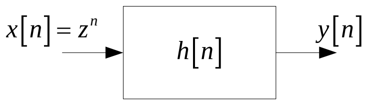
Chapter 4 Fourier Analysis for Discrete-Time Signals and Systems

4-1 Eigenfunctions and Eigenvalues of Discrete-Time LTI Systems



■ **Figure 4.1** Discrete-time LTI system with input $x[n] = z^n$, where z is a complex number.

$$\begin{aligned} y[n] &= x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = z^n H(z) \end{aligned} \tag{4.1}$$

$$\Rightarrow y[n] = H(z)x[n] \tag{4.2}$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ is the eigenvalue associated with the eigenfunction z^n.

Note:

- $$x[n] = \sum_k a_k z_k^n \rightarrow y[n] = \sum_k a_k H(z_k) z_k^n \quad \text{(superposition property)} \tag{4.3}$$
- In this chapter, we restrict ourself to the case of z^n with $|z| = 1$, i.e., the complex exponentials of the form $e^{j\Omega n}$.

4-2 Fourier Series Representation of Periodic Discrete-Time Signals: Discrete-Time Fourier Series (DTFS)

1. Discrete-time Fourier series representation

   Harmonically related discrete-time complex exponentials:

   $$\phi_k[n] = e^{jk\frac{2\pi}{N}n}, \ k = 0, \ \pm 1, \ \pm 2, \ldots \tag{4.4}$$

   where fundamental frequency = $\Omega_0 = 2\pi/N$

   fundamental period = N

   There are only N different signals in the set of $\phi_k[n], \ k = 0, \ \pm 1, \ \pm 2, \ldots$.

   ($\because e^{j(k+N)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n}$, $\therefore \phi_k[n] = \phi_{k+Nr}[n]$, r is an integer.)

If $x[n]$ is periodic with period N, then the discrete-time Fourier series representation of $x[n]$ is

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{4.5}$$

where $k =< N >$ means k varies over a range of N successive integers, e.g.,

$$\begin{cases} 0,\ 1,\ 2,\ 3,\ldots,\ N-1 \\ 1,\ 2,\ 3,\ 4,\ldots,\ N \\ 2,\ 3,\ 4,\ 5,\ldots,\ N+1 \\ \qquad\qquad \vdots \end{cases}$$

Note: Discrete-time Fourier series coefficients are the sampled values of the discrete-time Fourier transform.

2. Determination of the discrete-time Fourier series coefficients

If $x[n]$ is periodic with period N and its Fourier series representation is

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{4.6}$$

then $a_k = \dfrac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n}$.

Proof:

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{4.7}$$

$$\begin{aligned} \sum_{n=<N>} x[n] e^{-jr\frac{2\pi}{N}n} &= \sum_{n=<N>} \left(\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \right) e^{-jr\frac{2\pi}{N}n} \\ &= \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)\frac{2\pi}{N}n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)\frac{2\pi}{N}n} \end{aligned} \tag{4.8}$$

$$\because \sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n} = \begin{cases} N & ,\ k = mN \\ \dfrac{1-\left(e^{j\frac{2\pi}{N}}\right)^N}{1-e^{j\frac{2\pi}{N}}} = 0 & ,\ \text{otherwise} \end{cases} \tag{4.9}$$

$$\therefore \sum_{n=<N>} e^{j(k-r)\frac{2\pi}{N}n} = \begin{cases} N,\ k = r + mN \\ 0\ \ ,\ \text{otherwise} \end{cases} \tag{4.10}$$

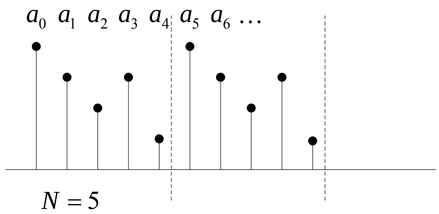
Thus,

$$\sum_{n=<N>} x[n] e^{-jr\frac{2\pi}{N}n} = N a_r \Rightarrow a_r = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jr\frac{2\pi}{N}n} \tag{4.11}$$

■

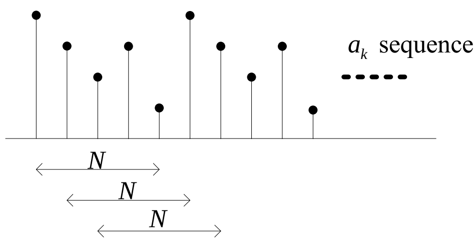
Note:

- The discrete-time Fourier series coefficients are often referred to as the spectral coefficients of $x[n]$.
- $a_k = a_{k+N}$



■ **Figure 4.2** The discrete-time Fourier series a_k will repeat periodically with period N.

