ_R; otherwise, the Laplace transform of $x(t)$ does not exist.

***Example* 6.8**:

$$x(t)=\begin{cases} e^{-at}, & 0<t<T \\ 0, & \text{otherwise} \end{cases}$$

$x(t)$ is a finite-duration sequence.

$\Rightarrow$the ROC of $X(s)$ in the entire *s*-plane

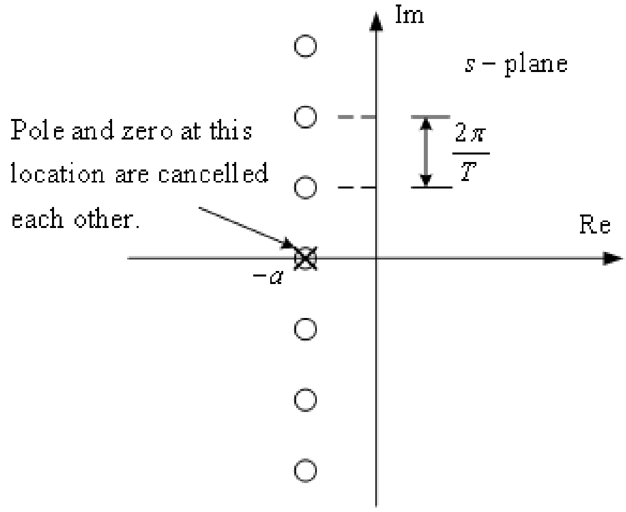
$$X(s)=\int_0^T e^{-at}e^{-st}dt=\frac{1}{s+a}\left[1-e^{-(s+a)T}\right]$$

$$\left(\begin{array}{l} s=-a\Rightarrow \left.\begin{array}{r} s+a\to 0 \\ 1-e^{-(s+a)T}\to 0 \end{array}\right\} \\ \therefore \lim_{s\to -a} X(s)=\lim_{s\to -a}\dfrac{\frac{d}{ds}\left[1-e^{-(s+a)T}\right]}{\frac{d}{ds}(s+a)}=\lim_{s\to -a} Te^{-aT}e^{-sT}=T \end{array}\right)$$

$$\begin{cases} X(s) \text{ has no poles.} \\ X(s) \text{ has an infinite number of zeros.} \end{cases}$$

$$1-e^{-(s+a)T}=0\Rightarrow (s+a)T=j2\pi k,\ k=0,\ \pm 1,\ \pm 2,\dots$$

$$\Rightarrow s=-a+j\frac{2\pi k}{T},\ k=0,\ \pm 1,\ \pm 2,\dots$$



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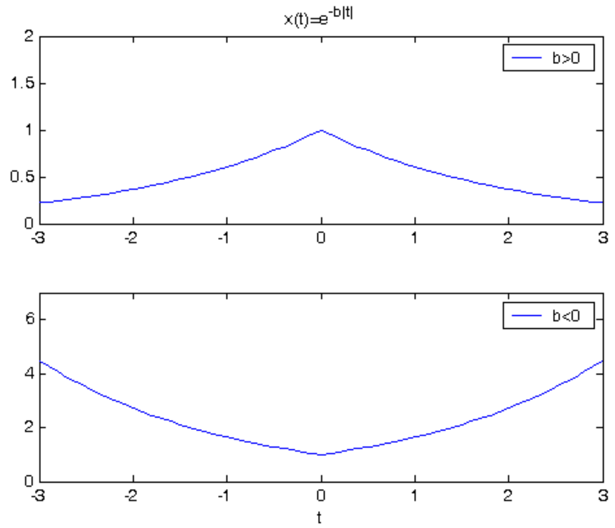
***Example* 6.9**: $x(t)=e^{-b|t|}$

$$x(t)=\underbrace{e^{-bt}u(t)}_{\text{right-sided}}+\underbrace{e^{bt}u(-t)}_{\text{left-sided}}$$

$$e^{-bt}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+b},\ \ \mathrm{Re}\{s\}>-b$$

$$e^{bt}u(\text{-}t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{-1}{s-b},\ \ \mathrm{Re}\{s\}<+b$$

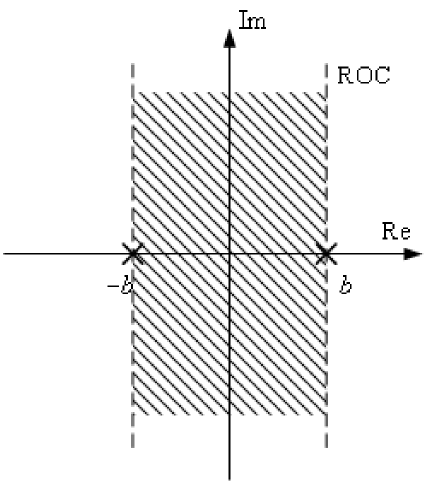


For b<0, there is no common region of convergence. $\Rightarrow x(t)$ has no Laplace transform if b<0.

For b> 0,

$$x(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+b}-\frac{1}{s-b}=\frac{-2b}{s^2-b^2},\ \ -b<\mathrm{Re}\{s\}<b$$

Pole-zero plot



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Summary for the ROC:

$$\text{Finite-duration signal} \rightarrow \begin{cases} \text{entire } s\text{-plane} \\ \text{does not exist} \end{cases}$$

$$\text{Right-sided signal} \rightarrow \begin{cases} \text{right-half } s\text{-plane} \\ \text{does not exist} \end{cases}$$

$$\text{Left-sided signal} \rightarrow \begin{cases} \text{left-half } s\text{-plane} \\ \text{does not exist} \end{cases}$$

$$\text{Two-sided signal} \rightarrow \begin{cases} \text{a strip} \\ \text{does not exist} \end{cases}$$

Note:

- The ROC is bounded by poles or extends to infinity.
- For a right-sided signal, the ROC is the region in the s-plane to the right of the rightmost pole.
- For a left-sided signal, the ROC is the region in the s-plane to the left of the leftmost pole.

6-3 The Inverse Laplace Transform

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} \left(x(t)e^{-\sigma t} \right) e^{-j\omega t} dt = \mathcal{F}\left\{ x(t)e^{-\sigma t} \right\} \tag{6.13}$$

$$\Rightarrow x(t)e^{-\sigma t} = \mathcal{F}^{-1}\left\{ X(\sigma + j\omega) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{6.14}$$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} e^{\sigma t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega \tag{6.15}$$

If we change the variable of integration from ω to s and use the fact that σ is constant so that $ds = jd\omega$, we obtain

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds \tag{6.16}$$

"The basic Inverse Laplace Transform."

Note:

- σ is any value in the ROC of $X(s)$.

***Example* 6.10**:

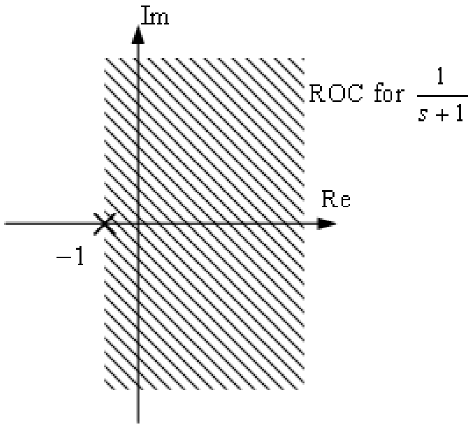
$$X(s)=\frac{1}{(s+1)(s+2)},\ \operatorname{Re}\{s\}>-1$$

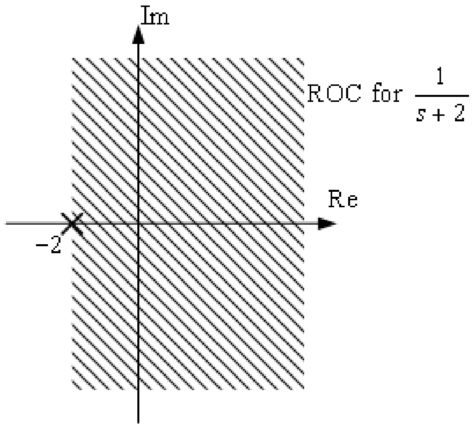
$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}=\frac{1}{s+1}-\frac{1}{s+2}$$

$$A=(s+1)X(s)\Big|_{s=-1}=1$$

$$B=(s+2)X(s)\Big|_{s=-2}=-1$$

Since the ROC for $X(s)$ is $\operatorname{Re}\{s\}>-1$, the ROC for the individual terms in the partial fraction includes $\operatorname{Re}\{s\}>-1$.





$$\Rightarrow \begin{matrix} e^{-t}u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s+1},\ \operatorname{Re}\{s\}>-1\ (\text{right-sided}) \\ e^{-2t}u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s+2},\ \operatorname{Re}\{s\}>-2\ (\text{right-sided}) \end{matrix}$$

$$\Rightarrow \left(e^{-t}-e^{-2t}\right)u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{(s+1)(s+2)},\ \operatorname{Re}\{s\}>-1$$

