Chapter 5 Time and Frequency Characterization of Linear Time-Invariant Systems

5-1 The Magnitude-Phase Representation of the Frequency Response of LTI Systems

The magnitude-phase representation of the continuous-time Fourier transform $X(j\omega)$ is

$$X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}. \tag{5.1}$$

Similarly the magnitude-phase representation of the discrete-time Fourier transform $X(e^{j\Omega})$ is

$$X(e^{j\Omega}) = |X(e^{j\Omega})| e^{j\angle X(e^{j\Omega})}. \tag{5.2}$$

From the convolution property for continuous-time Fourier transforms, the transform $Y(j\omega)$ of the output of an LTI system is related to the transform $X(j\omega)$ of the input to the system by the equation

$$Y(j\omega) = X(j\omega)H(j\omega), \tag{5.3}$$

where $H(j\omega)$ is the frequency response of the system, i.e., the Fourier transform of the system's impulse response. Similarly, in discrete time case, the Fourier transforms of the input $X(e^{j\Omega})$ and the output $Y(e^{j\Omega})$ of an LTI system with frequency response $H(e^{j\Omega})$ are related by

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}). \tag{5.4}$$

Specifically, in continuous time, we have

$$|Y(j\omega)| = |X(j\omega)||H(j\omega)| \tag{5.5}$$

and

$$\angle Y(j\omega) = \angle X(j\omega) + \angle H(j\omega), \tag{5.6}$$

and exactly analogous relationships hold in the discrete-time case.

Note:

- $|H(j\omega)|$ and $|H(e^{j\Omega})|$ are commonly referred to as the *gain* of the system.
- $\angle H(j\omega)$ and $\angle H(e^{j\Omega})$ are typically referred to as the *phase shift* of the system.

1. Linear and nonlinear phase

   When the phase shift at the frequency ω is a linear function of ω, there is a particularly straightforward interpretation of the effect in the time domain. Consider the continuous-time LTI system with frequency response

$$H(j\omega) = e^{-j\omega t_0}, \tag{5.7}$$

   so that the system has unit gain and linear phase, i.e.,

$$|H(j\omega)| = 1, \ \angle H(j\omega) = -\omega t_0 \tag{5.8}$$

The system with this frequency response characteristic produces an output that is simply a time shift of the input, i.e.,

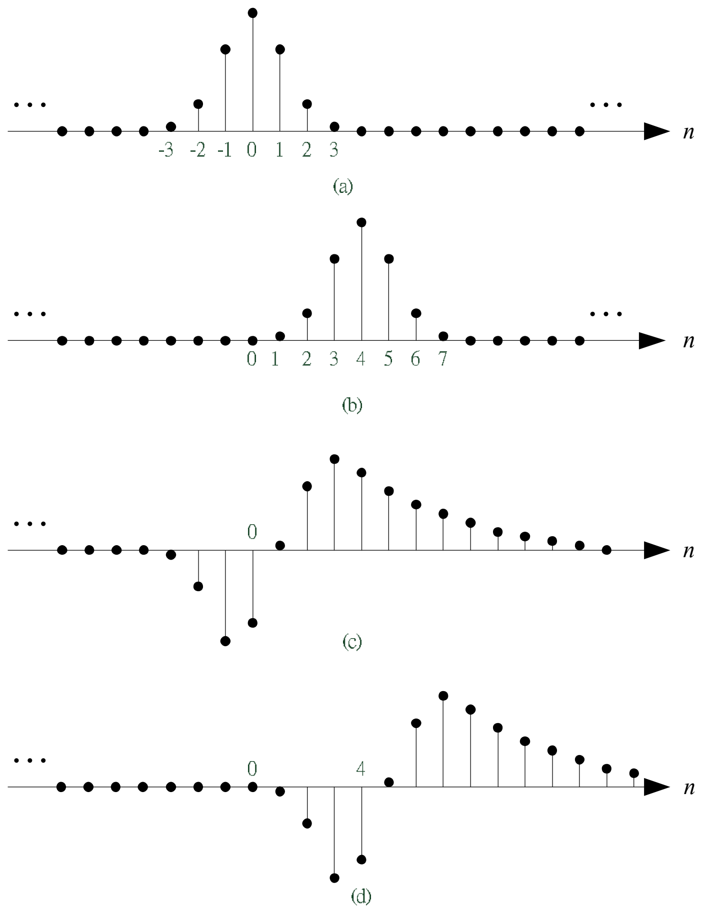
$$y(t) = x(t - t_0). \tag{5.9}$$

In the discrete-time case, the effect of linear phase is similar to that in the continuous-time case when the slope of the linear phase is an integer. That is, a linear phase shift with an integer slope corresponds to a shift of $x[n]$ by an integer number of samples.

Note:

- Informally, when the phase slope is not an integer, the effect is a time shift of the envelope of the sequence values, but the values themselves may change.

If an input signal is subjected to a phase shift that is a nonlinear function of frequency, we obtain a signal that may look considerably different from the input signal. This is illustrated in Fig. 5.1 in the discrete-time case.



■**Figure 5.1** (a) Discrete-time input signal; (b) response for a system with linear phase with slope of -4; (c) response for a system with nonlinear phase; and (d) response for a system whose phase characteristic as that of part (c) plus a linear phase with integer slope.

Note:

- All systems considered in Fig. 5.1 have unit gain (i.e., $\left|H\left(e^{j\Omega}\right)\right|=1$), such systems are commonly referred to as *all-pass* systems.
- A more general LTI system $H(j\omega)$ or $H\left(e^{j\Omega}\right)$, of course, imparts both magnitude shaping through the gain and phase shift that may or may not be linear.

2. Group delay

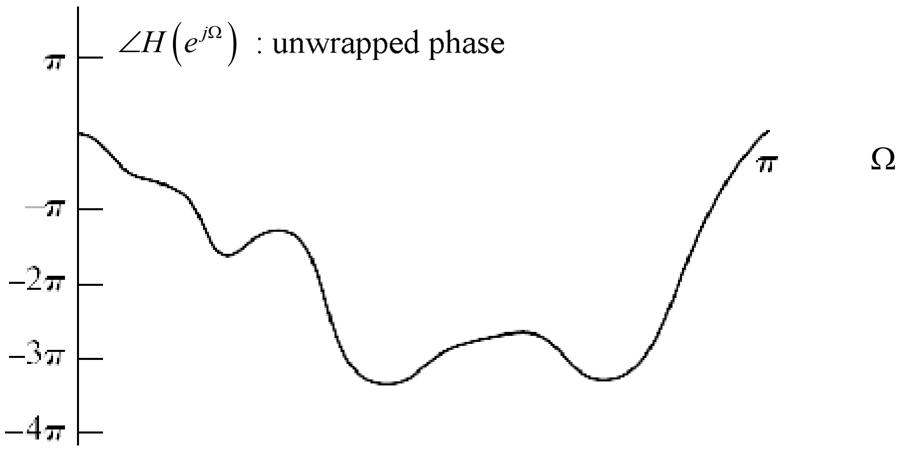
Group delay is a useful measure of phase distortion. The group delay at each frequency equals the negative of the slope of the phase at that frequency; i.e., the group delay for continuous-time case and discrete-time case are respectively defined as

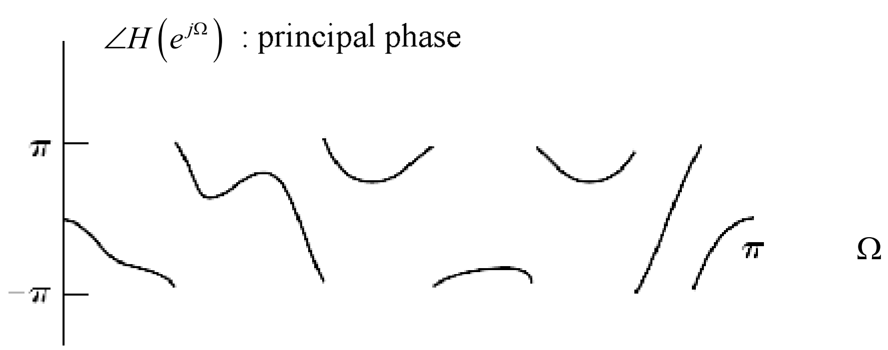
$$\tau_c(\omega) = -\frac{d}{d\omega}\left\{\angle H(j\omega)\right\} \tag{5.10}$$

and

$$\tau_d(\Omega) = -\frac{d}{d\Omega}\left\{\angle H\left(e^{j\Omega}\right)\right\}. \tag{5.11}$$

When the angle of a complex umber is computed, with the use of an arctangent subroutine on a calculator or with a computer system subroutine, the *principal phase* function is obtained. Note that this function contained





discontinuities of size 2π at various frequencies, making the phase function nondifferentiable at those points. Thus, by appropriately adding or subtracting such integer multiples of 2π from various points of the principal phase, we obtain the *unwrapped phase* function and the group delay as a function of frequency may now be computed by using this function.