- The discrete-time Fourier series representation is a finite series with N terms. (Only N successive elements of the a_k sequence are used in the Fourier series representation.)



■ **Figure 4.3** The discrete-time Fourier series are only N distinct terms that are periodic with period N.

***Example* 4.1**: $x[n] = \sin(\Omega_0 n)$, period = $2\pi/\Omega_0$

Three situations:

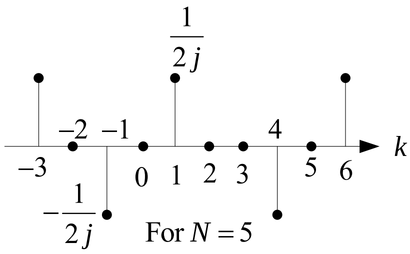
$$\begin{cases} 2\pi/\Omega_0 \text{ is an integer.} \\ 2\pi/\Omega_0 \text{ is a ratio of integers.} \end{cases} \Rightarrow \text{periodic}$$
$$2\pi/\Omega_0 \text{ is an irrational number.} \Rightarrow \text{aperiodic}$$

(1) $2\pi/\Omega_0 = N \Rightarrow x[n] = \sin\left(\frac{2\pi}{N}n\right) = \frac{1}{2j}\left(e^{j\frac{2\pi}{N}n} - e^{-j\frac{2\pi}{N}n}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$$

$$\Rightarrow a_1 = \frac{1}{2j} \text{ and } a_{-1} = -\frac{1}{2j}$$

and the remaining coefficients are zero.



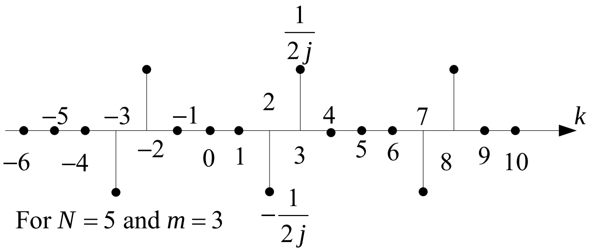
(2) $2\pi/\Omega_0 = m/N$, m and N have no common factors.

$$\Rightarrow \Omega_0 = 2\pi m/N$$

$$\Rightarrow x[n] = \sin\left(\frac{2\pi m}{N} n\right) = \frac{1}{2j}\left(e^{jm\frac{2\pi}{N}n} - e^{-jm\frac{2\pi}{N}n}\right)$$

$$\Rightarrow a_m = \frac{1}{2j} \text{ and } a_{-m} = -\frac{1}{2j}$$

and the remaining coefficients are zero.



$a_2 = a_{-3+5}$

$a_{-2} = a_{3-5}$

■

***Example* 4.2**: Discrete-time periodic square wave

For $k \neq 0,\ \pm N,\ \pm 2N, \ldots$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\frac{2\pi}{N}(m-N_1)} \quad (m = n + N_1)$$

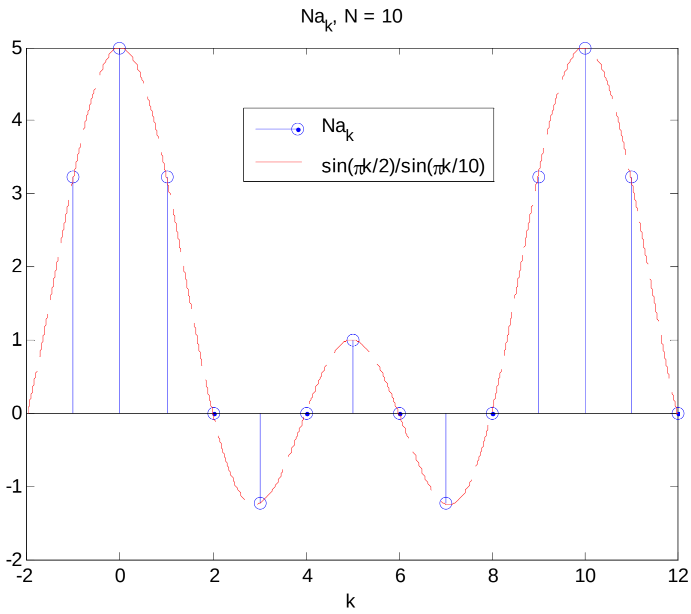
$$= \frac{1}{N} e^{jk\frac{2\pi}{N}N_1} \sum_{m=0}^{2N_1} e^{-jk\frac{2\pi}{N}m}$$

$$= \frac{1}{N} e^{jk\frac{2\pi}{N}N_1} \left(\frac{1-\left(e^{-jk\frac{2\pi}{N}}\right)^{2N_1+1}}{1-e^{-jk\frac{2\pi}{N}}} \right) = \frac{1}{N} \cdot \frac{\sin\left(2\pi k\left(N_1+\frac{1}{2}\right)/N\right)}{\sin\left(2\pi k/(2N)\right)}$$

For $k = 0, \pm N, \pm 2N, \ldots,\ a_k = \dfrac{2N_1+1}{N}$.