■

***Example* 6.11**:

$$X(s)=\frac{1}{(s+1)(s+2)},\ \operatorname{Re}\{s\}<-2\ (\text{left-sided})$$

$$=\frac{1}{(s+1)(s+2)}=\frac{1}{s+1}-\frac{1}{s+2}$$

$$x(t)=\left(-e^{-t}+e^{-2t}\right)u(-t)\xleftrightarrow{\mathcal{L}}\frac{1}{(s+1)(s+2)},\ \operatorname{Re}\{s\}<-2$$

$$\because e^{-bt}u(-t)\xleftrightarrow{\mathcal{L}}\frac{-1}{s-b},\ \operatorname{Re}\{s\}<b$$

$$\therefore \begin{cases} e^{-t}u(-t)\xleftrightarrow{\mathcal{L}}\frac{-1}{s+1},\ \operatorname{Re}\{s\}<-1 \\ e^{-2t}u(-t)\xleftrightarrow{\mathcal{L}}\frac{-1}{s+2},\ \operatorname{Re}\{s\}<-2 \end{cases}$$

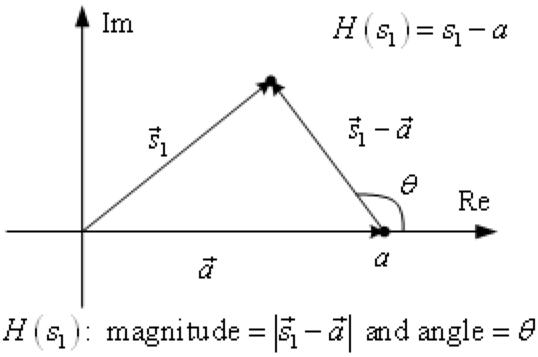
***Example* 6.12**: $X(s)=\frac{1}{(s+1)(s+2)},\ -2<\text{Re}\{s\}<-1$

$$x(t)=-e^{-t}u(-t)-e^{-2t}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{(s+1)(s+2)},\ -2<\text{Re}\{s\}<-1$$

■

6-4 Geometric Evaluation of the Fourier Transform from the Pole-Zero Plot

1. Consider $H(s_1)=s_1-a$



■ **Figure 6.7** Complex plane representation of the vectors $\vec{s}_1$, $\vec{a}$, and $(\vec{s}_1-\vec{a})$ representing the complex numbers s_1, a, and (s_1-a) respectively.

For $H(s)=1/(s-a)$, the denominator can be represented by the same vector as above and the value of $H(s_1)$ has a magnitude that is the reciprocal of the vector $(\vec{s}_1-\vec{a})$.

2. A system function described by a linear differential equation with constant coefficients is a rational fraction of the form

$$H(s)=\frac{B(s)}{A(s)}=\frac{\sum_{k=0}^{M}b_k s^k}{\sum_{k=0}^{N}a_k s^k}. \tag{6.17}$$

The numerator and denominator polynomials $B(s)$ and $A(s)$ can always be factored into products of *M* and *N* first-order terms, respectively, and thus $H(s)$ can also be written as

$$H(s) = C\frac{\prod_{k=1}^{M}(s-z_k)}{\prod_{k=1}^{N}(s-p_k)}, \tag{6.18}$$

where $C = b_M / a_N$, and z_k and p_k are the zeros and poles, respectively, of $H(s)$. Therefore the corresponding frequency response $H(j\omega)$ is simply

$$H(j\omega) = C\frac{\prod_{k=1}^{M}(j\omega-z_k)}{\prod_{k=1}^{N}(j\omega-p_k)}. \tag{6.19}$$

For a given frequency ω, each complex-valued numerator term $(j\omega - z_k)$ in Eq. (6.19) can be thought of as a vector in the complex (s) plane from the zero z_k to the point $j\omega$ on the imaginary axis; and likewise, each denominator term $(j\omega - p_k)$ is effectively a vector from the pole p_k to the point $j\omega$. Hence, via Eq. (6.19), the magnitude and phase responses of the system can be determined by the lengths and angles, respectively, of these pole/zero vectors as functions of the variable ω.

To evaluate $H(s)$ at $s = s_1$, each term in the product is represented by a vector from the zero or pole to the point s_1:

(1) The magnitude response $|H(j\omega)|$ is

$$\frac{|C|(\text{the product of the lengths of the zero vectors})}{(\text{the product of the lengths of the pole vectors})}$$

$$= |C|\frac{\prod_{k=1}^{M}|j\omega-z_k|}{\prod_{k=1}^{N}|j\omega-p_k|}. \tag{6.20}$$

The phase response $\angle H(j\omega)$ is

(the sum of the angles of the zero vectors)

$-$(the sum of the angles of the pole vectors) (6.21)

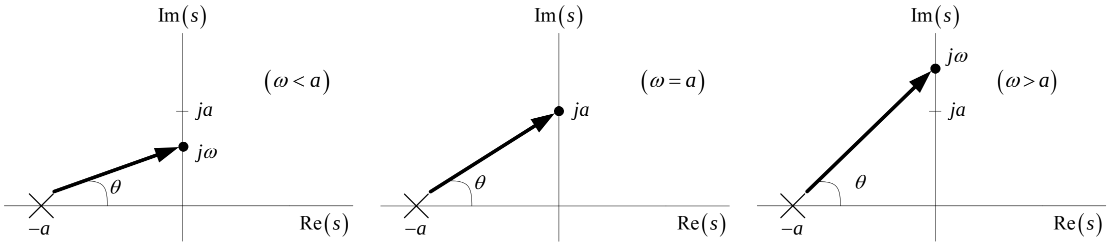
$$= \sum_{k=1}^{M}\angle(j\omega-z_k) - \sum_{k=1}^{n}\angle(j\omega-p_k).$$

If C is negative, then in additional angle of π would be included.

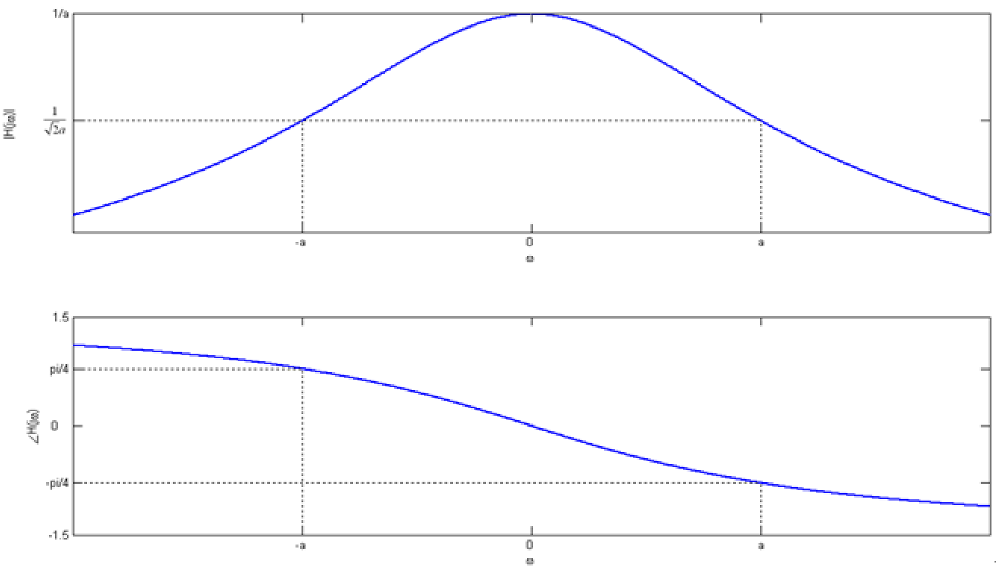
In many cases, this simple geometric approach is sufficient to enable us to sketch $|H(j\omega)|$ and $\angle H(j\omega)$ with adequate directly from the pole/zero diagram for $H(s)$ without having to evaluate $H(j\omega)$ itself.

***Example* 6.13**: Consider the simple first order response $h(t)=e^{-at}u(t)$, $a>0$, with system function $H(s)=1/(s+a)$ and frequency response $H(j\omega)=1/(j\omega+a)$.

The corresponding pole/zero plot showing the denominator vector $(j\omega+a)$ is drawn in the following with three cases:



The resulting sketch of $|H(j\omega)|$ and $\angle H(j\omega)$ are shown in the following.