***Example* 5.1**: $H\left(e^{j\Omega}\right)=1-re^{j\theta}e^{-j\Omega}$

The square of the magnitude of $H\left(e^{j\Omega}\right)$ is

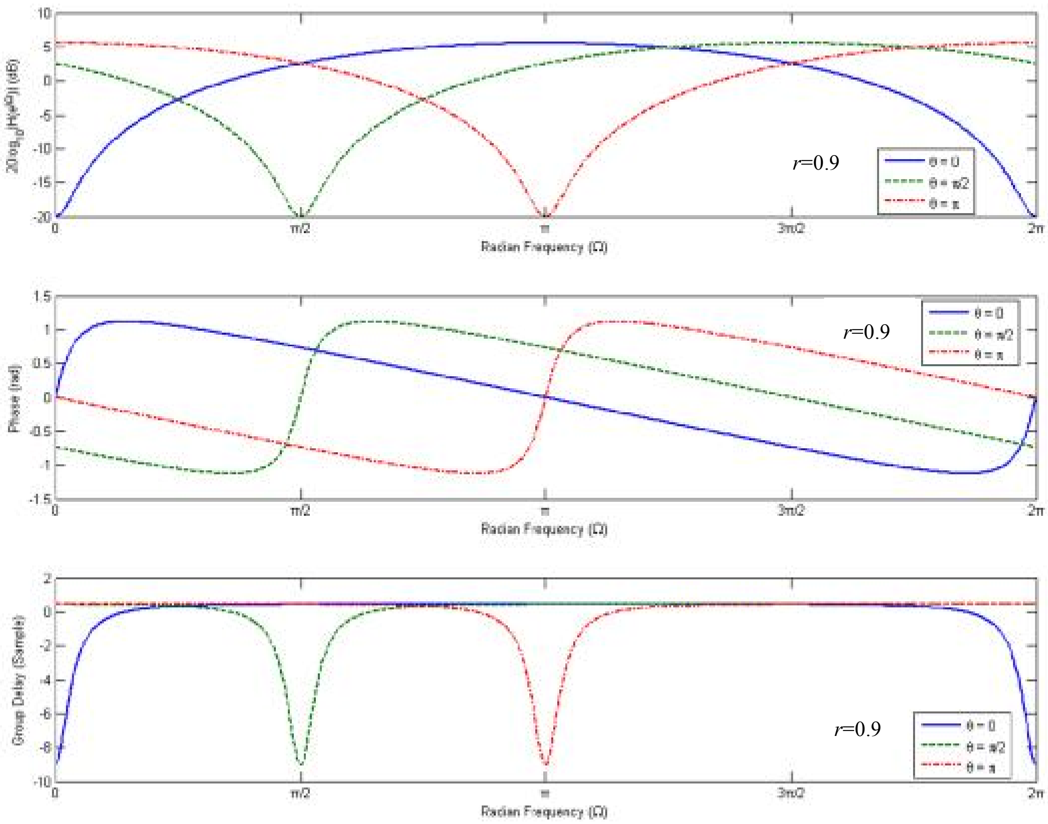
$$\left|H\left(e^{j\Omega}\right)\right|^2=\left|1-re^{j\theta}e^{-j\Omega}\right|^2=1+r^2-2r\cos\left(\Omega-\theta\right).$$

Thus the log-magnitude in *decibels* (dB) is

$$20\log_{10}\left|1-re^{j\theta}e^{-j\Omega}\right|=10\log_{10}\left[1+r^2-2r\cos\left(\Omega-\theta\right)\right].$$

The principal phase and group delay can be calculated respectively as

$$\angle H\left(e^{j\Omega}\right)=\tan^{-1}\left[\frac{r\sin\left(\Omega-\theta\right)}{1-r\cos\left(\Omega-\theta\right)}\right] \text{ and } \tau_d\left(\Omega\right)=\frac{r^2-r\cos\left(\Omega-\theta\right)}{\left|1-re^{j\theta}e^{-j\Omega}\right|^2}.$$



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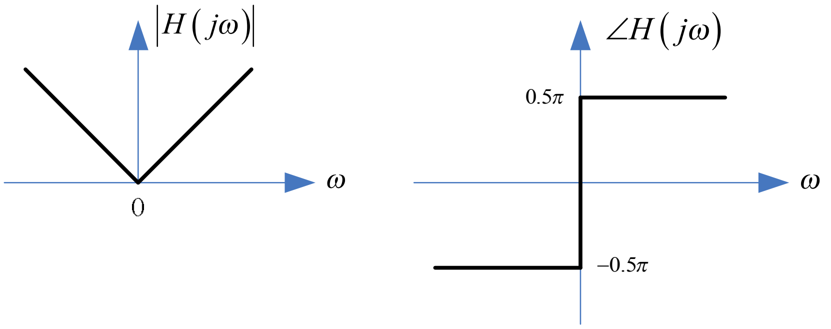
In a variety of applications, it is of interest to change the relative amplitudes of the frequency components in a signal or perhaps eliminate some frequency components entirely, a process referred to as *filtering*. Linear time-invariant systems that change the shape of the spectrum are often referred to as *frequency-shaping filters*. Systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others are referred to as *frequency-selective filters*.

5-2 Frequency-Shaping Filters

1. LTI filters are typically included in audio systems to permit the listener to modify the relative amounts of low-frequency energy (bass) and high-frequency (treble). These filters correspond to LTI systems whose frequency response can be changed by manipulating the tone controls.
2. Another class of frequency-shaping filters often encountered is that for which the filter output is the derivative of the filter input, i.e., $y(t) = dx(t)/dt$. With $x(t)$ of the form $x(t) = e^{j\omega t}$, $y(t)$ will be $y(t) = j\omega e^{j\omega t}$, from which it follows that the frequency response is $H(j\omega) = j\omega$.



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5-3 Ideal Frequency-Selective Filters

Frequency-Selective Filters:

Filters that pass signals in one or a set of frequency bands and attenuate or totally eliminate signals in the remaining frequency bands.

Ideal Frequency-Selective Filters:

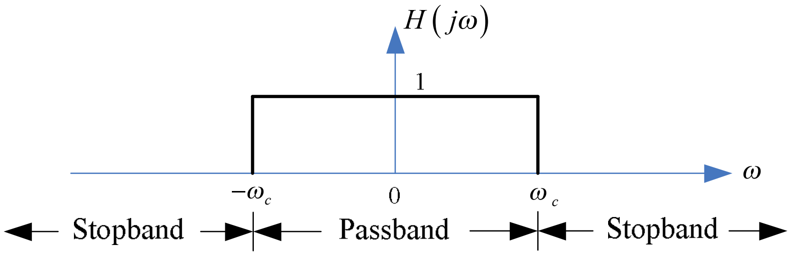
Filters that exactly pass complex exponentials at one set of frequencies and completely reject the rest.

$$\begin{cases} \text{Lowpass filter} \\ \text{Bandpass filter} \\ \text{Highpass filter} \\ \text{Band-reject filter (or Band-stop filter)} \end{cases}$$

1. Frequency-Domain Characteristics

(1) Lowpass filter:

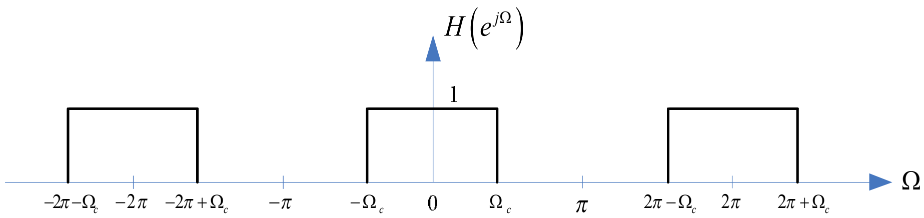
Continuous-time case:



$$H(\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

ω_c : cutoff frequency

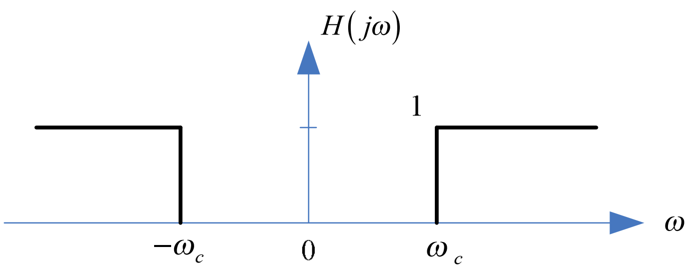
Discrete-time case:



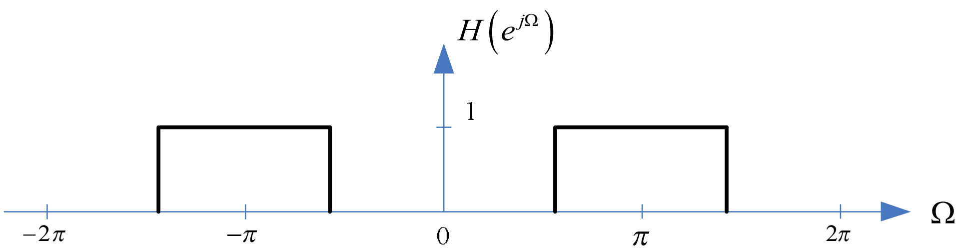
$$\begin{cases}\Omega \text{ near } \quad 0,\ \pm 2\pi,\ \pm 4\pi,\ \cdots \Rightarrow \text{ low frequencies}\\ \Omega \text{ near } \quad \pi,\ \pm 3\pi,\ \pm 5\pi,\ \cdots \Rightarrow \text{ high frequencies}\end{cases}$$

(2) Highpass filter:

Continuous-time case:

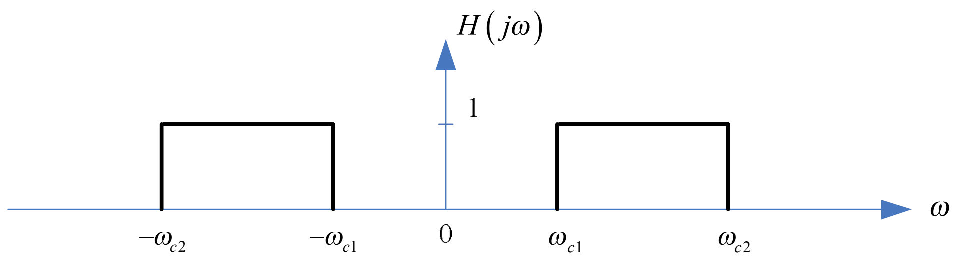


Discrete-time case:

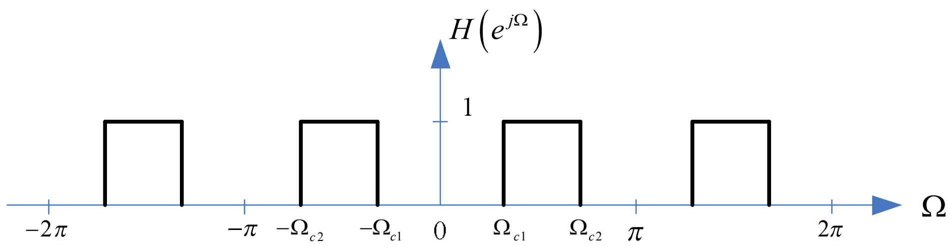


(3) Bandpass filter:

Continuous-time case:

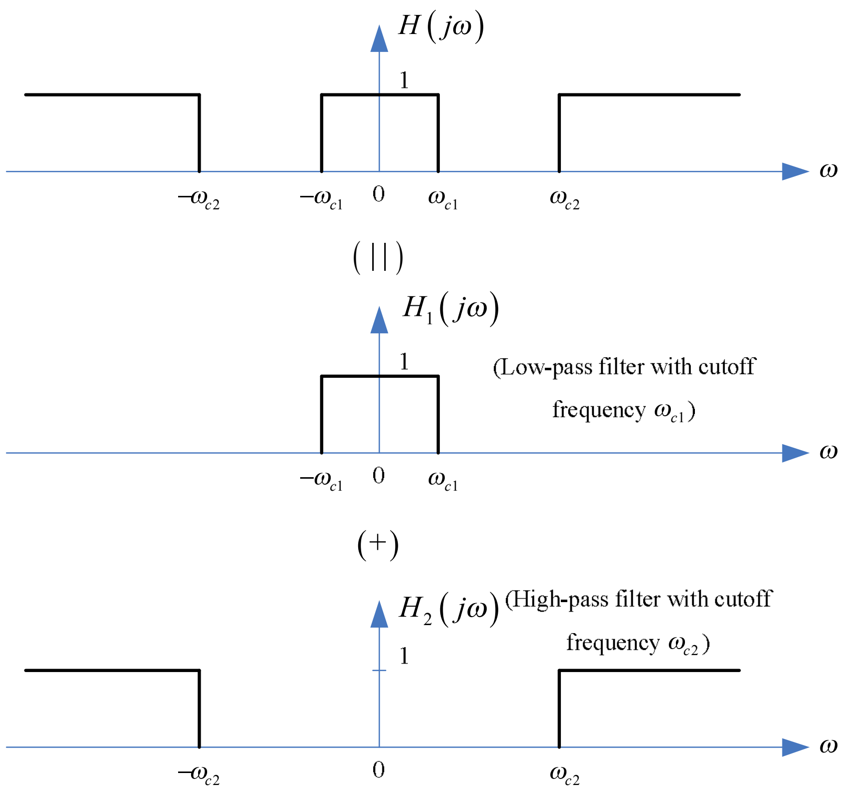


Discrete-time case:

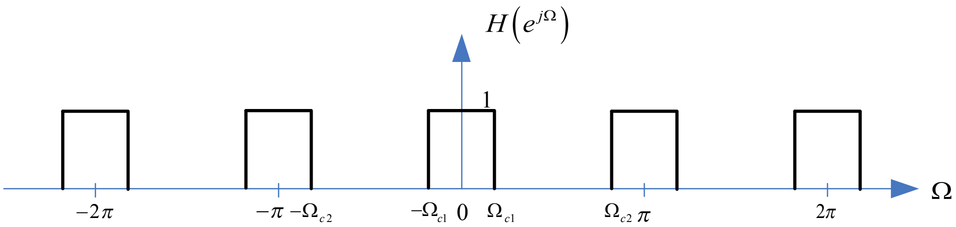


(4) Band-reject filter:

Continuous-time case:



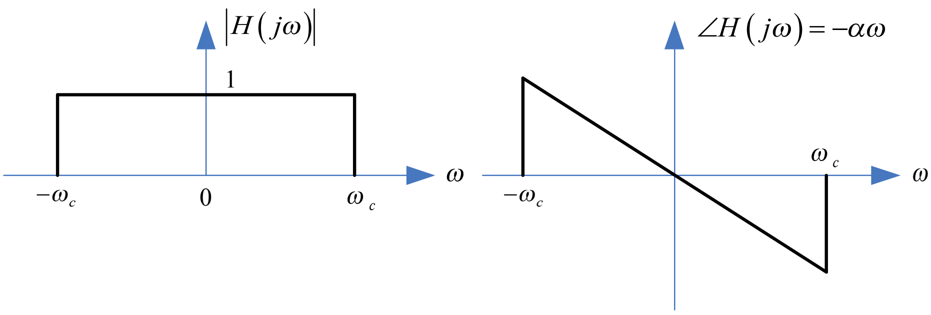
Discrete-time case:



Note:

- Each of the ideal filters must have a zero phase characteristic or a linear phase characteristic.