Note: Discrete-time counterpart of the sinc function is of the form $\sin(\beta x)/\sin(x)$.

The coefficients a_k for $2N_1 + 1 = 5$ are sketched for $N = 10$.



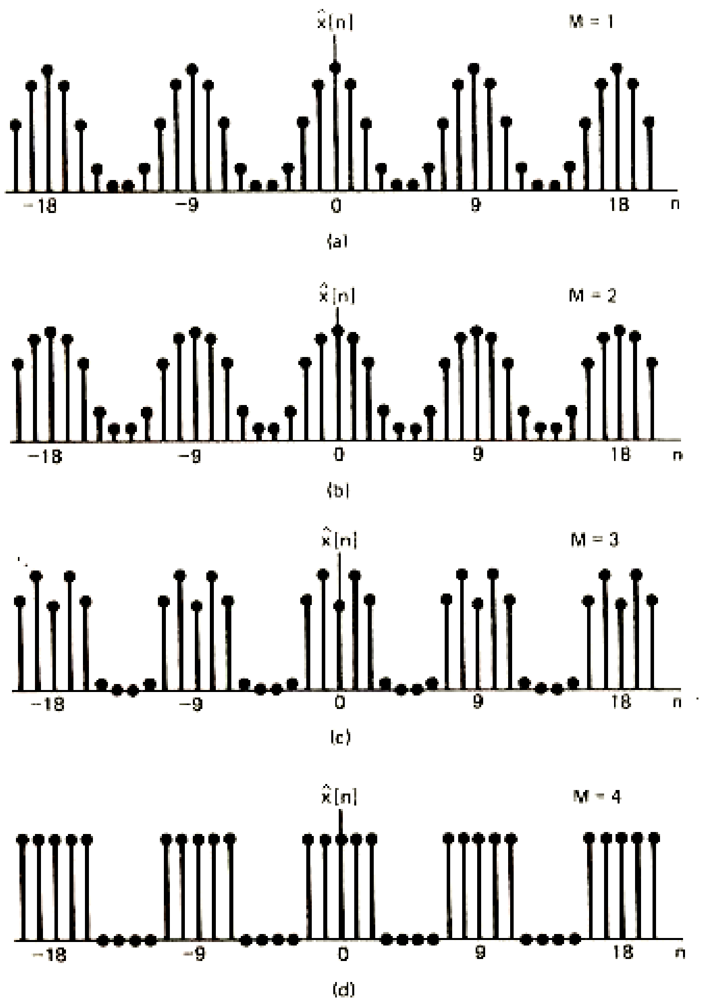
$$\Rightarrow Na_k = \frac{\sin\left((2N_1+1)\Omega/2\right)}{\sin(\Omega/2)}\Bigg|_{\Omega=2\pi k/N}$$ ■

3. Approximation of a discrete-time periodic signal using a truncated Fourier series

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} = \begin{cases} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} a_k e^{jk\frac{2\pi}{N}n}, & N \text{ is even} \\ \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} a_k e^{jk\frac{2\pi}{N}n}, & N \text{ is odd} \end{cases} \quad (4.12)$$

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk\frac{2\pi}{N}n}, \quad M < N/2 \text{ or } M < (N-1)/2 \quad (4.13)$$

(1) When $M \to N/2$ or $(N-1)/2$, the approximation of $x[n]$ by $\hat{x}[n]$ is shown in Fig. 4.4, where $x[n]$ is a square wave. There are no convergence issues and no **Gibbs phenomenon**.
(Gibbs phenomenon: it exists in the continuous-time case and the ripples in the discontinuity do not disappear with the increasing terms of summation.)



■ **Figure 4.4** Partial sums of Eq. (4.13) for the periodic square wave with $N = 9$ and $2N_1+1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

(2) In general, there are no convergence issues with the discrete-time Fourier series. (∵ Any discrete-time periodic sequence $x[n]$ is completely specified by a finite number of parameters, namely the values of the sequence over one period.)