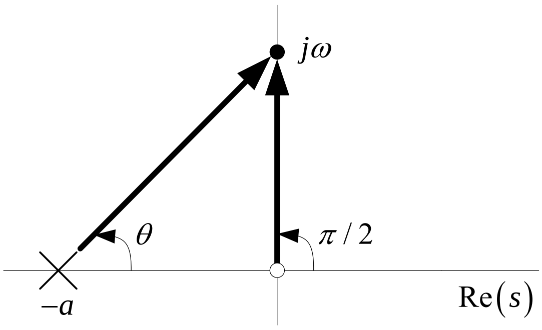


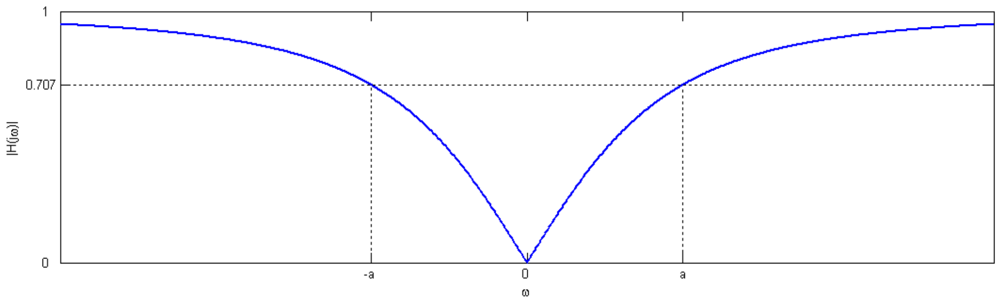
Note:

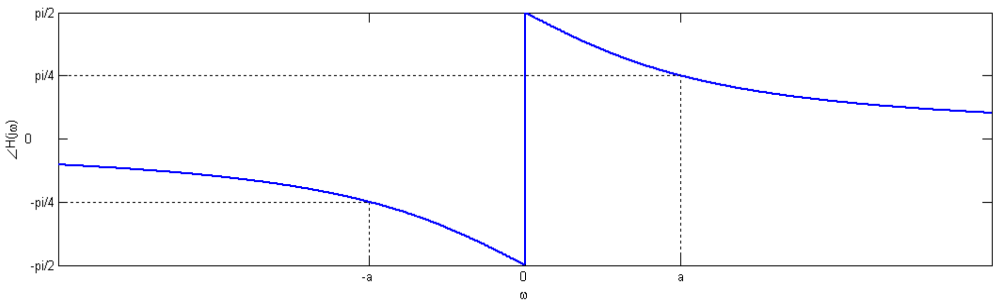
- $|j\omega+a|$ increases monotonically as increases from $\omega=0$.
- The 3dB point will be at $|\omega|=a$.
- $\angle H(j\omega)$ decreases monotonically with ω from a value at $\omega=0$ and approaches an asymptotic value of $-\pi/2$ for $\omega>>a$. Moreover, $\angle H(j\omega)$ is an odd function of ω.

■

***Example* 6.14**: Adding a zero at $s = 0$ to the preceding example, we have $H(s) = s/(s+a)$, $a > 0$, and thus $H(j\omega) = j\omega/(j\omega + a)$. The corresponding pole/zero diagram, magnitude response, and phase response are shown in the following.





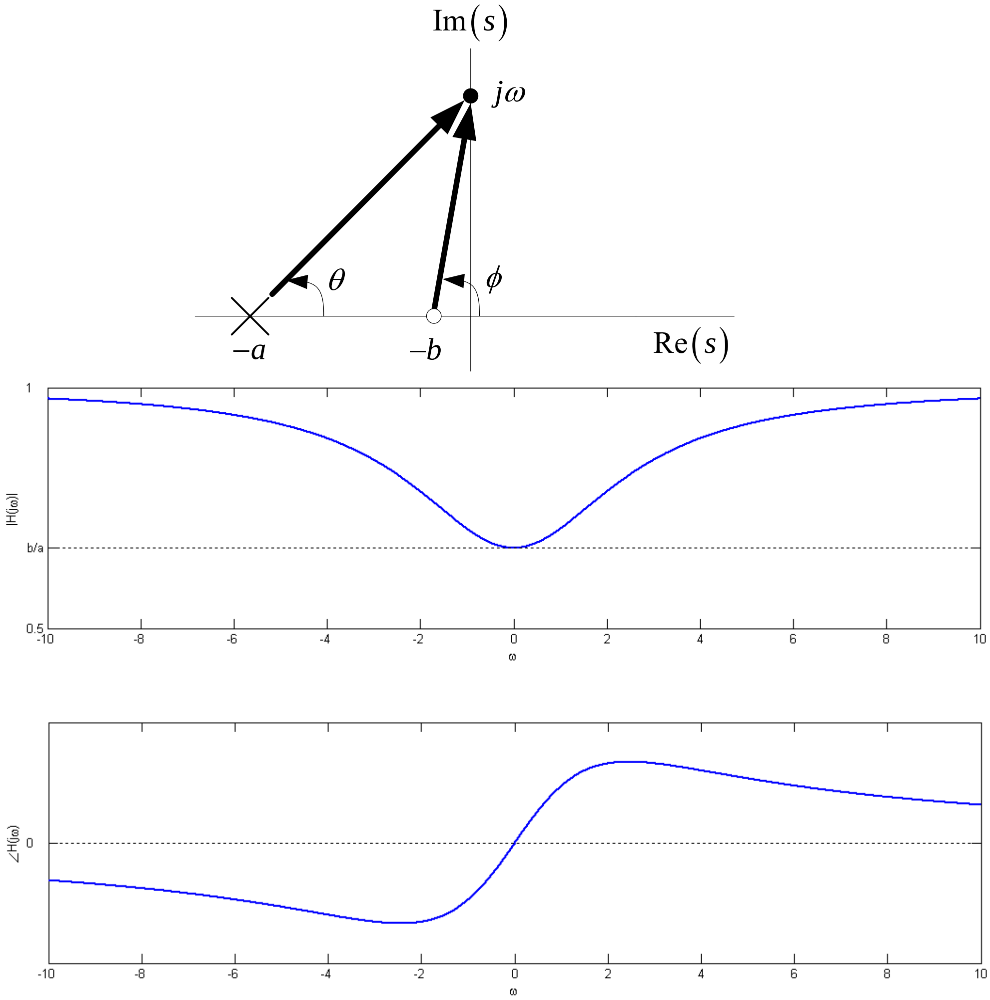


Note:

- $|H(j\omega)|$ is zero at $\omega = 0$ and decreases as $|\omega|$ increases. For $\omega >> a$, $|H(j\omega)| \approx 1$.
- The 3dB point will be at $\omega = a$.
- $\angle H(j\omega)$ discontinuity at $\omega = 0$ due to there is a zero on the $j\omega$ axis.

■

***Example* 6.15**: Letting $H(s)=(s+b)/(s+a)$, with $0<b<a$, we have the frequency response $H(j\omega)=(j\omega+b)/(j\omega+a)$. This case is almost the same as Example 6.14 except that the zero is moved to the left.



Note:

- $|H(j\omega)|=b/a<1$ at $\omega=0$ and increases as $|\omega|$ increases. For $\omega>>a$, $|H(j\omega)|\approx 1$. Therefore $|H(j\omega)|$ is again high-pass, but with less attenuation near $\omega=0$.

6-5 Properties of the Laplace Transform

1. Linearity

$$x_1(t)\overset{\mathcal{L}}{\longleftrightarrow}X_1(s),\ \text{with ROC}=R_1 \tag{6.22}$$

$$x_2(t)\overset{\mathcal{L}}{\longleftrightarrow}X_2(s),\ \text{with ROC}=R_2 \tag{6.23}$$

$$\Rightarrow ax_1(t)+bx_2(t)\overset{\mathcal{L}}{\longleftrightarrow}aX_1(s)+bX_2(s)\ \text{with ROC containing } R_1\cap R_2 \tag{6.24}$$

Note:

- The ROC can also be larger than $R_1\cap R_2$.

***Example* 6.16**: $X_1(s)=\frac{1}{s+1},\ \mathrm{Re}\{s\}>-1$

$$X_2(s)=\frac{1}{(s+1)(s+2)},\ \mathrm{Re}\{s\}>-1$$

$$x(t)=x_1(t)-x_2(t)$$

$$X(s)=X_1(s)-X_2(s)=\frac{1}{s+1}-\frac{1}{(s+1)(s+2)}$$

$$=\frac{s+1}{(s+1)(s+2)}=\frac{1}{s+2},\ \mathrm{Re}\{s\}>-2$$

In the combination of $x_1(t)$ and $x_2(t)$, the pole at $s=-1$ is cancelled by a zero at $s=-1$. ⇒"pole-zero cancellation"

2. Time shifting

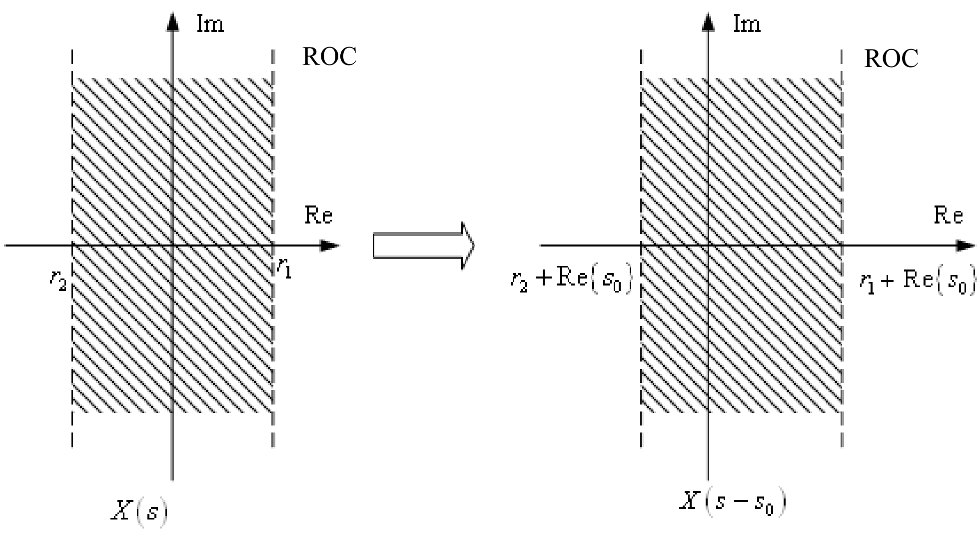
$$x(t)\xleftrightarrow{\mathcal{L}}X(s),\ \text{with ROC}=R \tag{6.25}$$

$$\Rightarrow x(t-t_0)\xleftrightarrow{\mathcal{L}}e^{-st_0}X(s),\ \text{with ROC}=R$$
$$\left(\int_{-\infty}^{\infty}x(t-t_0)e^{-st}dt=e^{-st_0}X(s)\right) \tag{6.26}$$