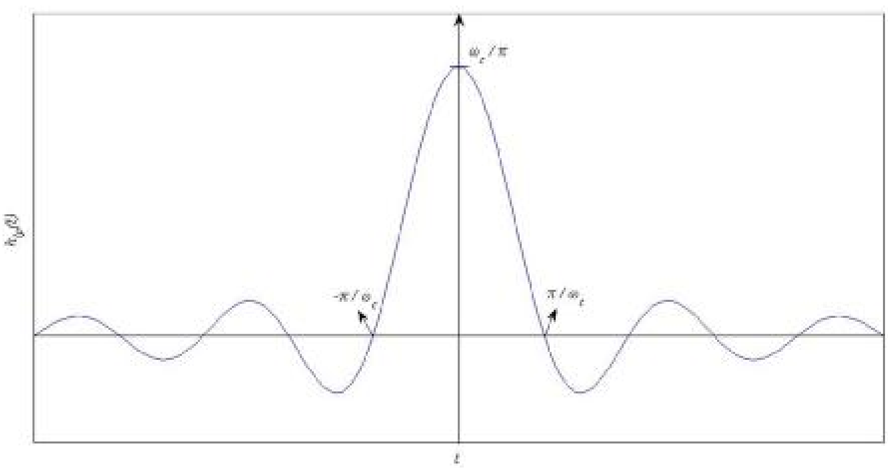
***Example* 5.2**:

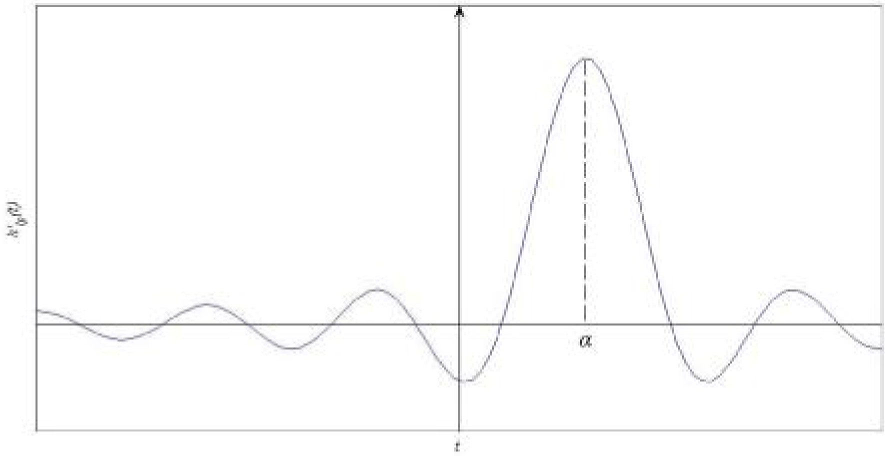


2. Time-Domain Characteristics of Lowpass filter

Impulse response of the continuous-time case:

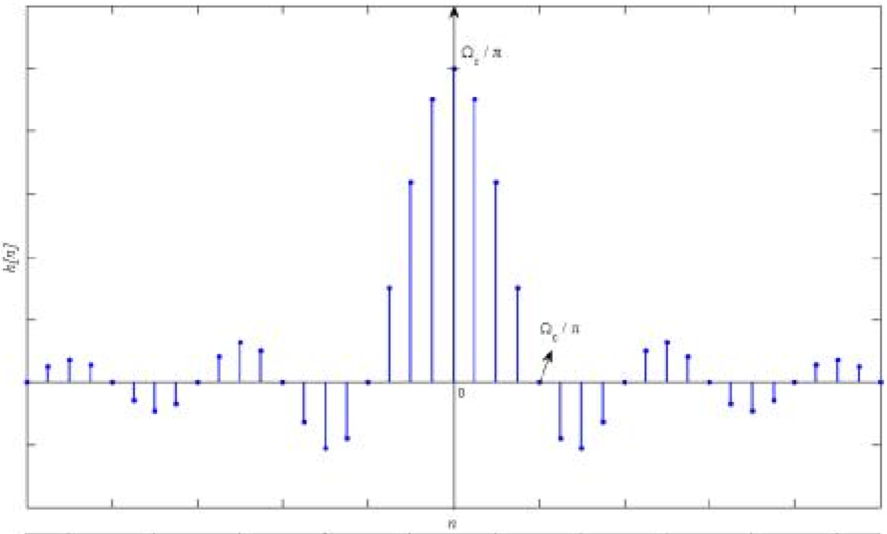
$$\begin{cases} \text{zero phase shift:} & h_{lp}(t) = \dfrac{\omega_c}{\pi}\operatorname{sinc}\left(\dfrac{\omega_c t}{\pi}\right), \quad \angle H_{lp}(j\omega) = 0 \\ \text{linear phase shift:} & h_{lp}^{'}(t) = h_{lp}(t-\alpha), \quad \angle H_{lp}^{'}(j\omega) = -\alpha\omega \end{cases}$$

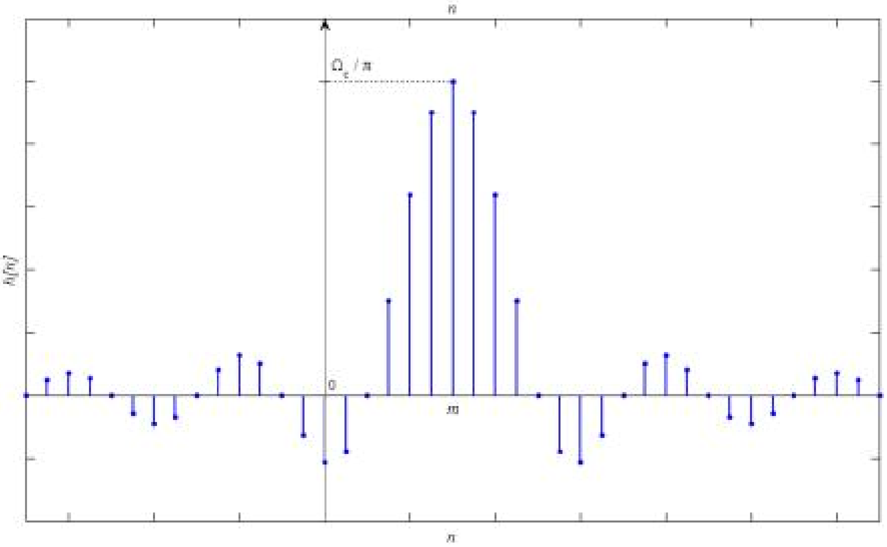




Impulse response of the discrete-time case:

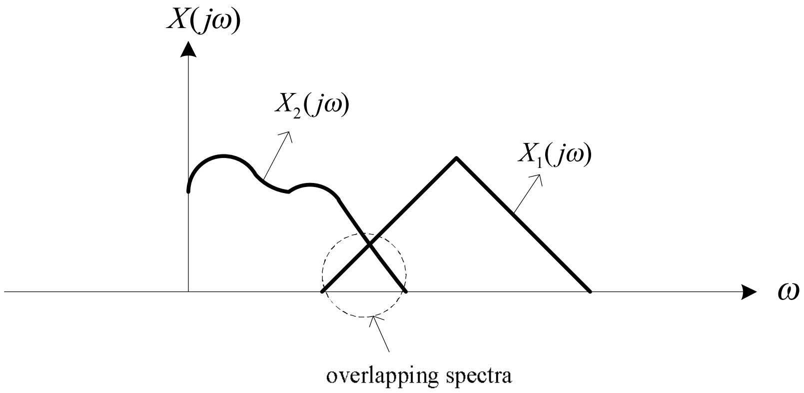
$$\begin{cases} \text{zero phase shift:} \quad h_{lp}[n] = \dfrac{\Omega_c}{\pi}\,\text{sinc}\left(\dfrac{\Omega_c n}{\pi}\right), \quad \angle H_{lp}\left(e^{j\Omega}\right) = 0 \\ \text{linear phase shift:} \quad h'_{lp}[n] = h_{lp}[n-m], \quad \angle H'_{lp}\left(e^{j\Omega}\right) = -m\Omega \end{cases}$$





5-4 Nonideal Frequency-Selective Filters

1. The ideal frequency-selective filters are noncausal.
   => They must be approximated by a causal system (nonideal frequency -selective filters) for real-time filtering.
2. In many filtering problems, the signals to be separated do not lie in totally disjoint frequency bands.