4.

$$x[n]=\sum_{k=<N>}a_k e^{jk\frac{2\pi}{N}n}\rightarrow y[n]=\sum_{k=<N>}a_k H\left(j\frac{2\pi k}{N}\right)e^{jk\frac{2\pi}{N}n} \quad (4.14)$$

where $H\left(j\frac{2\pi k}{N}\right)=\sum_{n=-\infty}^{\infty}h[n]e^{-jk\frac{2\pi}{N}n}$. $\left(\because\begin{cases}e^{jk\frac{2\pi}{N}n}\ \text{is an eigenfunction}\\ \text{superposition property}\end{cases}\right)$

***Example* 4.3**:

$$\begin{cases}h[n]=\alpha^n u[n],\ |\alpha|<1\\ x[n]=\cos(2\pi n/N)\end{cases}$$

$$x[n]=\frac{1}{2}\left(e^{j\frac{2\pi}{N}n}+e^{-j\frac{2\pi}{N}n}\right)$$

$$H\left(j\frac{2\pi k}{N}\right)=\sum_{n=0}^{\infty}\alpha^n e^{-jk\frac{2\pi}{N}n}=\sum_{n=0}^{\infty}\left(\alpha e^{-j\frac{2\pi k}{N}}\right)^n$$

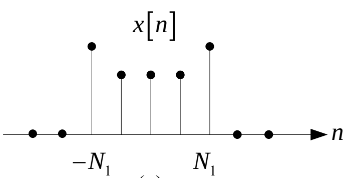
$$=\frac{1}{1-\alpha e^{-j\frac{2\pi k}{N}}}\quad\left(\because\left|\alpha e^{-j\frac{2\pi k}{N}}\right|<1\right)$$

$$\Rightarrow y[n]=\frac{1}{2}H\left(j\frac{2\pi}{N}\right)e^{j\frac{2\pi}{N}n}+\frac{1}{2}H\left(-j\frac{2\pi}{N}\right)e^{-j\frac{2\pi}{N}n}$$

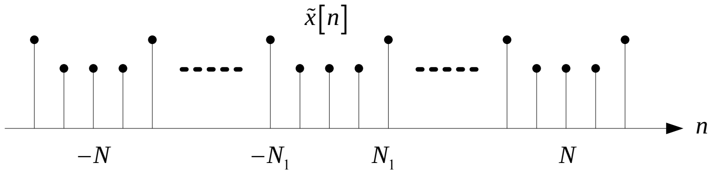
$$=\frac{1}{2}\left(\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}\right)e^{j\frac{2\pi}{N}n}+\frac{1}{2}\left(\frac{1}{1-\alpha e^{j\frac{2\pi}{N}}}\right)e^{-j\frac{2\pi}{N}n}$$ ■

4-3 Fourier Transform of Aperiodic Discrete-Time Signals: Discrete-Time Fourier Transform (DTFT)

1. Consider a general aperiodic sequence $x[n]$ which is of finite duration. From this aperiodic sequence, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is of one period.



(a)



(b)

■ **Figure 4.5** (a) Finite duration signal $x[n]$; (b) periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

Discrete-time Fourier series representation of $\tilde{x}[n]$ is

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{4.15}$$

$$a_k = \frac{1}{N}\sum_{k=<N>} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} \tag{4.16}$$

$$\because x[n] = \tilde{x}[n] \text{ for } |n| \le N_1 \tag{4.17}$$

$$\therefore a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n} \tag{4.18}$$

Defining the envelope of Na_k as $X\left(e^{j\Omega}\right)$, we have

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \tag{4.19}$$

$$Na_k = X\left(e^{j\Omega}\right)\Big|_{\Omega=\frac{2\pi k}{N}} \quad \left(\text{or } a_k = \frac{1}{N} X\left(e^{j\frac{2\pi}{N}k}\right)\right) \tag{4.20}$$

The coefficients a_k are proportional to equally spaced samples of the envelope function $X\left(e^{j\Omega}\right)$, where the sample spacing is equal to $2\pi/N$.

$$\Rightarrow \tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X\left(e^{j(k\Omega_0)}\right) e^{jk\Omega_0 n} \tag{4.21}$$

where $\Omega_0 = \dfrac{2\pi}{N}$.

$$\because N = \frac{2\pi}{\Omega_0}$$
$$\therefore \tilde{x}[n] = \frac{1}{2\pi}\sum_{k=<N>} X\left(e^{j(k\Omega_0)}\right) e^{jk\Omega_0 n}\Omega_0 \tag{4.22}$$

As $N \to \infty$, $\tilde{x}[n] \to x[n]$, and the above equation becomes a representation of $x[n]$ and the summation operator becomes the integration. $(\Omega_0 \to d\Omega,\ k\Omega_0 \to \Omega)$

Discrete-time Fourier transform pair

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega \quad \text{(synthesis equation)} \tag{4.23}$$

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \quad \text{(analysis equation)} \tag{4.24}$$

$X\left(e^{j\Omega}\right)$ is referred to as the discrete-time Fourier transform of $x[n]$ (or spectrum).