3. Shifting in the *s*-plane

$$x(t)\xleftrightarrow{\mathcal{L}}X(s),\ \text{with ROC}=R \tag{6.27}$$

$$\Rightarrow e^{s_0t}x(t)\xleftrightarrow{\mathcal{L}}X(s-s_0),\ \text{with ROC}=R+\mathrm{Re}\{s_0\}$$
$$(\because \text{pole } s_p\to s_p+s_0) \tag{6.28}$$

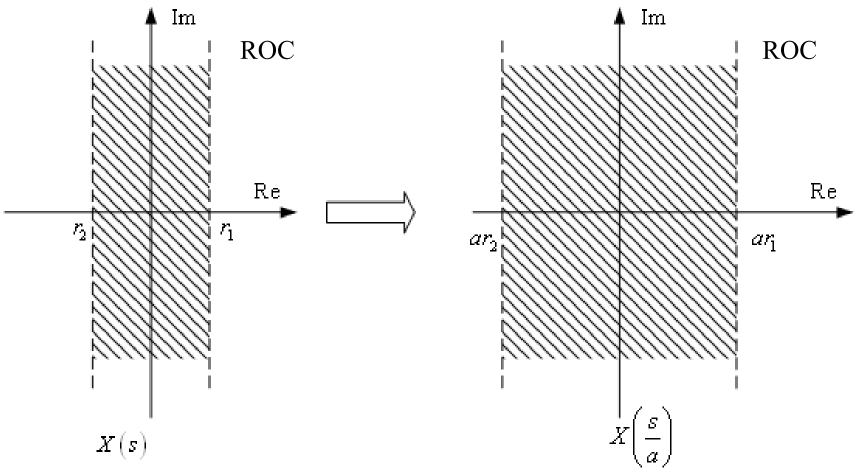


■ **Figure 6.8** Effects on the ROC of shifting in the *s*-domain.

4. Time scaling

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \text{ with ROC} = R \tag{6.29}$$

$$\Rightarrow x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right), \text{ with ROC} = aR \quad (\because \text{pole } s_p \to as_p) \tag{6.30}$$



■ **Figure 6.9** Effects on the ROC of time scaling.

$a > 0$,

$$\begin{aligned}\mathcal{L}\{x(at)\} &= \int_{-\infty}^{\infty} x(at)e^{-st}dt = \int_{-\infty}^{\infty} x(t')e^{-\frac{s}{a}t'}\frac{1}{a}dt' \\ &= \frac{1}{a}X\left(\frac{s}{a}\right) = \frac{1}{|a|}X\left(\frac{s}{a}\right)\end{aligned} \tag{6.31}$$

$a < 0$,

$$\begin{aligned}\mathcal{L}\{x(at)\} &= \int_{-\infty}^{\infty} x(at)e^{-st}dt = \int_{\infty}^{-\infty} x(t')e^{-\frac{s}{a}t'}\frac{1}{a}dt' \\ &= -\frac{1}{a}\int_{-\infty}^{\infty} x(t')e^{-\frac{s}{a}t'}dt' = \frac{1}{|a|}X\left(\frac{s}{a}\right)\end{aligned} \tag{6.32}$$

5. Convolution property

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s), \text{ with ROC} = R_1 \tag{6.33}$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s), \text{ with ROC} = R_2 \tag{6.34}$$

$$\Rightarrow x(t) = x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = X_1(s)X_2(s), \text{ with ROC containing } R_1 \cap R_2 \tag{6.35}$$

Note:

- The ROC of $X(s)$ may be larger than $R_1 \cap R_2$ if pole-zero cancellation occurs in the product.

***Example* 6.17**:

$$X_1(s) = \frac{s+1}{s+2},\ \text{Re}\{s\} > -2$$

$$X_2(s) = \frac{s+2}{s+1},\ \text{Re}\{s\} > -1$$

Then $X(s) = X_1(s)X_2(s) = 1$, with ROC = entire *s*-plane.

6. Differentiation in the time domain

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s),\ \text{with ROC} = R \tag{6.36}$$

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s),\ \text{with ROC containing } R \tag{6.37}$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \Rightarrow \frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds \tag{6.38}$$

***Example* 6.18**: $x(t) = \frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right)$

$$e^{-3t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+3},\ \text{with ROC Re}\{s\} > -3$$

$$e^{-3(t-2)}u(t-2) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+3}e^{-2s},\ \text{with ROC Re}\{s\} > -3$$

$$x(t) = \frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \frac{s^2}{s+3}e^{-2s},\ \text{with ROC Re}\{s\} > -3$$

■

***Example* 6.19**: $X(s) = \frac{2s^3 - 9s^2 + 4s + 10}{s^2 - 3s - 4}$, with $\text{Re}\{s\} < -1$

$$\begin{array}{r|l} & 2s - 3 \\ \hline s^2 - 3s - 4 & 2s^3 - 9s^2 + 4s + 10 \\ & \underline{2s^3 - 6s^2 - 8s} \\ & -3s^2 + 12s + 10 \\ & \underline{-3s^2 + 9s + 12} \\ & 3s - 2 \end{array}$$

$$X(s) = 2s - 3 + \frac{1}{s+1} + \frac{2}{s-4},\ \text{with Re}\{s\} < -1$$

$$x(t) = 2\delta^{(1)}(t) - 3\delta(t) - e^{-t}u(-t) - 2e^{4t}u(-t)$$

■

Note:

- The ROC of $sX(s)$ includes the ROC of $X(s)$ and may be larger if $X(s)$ has a first order pole at $s=0$ which is cancelled by the multiplication by *s*.

7. Differentiation in the *s*-domain

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \text{ with ROC} = R \tag{6.39}$$

$$\Rightarrow -tx(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds}, \text{ with ROC} = R \tag{6.40}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{6.41}$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} \left[-tx(t)\right] e^{-st} dt \tag{6.42}$$

8. Integration in the time domain

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \text{ with ROC} = R \tag{6.43}$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau) d\tau \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)/s, \text{ with ROC containing } R \cap \{\operatorname{Re}\{s\} > 0\} \tag{6.44}$$

$$\int_{-\infty}^{t} x(\tau) d\tau = u(t) * x(t) \tag{6.45}$$

$$u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} s^{-1}, \text{ with ROC} = \operatorname{Re}\{s\} > 0$$

$$\left(\begin{array}{l} e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{s+a}, \text{ with ROC} = \operatorname{Re}\{s\} > -a \\ \text{Let } a = 0. \end{array} \right) \tag{6.46}$$

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \text{ with ROC} = R \tag{6.47}$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau) d\tau \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} X(s), \text{ with ROC containing } R \cap \{\operatorname{Re}\{s\} > 0\} \tag{6.48}$$

9. The initial and final value theorems

$x(t)=0$ for $t<0$ and $x(t)$ contains no impulses or higher-order singularities at the origin.

$$\begin{cases} x(0^+) = \lim_{s\to\infty} sX(s) \cdots\cdots\cdots \text{The initial value theorem} \\ \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \cdots\cdots\cdots \text{The final value theorem} \end{cases} \tag{6.49}$$

Proof:

Expanding $x(t)$ as a Taylor series at $t=0^+$,

$$x(t)=\left[x(0^+)+x^{(1)}(0^+)t+\cdots+x^{(n)}(0^+)\frac{t^n}{n!}+\cdots\right]u(t) \tag{6.50}$$

where $x^{(n)}(0^+)$ denotes the nth derivative of $x(t)$ evaluated at $t=0^+$.

$$u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s} \tag{6.51}$$

$$tu(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s^2} \tag{6.52}$$

$$\vdots$$

$$\frac{t^n}{n!}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s^n} \tag{6.53}$$

$$\Rightarrow \mathcal{L}\{x(t)\}=\frac{1}{s}x(0^+)+\frac{1}{s^2}x^{(1)}(0^+)+\cdots+\frac{1}{s^n}x^{(n)}(0^+)+\cdots=X(s) \tag{6.54}$$

$$\Rightarrow sX(s)=x(0^+)+\frac{1}{s}x^{(1)}(0^+)+\cdots+\frac{1}{s^{n-1}}x^{(n)}(0^+)+\cdots \tag{6.55}$$

$$\Rightarrow \lim_{s\to\infty}sX(s)=x(0^+)\cdots\cdots\cdots\text{The initial value theorem} \tag{6.56}$$