   => A filter with a gradual transition from passband to stopband is generally preferable when filtering the sum of signals with overlapping spectra.
3. The step response of an ideal frequency-selective filter has overshoots and rings in the vicinity of the discontinuity.
   => In some case, this time-domain behavior may be undesirable.
   => It is often preferable to allow some flexibility in the behavior of the filter in the passband, stopband, and transition band.
4. Even in cases when ideal frequency-selective filters are desirable, they may not be attainable.

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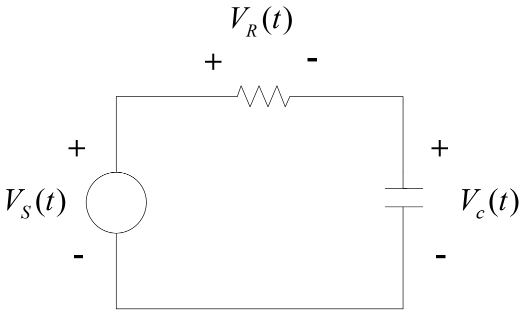
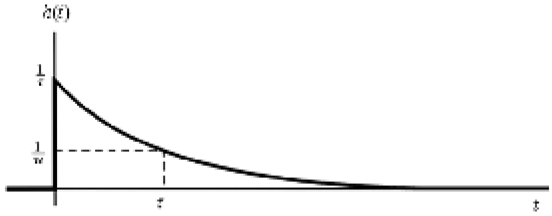
5-5 Examples of Continuous-Time Frequency-Selective Filters Described by Differential Equations

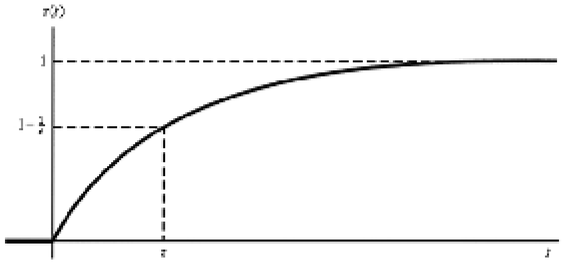
1. RC Lowpass and Highpass Filters



$$V_S(t) = V_R(t) + V_C(t)$$
$$= R \cdot C \frac{d}{dt} V_C(t) + V_C(t)$$
$$\Rightarrow \frac{d}{dt} V_C(t) + \frac{1}{RC} V_C(t) = V_S(t)$$

where $\tau = RC$ is time constant.





$$j\omega V_C(j\omega)+\frac{1}{RC}V_C(j\omega)=V_S(j\omega),\quad H(j\omega)=\frac{V_C(j\omega)}{V_S(j\omega)}=\frac{1}{j\omega+\frac{1}{RC}}$$

=> First-order RC lowpass filter with $\tau = RC$.

Choosing the resistor voltage as the output:

$$V_R(t)+V_C(t)=V_S(t)$$

$$\Rightarrow \frac{d}{dt}V_R(t)+\frac{d}{dt}V_C(t)=\frac{d}{dt}V_S(t)$$

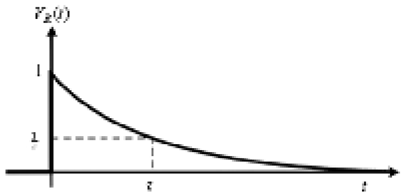
$$\Rightarrow RC\frac{d}{dt}V_R(t)+RC\frac{d}{dt}V_C(t)=RC\frac{d}{dt}V_S(t)$$

$$\Rightarrow RC\frac{d}{dt}V_R(t)+V_R(t)=RC\frac{d}{dt}V_S(t)$$

Let $V_S(t)=u(t)$

=> $RCj\omega V_R(j\omega)+V_R(j\omega)=RCj\omega V_S(j\omega)$

=> $G(j\omega)=\dfrac{V_R(j\omega)}{V_S(j\omega)}=\dfrac{j\omega RC}{1+j\omega RC}$



=> First-order RC highpass filter with $\tau = RC$.

Note:

- A simple RC circuit can serve as a lowpass or highpass filter, depending upon the choice of the physical output variable.
- If we need a sharp transition from passband to stopband, more complex filters with a sharper transition can be implemented by using more energy storage elements, leading to higher-order differential equations.

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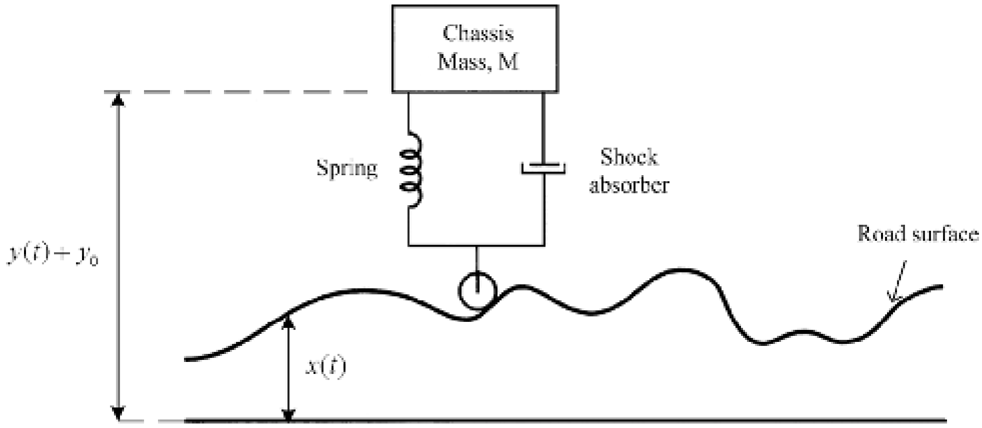
2. The Automobile Suspension System as a Lowpass Filter



y_0: The distance between the chassis and the road surface when the automobile is at rest.

$y(t) + y_0$: The position of the chassis above a reference evaluation.

$x(t)$: The elevation of the road surface above a reference elevation.

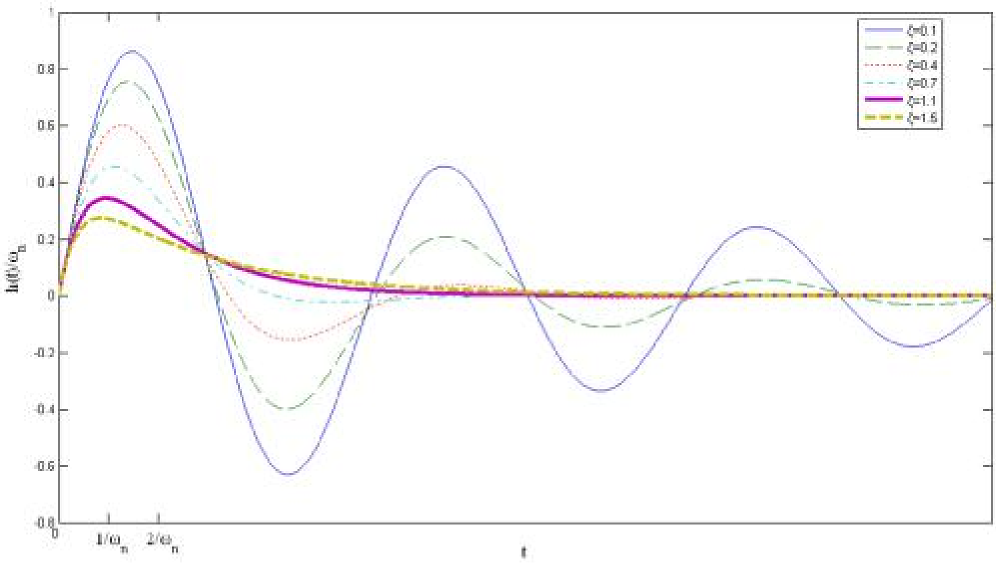
The difference equation governing the motion of the chassis is then:

$$M\frac{d^2y(t)}{dt}+b\frac{dy(t)}{dt}+Ky(t)=Kx(t)+b\frac{dx(t)}{dt}$$

where M is the mass of the chassis and K and b are the spring and shock absorber constants, respectively. The frequency response of the system is:

$$H(j\omega)=\frac{K+bj\omega}{(j\omega)^2M+b(j\omega)+K}=\frac{\omega_n^2+2\zeta\omega_n(j\omega)}{(j\omega)^2+\omega_n^2+2\zeta\omega_n(j\omega)}$$

where ω_n = natural frequency $=\sqrt{\frac{K}{M}}$, ζ = damping ratio, and $2\zeta\omega_n=\frac{b}{M}$.