2. Explanation of the concept of spectrum $x[n]$ is a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes $X\left(e^{j\Omega}\right)\left(d\Omega/2\pi\right)$.

3. The convergence of the discrete-time Fourier transform is guaranteed if $x[n]$ is absolutely summable or if the sequence has finite energy, i.e.,

$$\sum_{n=-\infty}^{\infty}\left|x[n]\right| < \infty \tag{4.25}$$

or

$$\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 < \infty \tag{4.26}$$

4. The major differences between the continuous-time Fourier transform and the discrete-time Fourier transform:
   (1) The discrete-time Fourier transform is periodic, and the continuous-time Fourier transform is aperiodic except for some special cases. (for example, the periodic impulse train)
   (2) The discrete-time Fourier transform has a finite interval of integration in the synthesis equation, while the continuous-time Fourier transform has an infinite interval of integration in the synthesis equation.

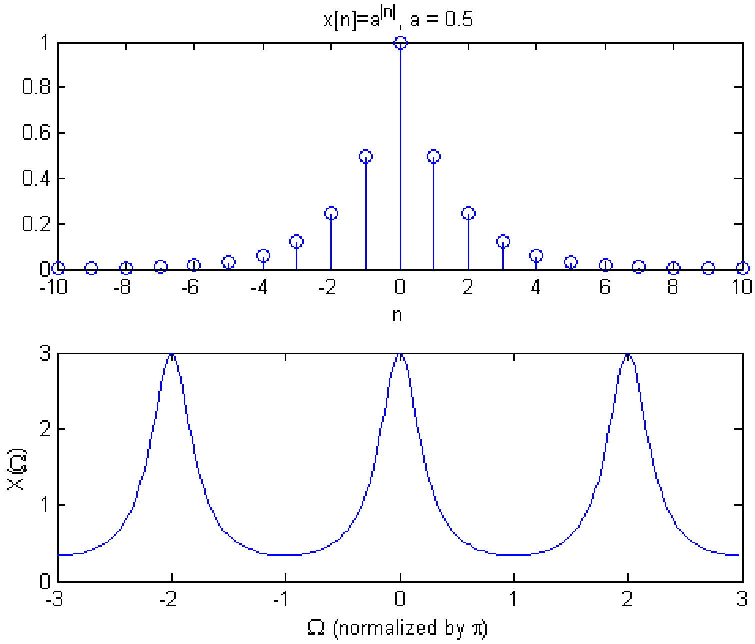
$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega & x[n] &= \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega \\ X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt & X\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \end{aligned} \tag{4.27}$$

***Example* 4.4**: $x[n]=a^{|n|},\ |a|<1$

$$X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=\sum_{n=-\infty}^{\infty}a^{|n|}e^{-j\Omega n}=\sum_{n=0}^{\infty}a^{n}e^{-j\Omega n}+\sum_{n=-\infty}^{-1}a^{-n}e^{-j\Omega n}$$

$$=\sum_{n=0}^{\infty}a^{n}e^{-j\Omega n}+\sum_{m=1}^{\infty}\left(ae^{j\Omega}\right)^{m}\quad(m=-n)$$

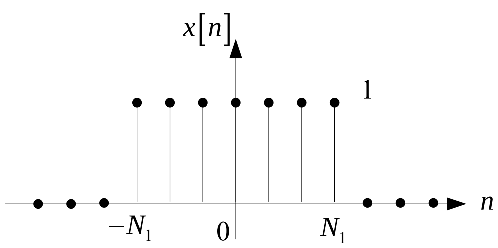
$$=\frac{1}{1-ae^{-j\Omega}}+\left(\frac{1}{1-ae^{j\Omega}}-1\right)=\frac{1-a^{2}}{1-2a\cos\Omega+a^{2}}$$



■

***Example* 4.5**:

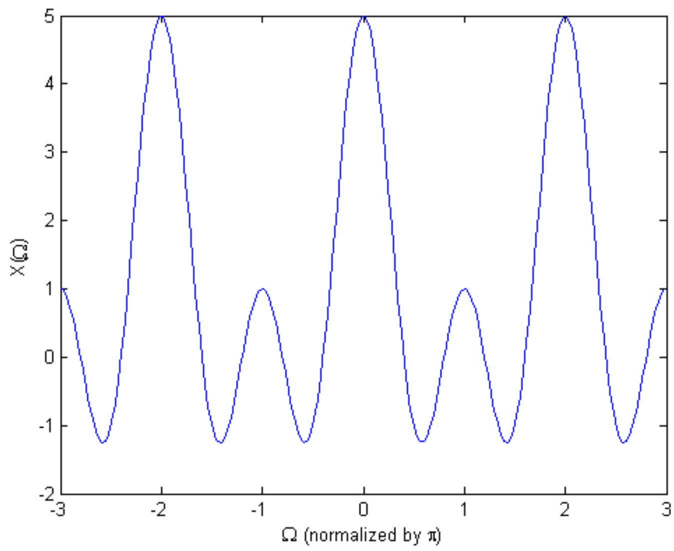
$$x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|>N_1\end{cases}\quad\text{(rectangular pulse)}$$