Let us consider the limit of the integral $\int_{0^+}^{\infty}\frac{dx(t)}{dt}e^{-st}dt$ as s approach 0. We have

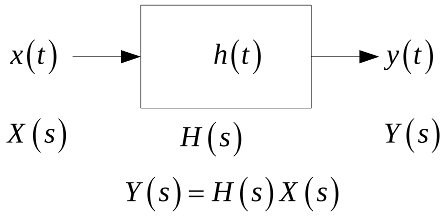
$$\lim_{s\to 0}\int_{0^+}^{\infty}\frac{dx(t)}{dt}e^{-st}dt=\int_{0^+}^{\infty}\frac{dx(t)}{dt}dt=x(t)\Big|_{0^+}^{\infty}=\lim_{t\to\infty}x(t)-x(0^+) \tag{6.57}$$

Also,

$$\begin{aligned}\lim_{s\to 0}\int_{0^+}^{\infty}\frac{dx(t)}{dt}e^{-st}dt&=\lim_{s\to 0}\left\{\left[x(t)e^{-st}\right]\Big|_{0^+}^{\infty}-\int_{0^+}^{\infty}x(t)\frac{de^{-st}}{dt}dt\right\}\\&=\lim_{s\to 0}\left\{\left[x(t)e^{-st}\right]\Big|_{0^+}^{\infty}-\int_{0^+}^{\infty}x(t)(-s)e^{-st}dt\right\}\\&=\lim_{s\to 0}\left[-x(0^+)+sX(s)\right]=-x(0^+)+\lim_{s\to 0}sX(s)\end{aligned} \tag{6.58}$$

$$\Rightarrow \lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)\cdots\cdots\cdots\text{The final value theorem} \tag{6.59}$$

6-6 Analysis and Characterization of LTI Systems Using the Laplace Transform



■ **Figure 6.10** Block diagram of a system.

$H(s)$: the system function or transfer function

$s = j\omega$, $H(j\omega)$ is the frequency response of the LTI system.

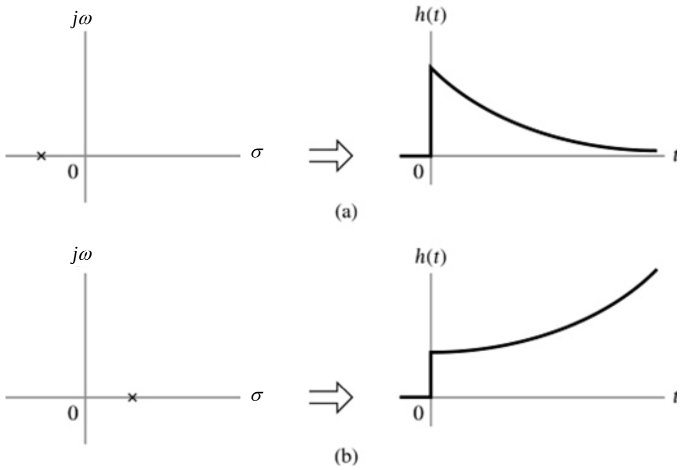
For a causal system, $h(t) = 0$ for $t < 0$ (Fig. 6.11).

$\Rightarrow h(t)$ is a right-sided signal.

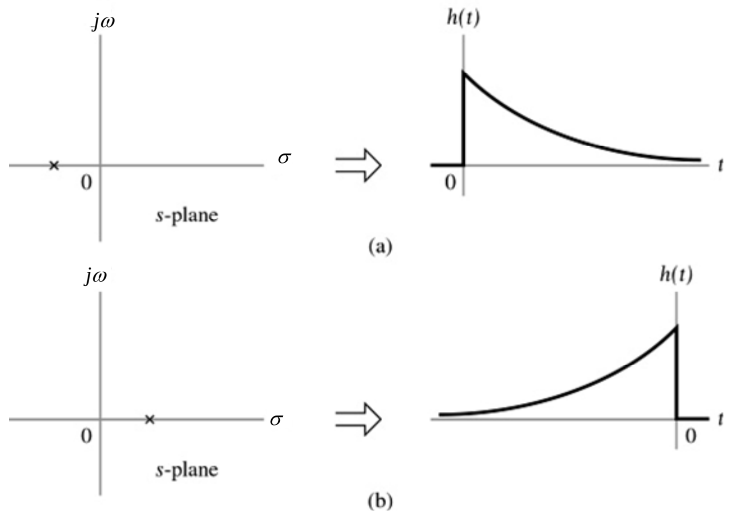
$\Rightarrow$ The ROC is the entire region in the *s*-plane to the right of the rightmost pole.

Note:

- Anticausal system $h(t)$
  $\Rightarrow$Its ROC is the region in the *s*-plane to the left of the leftmost pole.
- An ROC to the right of the rightmost pole does not guarantee that the system is causal, only that the impulse response is right-sided.
- The Fourier transform of the impulse response for a stable LTI system exists.
  $\Rightarrow$ For a stable system, the ROC of *H*(*s*) must include the $j\omega$-axis (Fig. 6.12).
- For a causal and stable LTI system with a rational system function, all poles must lie in the left half of the *s*-plane.
  causal $\rightarrow$ ROC is to the right of the rightmost pole.
  stable$\rightarrow$ ROC must include the $j\omega$-axis.



■ **Figure 6.11** The relationship between the locations of poles and the impulse response in a causal system. (a) A pole in the left half of the *s*-plane corresponds to an exponentially decaying impulse response. (b) A pole in the right half of the *s*-plane corresponds to an exponentially increasing impulse response. The system is unstable in this case.



■ **Figure 6.12** The relationship between the locations of poles and the impulse response in a stable system. (a) A pole in the left half of the s-plane corresponds to a right-sided impulse response. (b) A pole in the right half of the s-plane corresponds to an left-sided impulse response. In this case, the system is noncausal.

***Example* 6.20**: $h(t)=e^{-t}u(t)\Rightarrow H(s)=\dfrac{1}{s+1},\ \mathrm{Re}\{s\}>-1$

$\Rightarrow$ causal and stable

***Example* 6.21**: $H(s)=\dfrac{e^{s}}{s+1},\ \mathrm{Re}\{s\}>-1$

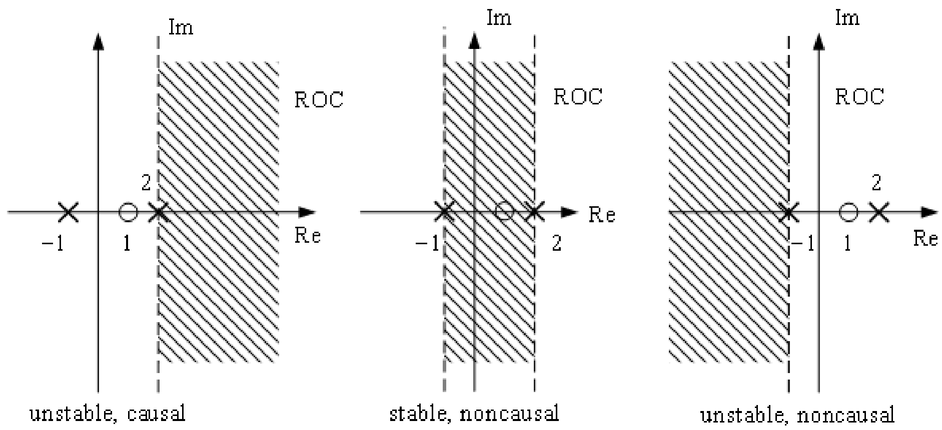
$$e^{-t}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+1},\ \mathrm{Re}\{s\}>-1$$

$$e^{-(t+1)}u(t+1)\overset{\mathcal{L}}{\longleftrightarrow}\frac{e^{s}}{s+1},\ \mathrm{Re}\{s\}>-1$$

$\Rightarrow h(t)=e^{-(t+1)}u(t+1)$, zero for $t<-1$ but not for $t<0$

$\Rightarrow$ not causal but is stable.

***Example* 6.22**: $H(s)=\dfrac{s-1}{(s+1)(s-2)}$



***Example* 6.23**: Inverse Laplace transform with stability and causality constraints

$$H(s) = \frac{2}{s+3} + \frac{1}{s-2}$$

If the system is stable, then the pole at $s = -3$ contributes a right-sided term to the impulse response, while the pole at $s = 2$ contributes a left-sided term.

$$h(t) = 2e^{-3t}u(t) - e^{2t}u(-t)$$

If the system is causal, then both poles must contribute right-sided terms to the impulse response

$$h(t) = 2e^{-3t}u(t) + e^{2t}u(t)$$

■

1. System characterized by linear constant-coefficient differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$
$$\updownarrow \mathcal{L} \qquad\qquad \updownarrow \mathcal{L} \tag{6.60}$$
$$\left\{\sum_{k=0}^{N} a_k s^k\right\} Y(s) = \left\{\sum_{k=0}^{M} b_k s^k\right\} X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\left\{\sum_{k=0}^{M} b_k s^k\right\}}{\left\{\sum_{k=0}^{N} a_k s^k\right\}} \tag{6.61}$$

The system function has zeros at the solutions of

$$\sum_{k=0}^{M} b_k s^k = 0 \tag{6.62}$$

and poles at the solutions of

$$\sum_{k=0}^{N} a_k s^k = 0 \tag{6.63}$$

Note:

- With additional information such as stability or causality of the system, the ROC can be inferred and the corresponding impulse response can be obtained.