■**Figure 5.2** Impulse response of continuous-time second-order systems with different values of the damping ratio ζ.

$\omega_n \downarrow \ \Rightarrow \ \zeta \uparrow \ \Rightarrow$ the suspension will tend to filter out slower road variation.

$\Rightarrow$ provide a smoother ride.

$\omega_n \uparrow \ \Rightarrow \ \zeta \downarrow \ \Rightarrow$ rise time of the system $\downarrow$.

$\Rightarrow$ overshoot and ringing in the step response tend to increase.

5-6 Examples of Discrete-Time Frequency-Selective Filters Described by Difference Equations

1. Nonrecursive Discrete-time Filters

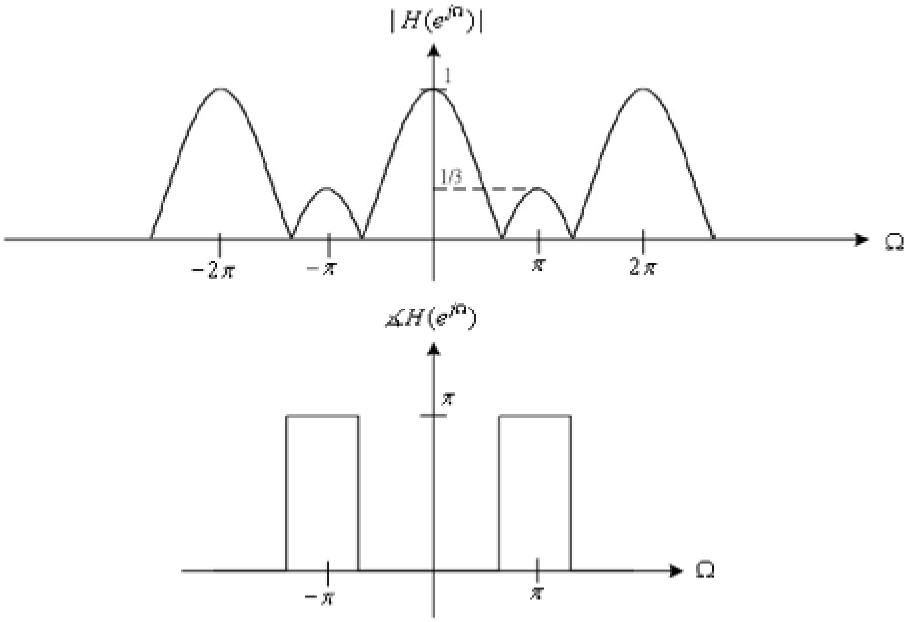
(1) Lowpass filtering :

For discrete-time sequences, a common smoothing operation is one referred to as a moving average.

$x[n] \rightarrow y[n]$: $y[n_0]$ is an average value of $x[n]$ in the vinciuity of n_0.

***Example* 5.3**: Moving average lowpass filter

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right) \ \Rightarrow \ H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{1}{3}\left(1 + 2\cos\Omega\right)$$

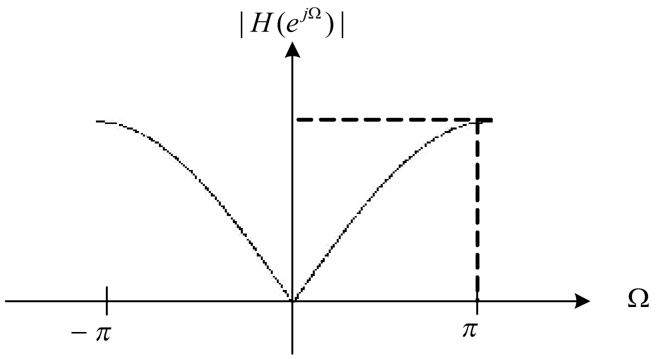


(2) Highpass filtering:

***Example* 5.4**: Moving average highpass filter

$$y[n] = \frac{x[n] - x[n-1]}{2}$$

$$\Rightarrow H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{1}{2}\left(1 - e^{j\Omega}\right) = je^{-j\Omega/2}\sin\left(\Omega/2\right)$$



(3) Generalization of the moving average filter:

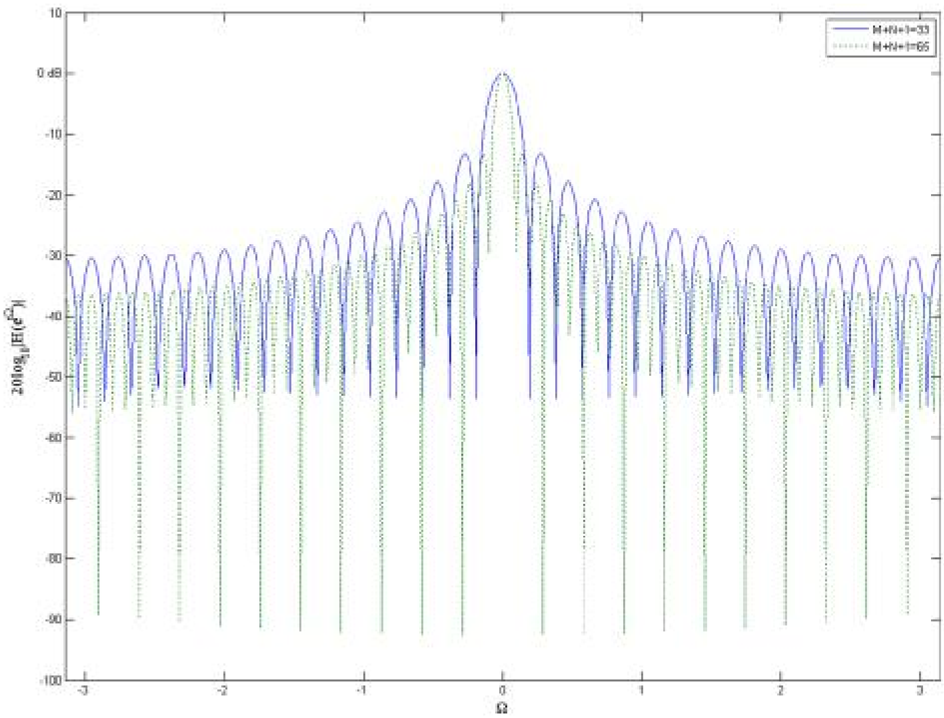
For this class of filters, the output is the average of the values of the input over a finite window:

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k].$$

The corresponding impulse response is a rectangular pulse, and the frequency response is

$$H\left(e^{j\Omega}\right) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{-j\Omega k}$$

$$= \frac{1}{N+M+1} e^{j\Omega[(N-M)/2]} \frac{\sin\left[\Omega\left[(N+M+1)/2\right]\right]}{\sin\left(\Omega/2\right)}$$



■**Figure 5.3** Log-magnitude plots for the moving average filter.

Note:

- For $N \le 0$, the filter is causal.