$$X\left(e^{j\Omega}\right)=\sum_{n=-N_1}^{N_1}e^{-j\Omega n}=\sum_{m=0}^{2N_1}e^{-j\Omega(m-N_1)}\quad(m=n+N_1)$$

$$=e^{j\Omega N_1}\sum_{m=0}^{2N_1}e^{-j\Omega m}=e^{j\Omega N_1}\frac{1-e^{-j\Omega(2N_1+1)}}{1-e^{-j\Omega}}$$

$$=e^{j\Omega N_1}\frac{e^{-j\Omega(2N_1+1)/2}}{e^{-j\Omega/2}}\left(\frac{e^{j\Omega(2N_1+1)/2}-e^{-j\Omega(2N_1+1)/2}}{e^{j\Omega/2}-e^{-j\Omega/2}}\right)=\frac{\sin\left(\Omega(2N_1+1)/2\right)}{\sin(\Omega/2)}$$



"Discrete-time counterpart of the sinc function": periodic with period 2π.

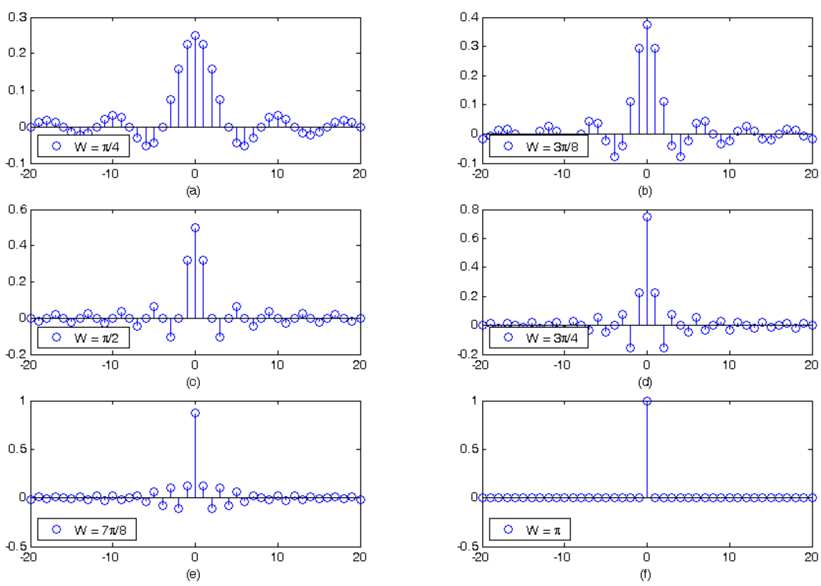
■

***Example* 4.6**: Let $x[n]=\delta[n]$, then $X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty}\delta[n]e^{-j\Omega n}=1$

Let $\hat{x}[n]=\dfrac{1}{2\pi}\int_{-W}^{W}e^{j\Omega n}d\Omega=\dfrac{1}{2\pi}\cdot\dfrac{1}{jn}e^{j\Omega n}\Big|_{-W}^{W}=\dfrac{1}{j2\pi n}\left(e^{jWn}-e^{-jWn}\right)=\dfrac{1}{\pi n}\sin(Wn)$

The approximation of $x[n]$ by $\hat{x}[n]$ is shown in the figure below. As $W\to\pi$, $\hat{x}[n]\to x[n]$ with no Gibbs phenomenon.

Note: There are no convergence problems in the discrete-time Fourier transform synthesis equation.



■

4-4 Periodic Signals and the Discrete-Time Fourier Transform

1. Fourier series coefficients as samples of the Fourier transform of one period

Let $\tilde{x}[n]$ be a periodic signal with period N, and let $x[n]$ represent one period of $\tilde{x}[n]$, i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & M \le n \le M + N - 1 \\ 0 & \text{, otherwise} \end{cases} \tag{4.28}$$

where M is arbitrary. Then

$$Na_k = X\left(e^{jk\frac{2\pi}{N}}\right) \tag{4.29}$$

where a_k is the discrete-time Fourier series coefficients of $\tilde{x}[n]$ and $X\left(e^{j\Omega}\right)$ is the discrete-time Fourier transform of $x[n]$.