***Example* 6.24**:

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$\Rightarrow sY(s) + 3Y(s) = X(s) \Rightarrow H(s) = \frac{1}{s+3}$$

If the system is causal, the ROC is $\mathrm{Re}\{s\} > -3$, and the corresponding impulse response is

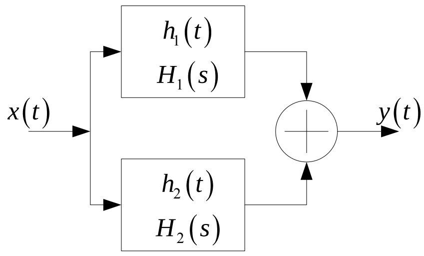
$$h(t) = e^{-3t}u(t)$$

If the system is noncausal, then the ROC is $\text{Re}\{s\} < -3$, and the corresponding impulse response is

$$h(t) = -e^{-3t}u(-t)$$

2. System function for interconnections of LTI systems

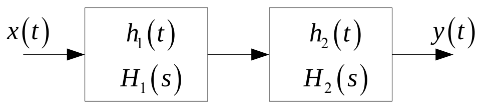
(1) Parallel interconnection



■ **Figure 6.13** Parallel connection of two LTI systems.

$$h(t) = h_1(t) + h_2(t) \Rightarrow H(s) = H_1(s) + H_2(s) \tag{6.64}$$

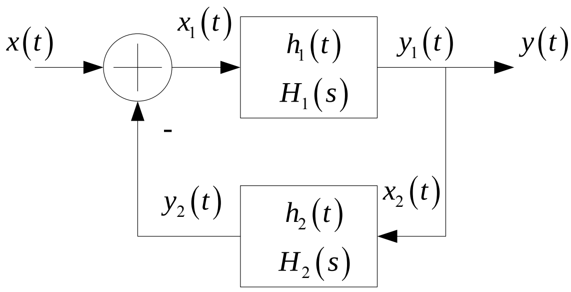
(2) Cascade interconnection



■ **Figure 6.14** Series connection of two LTI systems.

$$h(t) = h_1(t) * h_2(t) \Rightarrow H(s) = H_1(s)H_2(s) \tag{6.65}$$

(3) Feedback interconnection



■ **Figure 6.15** Feedback interconnection of two LTI systems.

$$Y_2(s) = H_2(s)X_2(s) = H_2(s)Y_1(s) = H_2(s)Y(s) \tag{6.66}$$

$$\begin{aligned} Y(s) &= H_1(s)X_1(s) = H_1(s)\left[X(s) - Y_2(s)\right] \\ &= H_1(s)X(s) - H_1(s)H_2(s)Y(s) \end{aligned} \tag{6.67}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \tag{6.68}$$

3. Butterworth filters

$$\left|B(\omega)\right|^2 = \frac{1}{1+\left(\omega/\omega_c\right)^{2N}} = \frac{1}{1+\left(j\omega/j\omega_c\right)^{2N}} = \left|B(j\omega)\right|^2 \qquad (6.69)$$

$$\left|B(j\omega)\right|^2 = B(j\omega)B^*(j\omega) \qquad (6.70)$$

Restricting the impulse response of the Butterworth filter to be real, we have

$$B^*(j\omega) = B(-j\omega) \qquad (6.71)$$

$$B(j\omega)B(-j\omega) = 1\Big/\left[1+\left(j\omega/j\omega_c\right)^{2N}\right] \qquad (6.72)$$

$$\because B(s)\big|_{s=j\omega} = B(j\omega) \qquad (6.73)$$

$$\therefore B(s)B(-s) = 1\Big/\left[1+\left(s/j\omega_c\right)^{2N}\right] \qquad (6.74)$$

The poles of $B(s)B(-s)$ are the solutions of

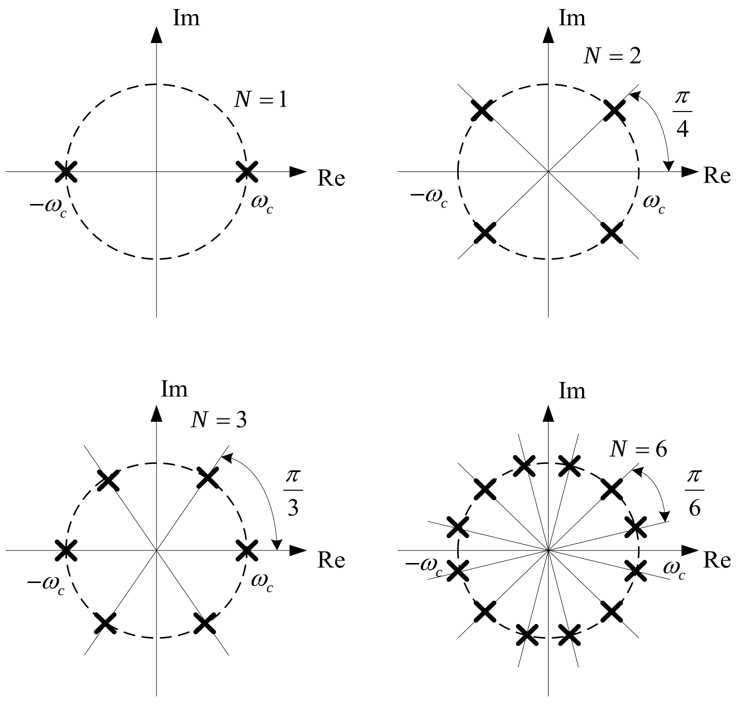
$$1+\left(s/j\omega_c\right)^{2N} = 0 \qquad (6.75)$$

$$\Rightarrow s_p = (-1)^{1/2N}\left(j\omega_c\right) \qquad (6.76)$$

$$\Rightarrow \left|s_p\right| = \omega_c,\ \angle s_p = \frac{\pi(2k+1)}{2N} + \frac{\pi}{2},\ k \text{ is an integer} \qquad (6.77)$$

$$\Rightarrow s_p = \omega_c e^{j\left[\frac{\pi(2k+1)}{2N}+\frac{\pi}{2}\right]} \qquad (6.78)$$

The positions of the poles of $B(s)B(-s)$ for N = 1, 2, 3, and 6 are shown in Fig. 6.16.



■ **Figure 6.16** Position of $B(s)B(-s)$ the poles of for N = 1, 2, 3, and 6.

(1) The poles of $B(s)B(-s)$ occurs in pairs, so that if there is a pole at $s = s_p$, then there is also a pole at $s = -s_p$.

(2) To construct $B(s)$, we choose one pole from each pair of poles.

(3) If we restrict the system to be stable and causal, then the poles of $B(s)$ should be in the left-half plane.

(4) $B^2(s)\big|_{s=0} = 1$

$$N = 1:\ B(s) = \frac{\omega_c}{s + \omega_c} \tag{6.79}$$

$$N = 2:\ B(s) = \frac{\omega_c^2}{\left(s + \omega_c e^{j\frac{\pi}{4}}\right)\left(s + \omega_c e^{-j\frac{\pi}{4}}\right)} = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} \tag{6.80}$$

$$N = 3:\ B(s) = \frac{\omega_c^3}{(s + \omega_c)\left(s + \omega_c e^{j\frac{\pi}{3}}\right)\left(s + \omega_c e^{-j\frac{\pi}{3}}\right)} = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3} \tag{6.81}$$

The corresponding differential equations for the above three cases are:

$$N = 1:\ \frac{dy(t)}{dt} + \omega_c y(t) = \omega_c x(t) \tag{6.82}$$

$$N = 2:\ \frac{d^2 y(t)}{dt^2} + \sqrt{2}\omega_c \frac{dy(t)}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t) \tag{6.83}$$

$$N = 3:\ \frac{d^3 y(t)}{dt^3} + 2\omega_c \frac{d^2 y(t)}{dt^2} + 2\omega_c^2 \frac{dy(t)}{dt} + \omega_c^3 y(t) = \omega_c^3 x(t) \tag{6.84}$$

$$\frac{\omega_c}{s + \omega_c} = \frac{Y(s)}{X(s)} \Rightarrow \omega_c X(s) = sY(s) + \omega_c Y(s) \tag{6.85}$$

$$\frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} = \frac{Y(s)}{X(s)} \Rightarrow \omega_c^2 X(s) = s^2 Y(s) + \sqrt{2}\omega_c s Y(s) + \omega_c^2 Y(s) \tag{6.86}$$

$$\frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3} = \frac{Y(s)}{X(s)} \tag{6.87}$$

$$\Rightarrow \omega_c^3 X(s) = s^3 Y(s) + 2\omega_c s^2 Y(s) + 2\omega_c^2 s Y(s) + \omega_c^3 Y(s)$$

6-7 The Unilateral Laplace Transform

1. The unilateral Laplace transform of $x(t)$ is defined as

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t) e^{-st} dt \tag{6.88}$$

The bilateral Laplace transform

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{6.89}$$

when $x(t) = 0$ for $t < 0$, the unilateral and bilateral Laplace transforms are identical.