(4) Weighted moving average filter (generalized nonrecursive filter):

The more general form of a discrete-time nonrecursive filter is

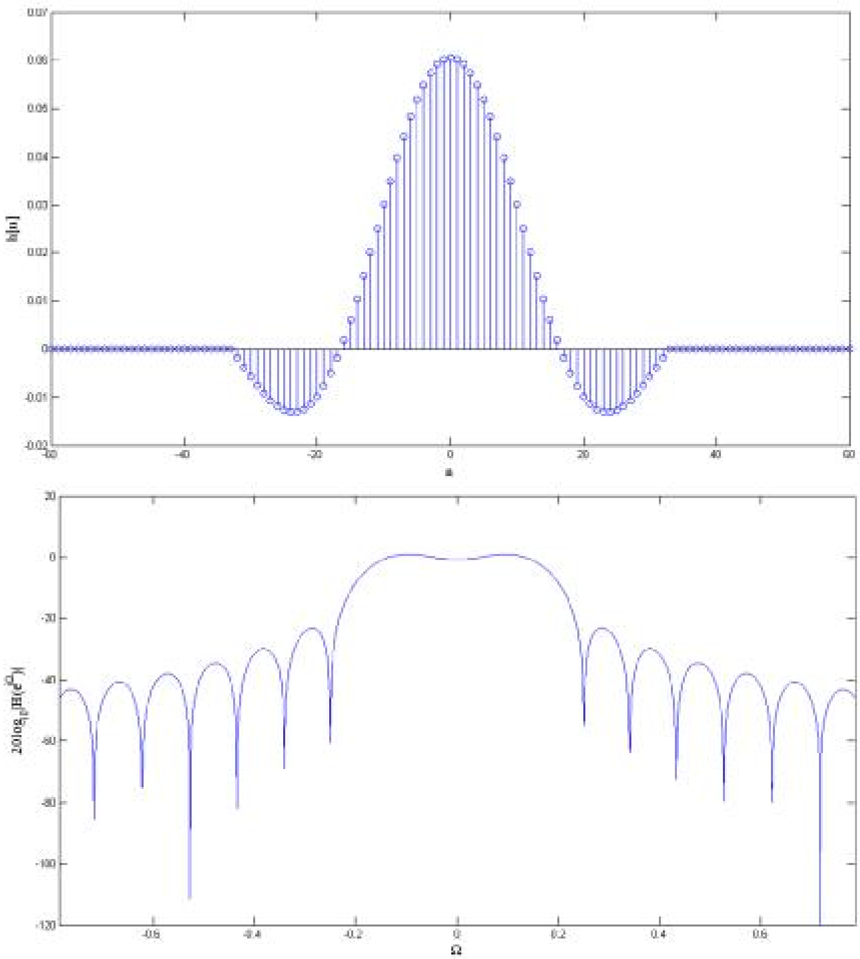
$$y[n]=\sum_{k=-N}^{M} b_k x[n-k]=\sum_{k=-N}^{M} h[k]x[n-k]=h[n]*x[n],$$

where the coefficients b_k can be selected to achieve the prescribe filter characteristics.

***Example* 5.5**: $N = M = 16$

$$\text{Let } b_k=\begin{cases}\dfrac{2}{33}\text{sinc}\left(\dfrac{2k}{33}\right) & |k|\le 32\\ 0 & |k|>32\end{cases} \Rightarrow h[n]=\begin{cases}\text{sinc}\left(\dfrac{2\pi n/33}{\pi n}\right) & |n|\le 32\\ 0 & |n|>32\end{cases}$$

Comparing this impulse response with ideal lowpass filter, it can be seen that this example corresponds to truncating, for $|n|\ge 32$, the impulse response for the ideal lowpass filter with cutoff frequency $\omega_c = 2\pi/32$.



■**Figure 5.4** Impulse response and log-magnitude of the frequency response of Example 5.5.

Note:

- Comparing this frequency response to Fig. 5.3, we observe that the bandwidth of the filter has approximately the same width, but that the transition to the stopband is sharper.
- It is clear that by intelligent choice of the weighting coefficients, the transition band can be sharpened.

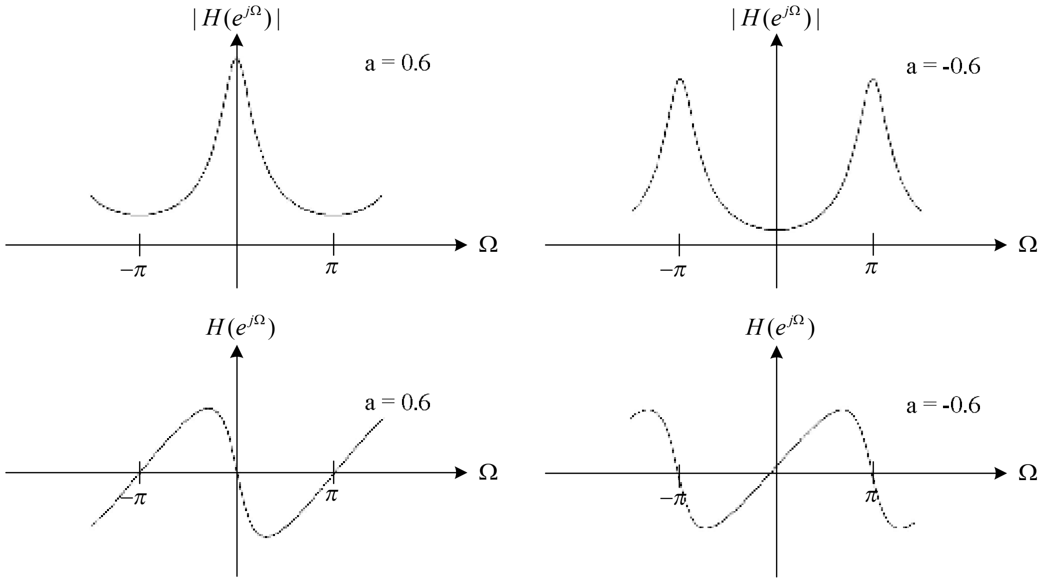
2. Recursive Discrete-Time Filters

$$y[n]-ay[n-1]=x[n] \;\Rightarrow\; H\left(e^{j\Omega}\right)=\frac{1}{1-ae^{-j\Omega}}$$

$\therefore\; h[n]=a[n]u[n] \;\Rightarrow$ Infinite Impulse Response Filters (IIR)

$\Rightarrow$ Acts as a lowpass filter for $a>0$.

$\Rightarrow$ Acts as a highpass filter for $a<0$.



Note:

- There is always tradeoff in the filter design between time-domain and frequency-domain characteristics.

$$\begin{cases} \text{cutoff frequency}\uparrow \;\Rightarrow\; \text{rise time}\downarrow \\ \text{cutoff frequency}\downarrow \;\Rightarrow\; \text{rise time}\uparrow \end{cases}$$

- Higher-order recursive difference equations can be used to provide sharper filter characteristics and to provide more flexibility in balancing time-domain and frequency-domain constraints.

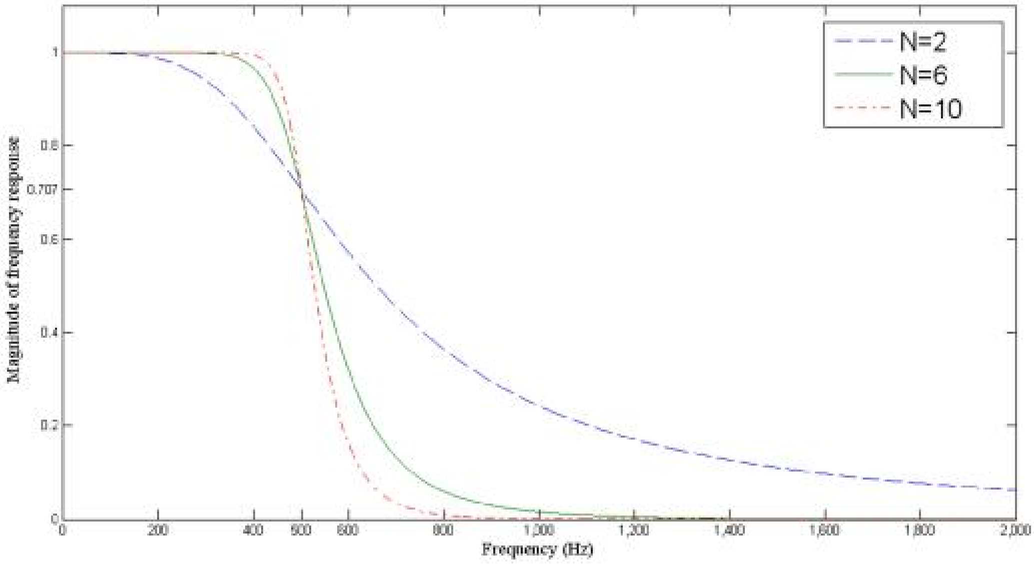
■

5-7 The Class of Butterworth Frequency-Selective Filters

There are a number of specific classes of continuous-time and discrete-time filters for which standard procedures have been developed to determine the coefficients of the associated differential or difference equation.