$\Rightarrow$ Na_k correspond to samples of the Fourier transform of one period.

When M is varied, $X\left(e^{j\Omega}\right)$ is changed. But the values of $X\left(e^{j\Omega}\right)$ at the sample frequencies $2\pi k/N$ do not depend on M.

***Example* 4.7**:

Let $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$,

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}$$

Let $x_1[n] = \delta[n]$ (i.e., $M = 0$). Then, $X_1\left(e^{j\Omega}\right) = 1$.

Let $x_2[n] = \delta[n-N]$ (i.e., $0 < M < N$). Then, $X_2\left(e^{j\Omega}\right) = e^{-j\Omega N}$.

Clearly, $X_1\left(e^{j\Omega}\right) \ne X_2\left(e^{j\Omega}\right)$. However, at the set of sample frequencies $\Omega = 2\pi k/N$, $X_1\left(e^{j\Omega}\right)$ and $X_2\left(e^{j\Omega}\right)$ are identical. ■

2. The discrete-time Fourier transform for periodic signals

Consider the signal

$$x[n] = e^{j\Omega_0 n} \tag{4.30}$$

$$X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=\sum_{n=-\infty}^{\infty}e^{j\Omega_0 n}e^{-j\Omega n} \\ =\sum_{n=-\infty}^{\infty}e^{-j(\Omega-\Omega_0)n}=? \tag{4.31}$$

We consider the discrete-time Fourier transform

$$X\left(e^{j\Omega}\right)=\sum_{l=-\infty}^{\infty}2\pi\delta\left(\Omega-\Omega_0-2\pi l\right) \tag{4.32}$$

Then the inverse discrete-time Fourier transform $X\left(e^{j\Omega}\right)$ is

$$x[n]=\frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{\infty}2\pi\delta\left(\Omega-\Omega_0-2\pi l\right)e^{j\Omega n}d\Omega \\ =e^{j\Omega_0 n+j2\pi rn}=e^{j\Omega_0 n}\ \left(\Omega=\Omega_0+2\pi r,\ \text{with } l=r\right) \tag{4.33}$$

($\because$ Any interval of length includes exactly one impulse in the summation.)

More generally, if $x[n]$ is the sum of an arbitrary set of complex exponentials, i.e.,

$$x[n]=b_1e^{j\Omega_1 n}+b_2e^{j\Omega_2 n}+\cdots+b_Me^{j\Omega_M n} \tag{4.34}$$

then

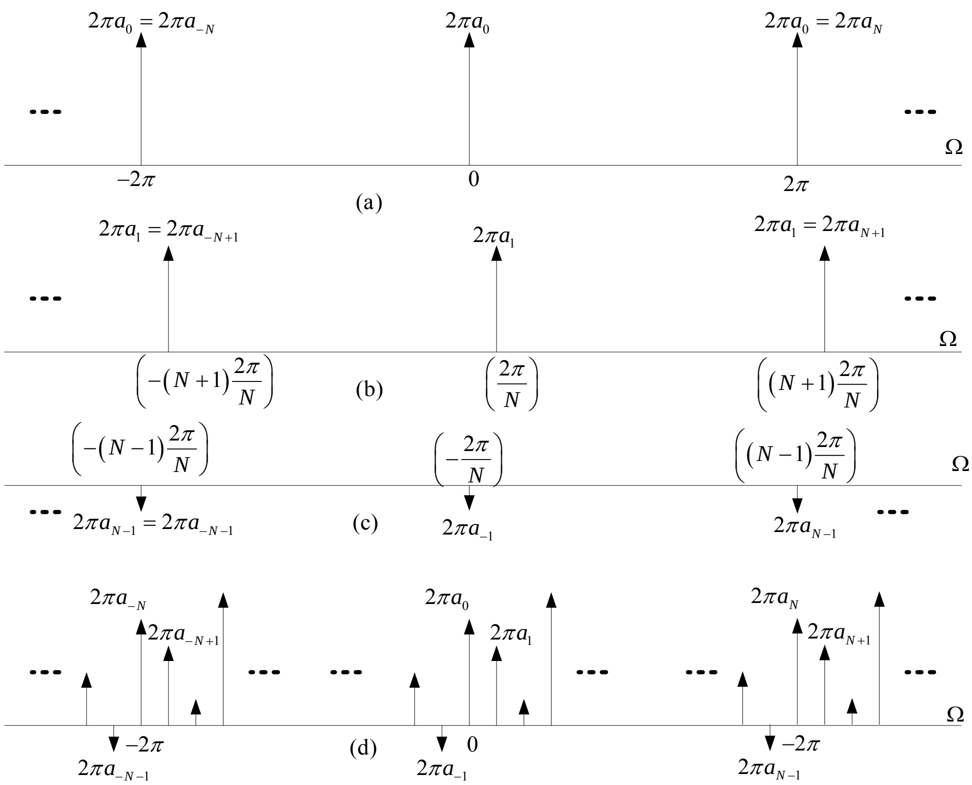
$$X\left(e^{j\Omega}\right)=b_1\sum_{l=-\infty}^{\infty}2\pi\delta\left(\Omega-\Omega_1-2\pi l\right)+b_2\sum_{l=-\infty}^{\infty}2\pi\delta\left(\Omega-\Omega_2-2\pi l\right) \\ +\cdots+b_M\sum_{l=-\infty}^{\infty}2\pi\delta\left(\Omega-\Omega_M-2\pi l\right) \tag{4.35}$$