Note:

- The ROC for the unilateral Laplace transform is always a right-half plane (causal).

***Example* 6.25**:

$$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$$

$$\mathcal{X}(s) = \frac{1}{(s+a)^n}, \ \text{Re}\{s\} > -a$$

***Example* 6.26**:

$$x(t) = e^{-a(t+1)} u(t+1)$$

$$X(s) = \frac{e^s}{s+a}, \ \text{Re}\{s\} > -a$$

$$\mathcal{X}(s) = \int_{0^-}^{\infty} e^{-a(t+1)} u(t+1) e^{-st} dt = \int_{0^-}^{\infty} e^{-a} e^{-(s+a)t} dt = e^{-a} \int_{\infty}^{\infty} \{e^{-at} u(t)\} e^{-st} dt$$

$$= \frac{e^{-a}}{s+a}, \ \text{Re}\{s\} > -a$$

The unilateral and bilateral Laplace transforms are distinctly different.

2. Most of the properties of the unilateral Laplace transform are the same as for the bilateral Laplace transform.

3. Differentiation property of the unilateral Laplace transform

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = x(t) e^{-st} \Big|_{0^-}^{\infty} + s \int_{0^-}^{\infty} x(t) e^{-st} dt = s\mathcal{X}(s) - x(0^-) \tag{6.90}$$

(Integration by parts)

$\mathcal{X}(s)$ is the unilateral Laplace transform of $x(t)$.

Similarly,

$$\begin{aligned}\int_{0^-}^{\infty}\frac{d^2x(t)}{dt^2}e^{-st}dt &= s\int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}dt - x^{(1)}(0^-)\\ &= s\left[s\mathcal{X}(s)-x(0^-)\right]-x^{(1)}(0^-)\\ &= s^2\mathcal{X}(s)-sx(0^-)-x^{(1)}(0^-)\end{aligned} \tag{6.91}$$

The general form for the differentiation property is

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{\mathcal{L}_u} \begin{array}{l} s^n\mathcal{X}(s)-\left.\frac{d^{n-1}}{dt^{n-1}}x(t)\right|_{t=0^-} - s\left.\frac{d^{n-2}}{dt^{n-2}}x(t)\right|_{t=0^-} \\ -\cdots - s^{n-2}\left.\frac{d}{dt}x(t)\right|_{t=0^-} - s^{n-1}x(0^-)\end{array} \tag{6.92}$$

where the subscript u in $\mathcal{L}_u$ denotes the unilateral transform.

***Example* 6.27**: $x(t)=e^{at}u(t)$

Apply the product rule for differentiation to obtain the derivative of $x(t)$, $t>0^-$:

$$\frac{d}{dt}x(t)=\frac{d}{dt}e^{at}u(t)=ae^{at}u(t)+\delta(t) \xleftrightarrow{\mathcal{L}} \frac{a}{s-a}+1=\frac{s}{s-a}$$

Using Eq. (6.90),

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}_u} s\frac{1}{s-a}+0=\frac{s}{s-a}$$

■

4. Integration property

$$\int_{-\infty}^{t}x(\tau)d\tau \xleftrightarrow{\mathcal{L}_u} \frac{\int_{-\infty}^{0^-}x(\tau)d\tau}{s}+\frac{\mathcal{X}(s)}{s} \tag{6.93}$$

Proof: Let $y(t)=\int_{-\infty}^{t}x(\tau)d\tau$. Then,

$$\frac{d}{dt}y(t)=x(t)$$

$$s\mathcal{Y}(s)-y(0^-)=\mathcal{X}(s)$$

$$\mathcal{Y}(s)=\frac{\mathcal{X}(s)}{s}+\frac{\int_{-\infty}^{0^-}x(\tau)d\tau}{s}$$

■

5. A primary use of the unilateral Laplace transform is in obtaining the solution of linear constant-coefficient differential equations with nonzero initial conditions.

***Example* 6.28**: $\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t),\ y(0^-)=3,\ \frac{dy(0^-)}{dt}=-5$

Let $x(t)=2u(t)$. Then we obtain

$$s^2\mathcal{Y}(s)-sy(0^-)-y^{(1)}(0^-)+3s\mathcal{Y}(s)-3y(0^-)+2\mathcal{Y}(s)=\frac{2}{s}$$

$$\mathcal{Y}(s)=\frac{3s+4}{(s+1)(s+2)}+\frac{2}{s(s+1)(s+2)}$$

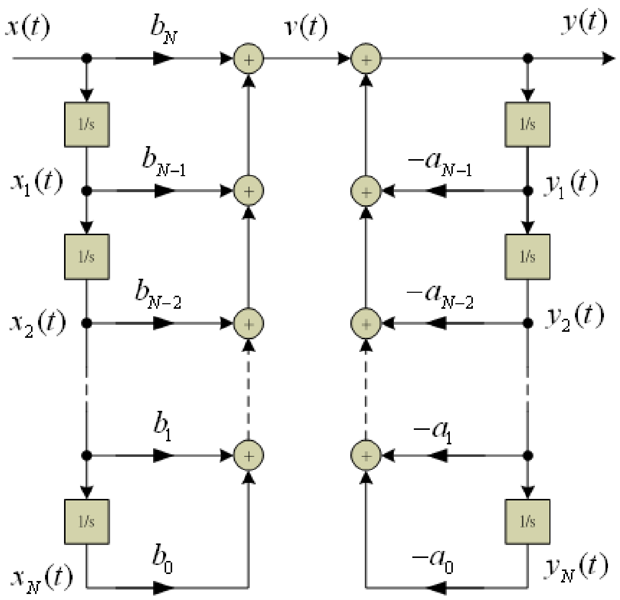
where $\mathcal{Y}(s)$ is the unilateral Laplace transform of $y(t)$.

$$\Rightarrow \mathcal{Y}(s)=\frac{1}{s}-\frac{1}{s+1}+\frac{3}{s+2}\Rightarrow y(t)=\left[1-e^{-t}+3e^{-2t}\right]u(t)$$

■

6-8 Structures for Continuous-Time Filters

In Section 2-8 block diagram were employed to show the structure of continuous-time filter implementations as described by the corresponding differential (or integral) equations. The structure corresponding directly to the general differential equation in Eq. (6.60), with $a_N=1$, was called the direct form and is shown in Fig. 6.17 for $M=N$, with $1/s$ denoting each integrator. Note that this structure consists effectively of the cascade of two subsystems.



■ **Figure 6.17** General continuous-time direct-form structure.

The first subsystem implements the differential equation

$$\frac{d^N v(t)}{dt^N} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}, \tag{6.94}$$

while the second subsystem realizes the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \frac{d^N v(t)}{dt^N}, \tag{6.95}$$

with $a_N = 1$. Calling these subsystems $H_1(s)$ and $H_2(s)$, respectively, we thus have

$$H(s) = H_1(s)H_2(s) = B(s)/A(s), \tag{6.96}$$

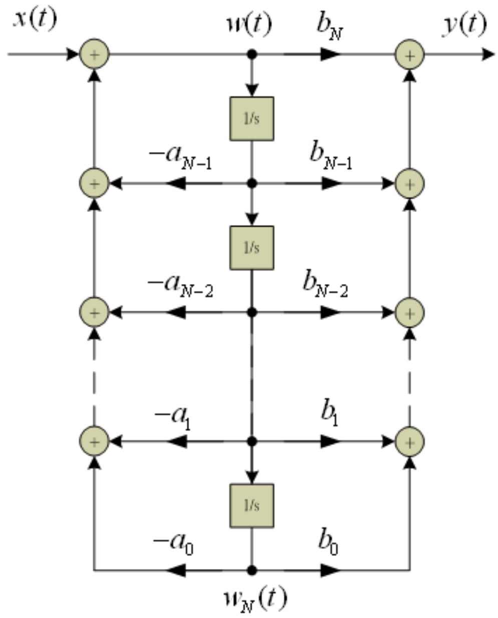
where

$$H_1(s) = \frac{B(s)}{s^N} = \frac{\sum_{k=0}^{N} b_k s^k}{s^N} \tag{6.97}$$

and

$$H_2(s) = \frac{s^N}{A(s)} = \frac{s^N}{\sum_{k=0}^{N} a_k s^k} \tag{6.98}$$

As noted in Section 2-8, the direct form in Fig. 6.17 is not canonical because the number of integrators (2*N*) is not minimum. Reversing the order of $H_1(s)$ and $H_2(s)$ and eliminating the *N* redundant integrators, we produce the canonical direct-form-II, as shown in Fig. 6.18 Note that, in addition to *N* integrators, this canonical form includes 2*N*+1 multipliers (amplifiers), in general, for an *N*th-order filter with *M* = *N*.