The class of Butterworth filters is that for which magnitude squared of the frequency response is of the form: $\left|B(j\omega)\right|^2 = \left[1+(\omega/\omega_c)^{2N}\right]^{-1}$

where N is referred to as the filter order and ω_c is 3dB frequency.



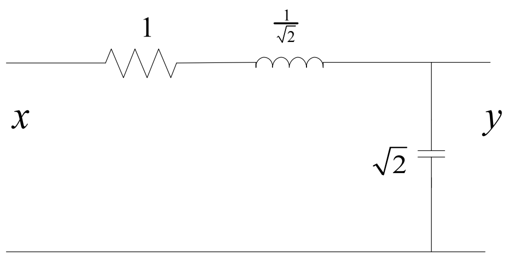
■**Figure 5.5** Magnitude of frequency response versus frequency for different N-order continuous-time Butterworth filters with cutoff frequency 500Hz.

1. The magnitude gain at $\omega = \omega_c$ is independent of the filter order.
2. The higher the order is, the sharper the transition from the passband to the stopband be.
3. First-order Butterworth filter

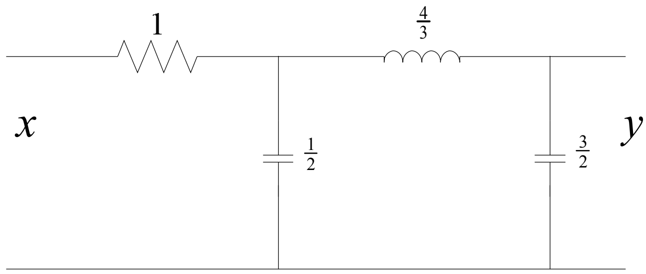
$$RC = \frac{1}{\omega_c}$$

$$\left|B(j\omega)\right|^2 = \frac{1}{1+(\omega/\omega_c)^2}$$

4. Second-order Butterworth filter

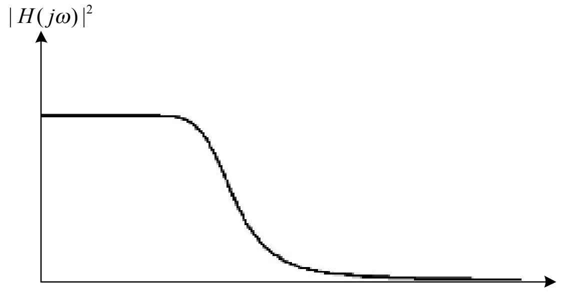


5. Third-order Butterworth filter



6. Commonly used frequency response types of continuous-time filters

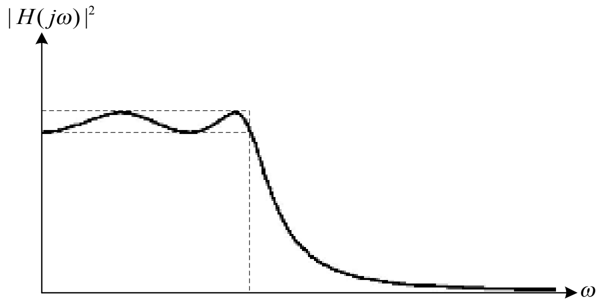
(1) Monotonic passband and stopband: Butterworth filters.



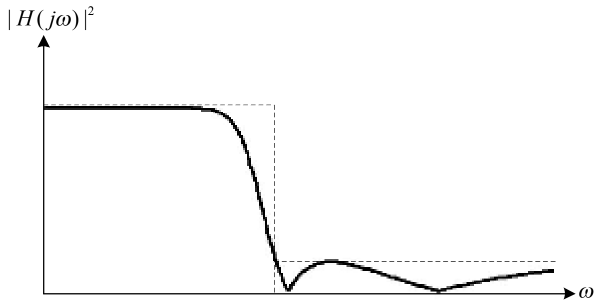
$$\left|B(j\omega)\right|^2 = \frac{1}{1+\left(\omega/\omega_c\right)^{2N}}$$

Maximally flat frequency response: The first (2N-1) derivations of $\left|B(j\omega)\right|^2$ are zero at $\omega = 0$.

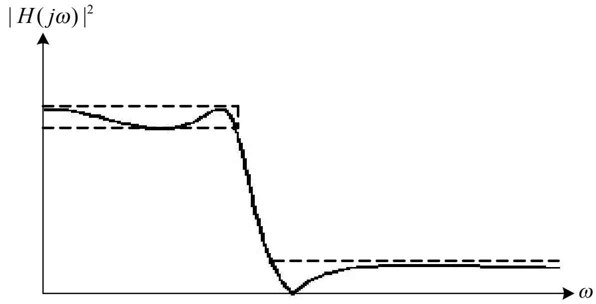
(2) Equiripple passband and monotonic stopband: Chebyshev filters.



(3) Monotonic passband and equiripple stopband: Chebyshev filters.

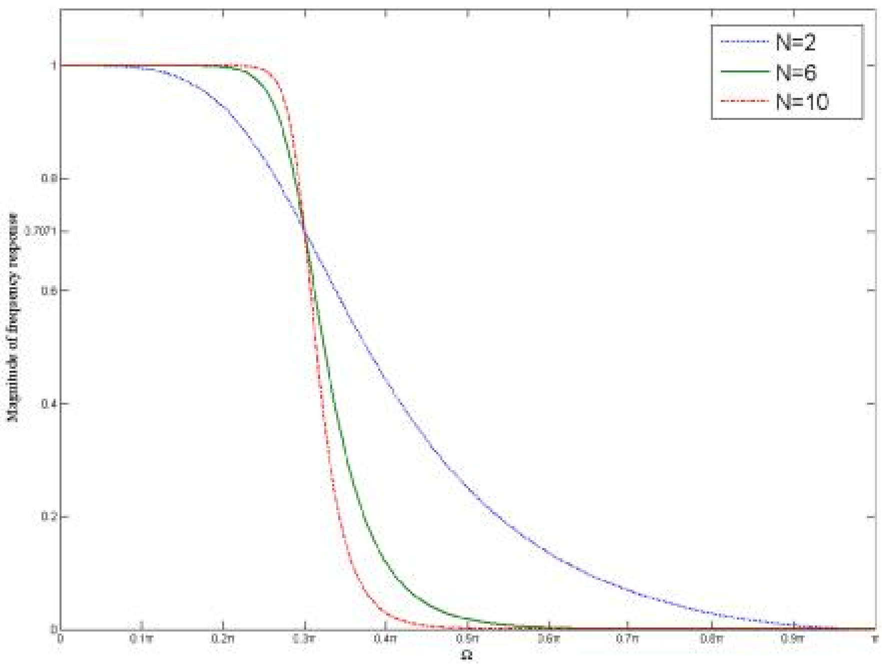


(4) Equiripple passband and stopband: Elliptic filters.



(5) Each of the above continuous-time filters has a direct counterpart in discrete-time filters.

***Example* 5.6**: The class of discrete-time Butterworth filters has a frequency response $B\left(e^{j\Omega}\right)$ for which the magnitude squared is of the form: $\left|B\left(e^{j\Omega}\right)\right|^2 = \left[1+\left(\dfrac{\tan\left(\Omega/2\right)}{\tan\left(\Omega_c/2\right)}\right)^{2N}\right]^{-1}$



■**Figure 5.6** Magnitude of frequency response versus frequency for different *N*-order discrete-time Butterworth filters with $\Omega_c = 0.3\pi$.

References:

[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, *Signals and Systems*, 2nd Ed., Prentice-Hall, 1997.

[2] Alan V. Oppenheim and Ronald W. Schafer, with John R. Buck, *Discrete-Time Signal Processing*, 2nd Ed., Prentice-Hall, 1999.