■ **Figure 6.18** General continuous-time direct-form-II.

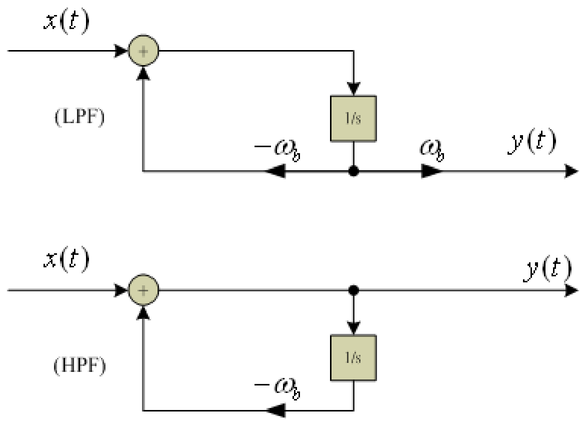
Direct-form-II implementations of the first-order LPF and HPF from

$$H(s) = \frac{1}{1 + s/\omega_b} = \frac{\omega_b}{s + \omega_b}, \tag{6.99}$$

and

$$H(s) = \frac{s}{s + \omega_b}, \tag{6.100}$$

are shown in Fig. 6.19 Note that the signs of the feedback multipliers $(-\omega_b)$ and the corresponding terms in the denominators of $H(s)$ are different, as opposed to the feedforward coefficients (ω_b and 1) and the corresponding numerator terms.



■ **Figure 6.19** First-order LPF and HPF direct-form-II structures.

There are many other structures (canonical and otherwise) that are useful for implementing continuous-time (analog) filters. These structures have various desirable properties such as modularity, reduced sensitivity to component variation, and/or suitability for integrated-circuit realization. Although we will consider only active filter implementations in this section, passive-circuit realizations are possible. Two basic modulator structures are the parallel form and cascade form. To derive the parallel form, we expand $H(s)$ in the partial-fraction expansion (assuming no multiple poles)

$$H(s) = b_N + \sum_{k=1}^{N} \frac{r_k}{s + s_k}, \tag{6.101}$$

where $b_N = 0$ if $M < N$. This form for $H(s)$ implies a parallel combination of N first-order filters. In general, the poles $-s_k$ and reduces r_k are complex-valued, complex multipliers would be required in the corresponding implementation. In particular, assuming that $h(t)$ is real-valued, $H(s)$ can be written as

$$H(s) = b_N + \sum_{k=1}^{L} \left(\frac{r_k}{s + s_k} + \frac{r_k^*}{s + s_k^*} \right) + \sum_{k=2L+1}^{N} \frac{r_k}{s + s_k}, \tag{6.102}$$

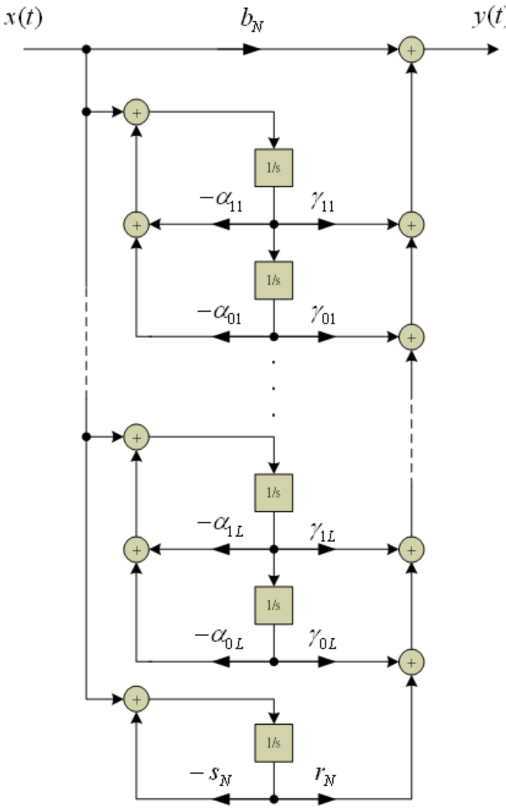
where s_k, k=1,…,L ($L \le N/2$) are complex-valued and s_k, k=2L+1,…,N, are real-valued. To avoid the unnecessary complication of complex multipliers, we combine the terms in the first summation to obtain

$$H(s) = b_N + \sum_{k=1}^{L} \frac{\gamma_{1k}s + \gamma_{0k}}{s^2 + \alpha_{1k}s + \alpha_{0k}} + \sum_{k=2L+1}^{N} \frac{r_k}{s + s_k}, \tag{6.103}$$

where

$$\begin{aligned} \alpha_{1k} &= 2\,\mathrm{Re}\{s_k\} & \alpha_{0k} &= |s_k|^2 \\ \gamma_{1k} &= 2\,\mathrm{Re}\{r_k\} & \gamma_{0k} &= 2\,\mathrm{Re}\{s_k^* r_k\} \end{aligned}.$$

Hence all of the coefficients in Eq. (6.103) are real-valued. Using direct-form-II networks to realize each of the terms in Eq. (6.103), we produce parallel form II, which is shown in Fig. 6.20 for N odd and L=(N-1)/2. Note that the parallel form is also canonical since it has N integrators and 2N-1 multipliers, in general, for M = N. If several poles are real-valued (that is, if N-2L ≥ 2), some or all of the associated first-order terms in Eq. (6.103) are often combined into second-order terms to produce additional second-order sections in the parallel form, which increases the modularity of the corresponding circuit realization.



■ **Figure 6.20** Nth-order parallel-form-II structure for N odd.

To obtain the cascade form, we instead factor the numerator and denominator of $H(s)$ into products of first-order terms of the form

$$H(s) = b_M \frac{\prod_{k=1}^{M}(s+v_k)}{\prod_{k=1}^{N}(s+s_k)}. \tag{6.104}$$

Assuming that $a_N = 1$, as before. (Multiple poles and/or zeros are allowed.) This form for $H(s)$ implies a cascade combination of first-order subfilters, but again complex-valued poles $-s_k$ and/or zeros $-v_k$ would necessitate complex multipliers. Therefore we rewrite $H(s)$ as

$$H(s) = b_M \frac{\prod_{k=1}^{K}(s+v_k)(s+v_k^*) \prod_{k=2K+1}^{M}(s+v_k)}{\prod_{k=1}^{L}(s+s_k)(s+s_k^*) \prod_{k=2L+1}^{N}(s+s_k)}, \tag{6.105}$$

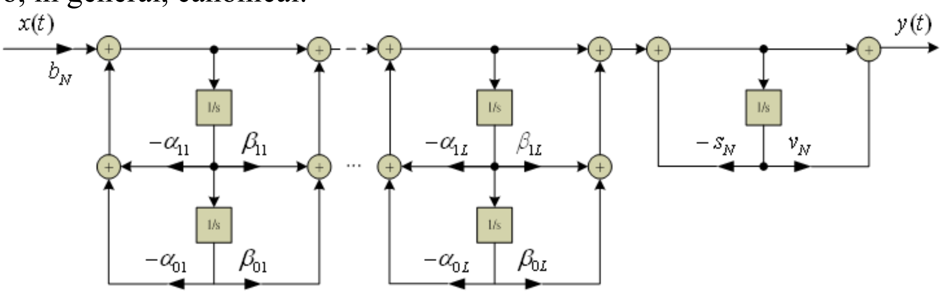
where v_k, k=1,…,K ($K \le M/2$), and s_k, k=1,…,L ($L \le N/2$), are complex-valued and v_k, k = 2K+1,…,M, and s_k, k = 2L+1,…,N, are real-valued. The complex factors are then combined to produce

$$H(s) = b_M \frac{\prod_{k=1}^{K}(s^2+\beta_{1k}s+\beta_{0k}) \prod_{k=2K+1}^{M}(s+v_k)}{\prod_{k=1}^{L}(s^2+\alpha_{1k}s+\alpha_{0k}) \prod_{k=2L+1}^{N}(s+s_k)}, \tag{6.106}$$

where

$$\begin{aligned} \alpha_{1k} &= -2\,\mathrm{Re}\{s_k\} & \alpha_{0k} &= |s_k|^2 \\ \beta_{1k} &= -2\,\mathrm{Re}\{r_k\} & \beta_{0k} &= |v_k|^2 \end{aligned}.$$

Therefore, since all of the coefficients in this expression are real-valued, $H(s)$ can be implemented as a cascade of first- and second-order sections with real multipliers. Again, if there are several real-valued poles and/or zeros (i.e., if N-2L≥2 and/or M- 2K≥ 2), the corresponding factors are usually combined in pairs to produce additional second-order sections. Realizing the resulting first- and second-order sections using direct-form-II networks, we produce cascade form II, shown in Fig. 6.21 for $M = N$ (odd) and L=(N-1)/2. Note that the cascade form is also, in general, canonical.



■ **Figure 6.21** Nth-order cascade-form-II structure for N odd.

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