Note:

- $e^{j\Omega_0 n}$ is periodic when $2\pi/\Omega_0=m/N$ is a rational number or integer.
- $x[n]=b_1e^{j\Omega_1 n}+b_2e^{j\Omega_2 n}+\cdots+b_Me^{j\Omega_M n}$ is periodic only when all of the $2\pi/\Omega_i=m/N$ are rational numbers or integers.
- If $x[n]$ is a periodic sequence with period N, then $x[n]$ can be represented as

$$x[n]=a_0+a_1e^{j\frac{2\pi}{N}n}+a_2e^{j2(\frac{2\pi}{N})n}+\cdots+a_{N-1}e^{j(N-1)(\frac{2\pi}{N})n} \tag{4.36}$$

$$X\left(e^{j\Omega}\right)=a_0\sum_{l=-\infty}^{\infty}2\pi\delta\left(\Omega-2\pi l\right)+a_1\sum_{l=-\infty}^{\infty}2\pi\delta\left(\Omega-\frac{2\pi}{N}-2\pi l\right) \\ +\cdots+a_{N-1}\sum_{l=-\infty}^{\infty}2\pi\delta\left(\Omega-(N-1)\frac{2\pi}{N}-2\pi l\right) \tag{4.37}$$



■ **Figure 4.6** Fourier transform of a discrete-time periodic signal. (a) the first summation on the right-hand side of Eq. (4.37); (b) the second summation on the right-hand side of Eq. (4.37); (c) the final summation on the right-hand side of Eq. (4.37); (d) the entire expression of $X(\Omega)$.

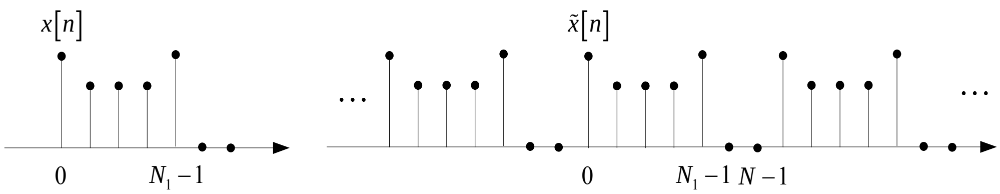
3. The discrete Fourier transform (DFT)

Let

$$x[n] = 0, \text{ outside the interval } 0 \le n \le N_1 - 1 \tag{4.38}$$

$$\tilde{x}[n] = x[n],\ 0 \le n \le N-1 \tag{4.39}$$

where $\tilde{x}[n]$ is periodic with period N and $N \ge N_1$.



■ **Figure 4.7** A nonperiodic signal $x[n]$ with finite duration and periodic signal $\tilde{x}[n]$ with period N.

The Fourier series representation of $\tilde{x}[n]$ is

$$\tilde{x}[n]=\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{4.40}$$

where

$$a_k=\frac{1}{N}\sum_{k=<N>}\tilde{x}[n]e^{-jk\frac{2\pi}{N}n}=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\frac{2\pi}{N}n} \tag{4.41}$$

Let $X[k]=Na_k$. Then we can define the N-point discrete Fourier transform (DFT) of $x[n]$ as

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-jk\frac{2\pi}{N}n} \qquad k=0,1,2,\ldots,N-1 \cdots\cdots\cdots \text{DFT} \tag{4.42}$$

with

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{jk\frac{2\pi}{N}n} \qquad n=0,1,2,\ldots,N-1 \cdots\cdots \text{Inverse DFT} \tag{4.43}$$

Note:

- The original finite duration signal can be reconstructed from its DFT.
- The length of DFT is chosen approximately so that fast algorithms can easily be used for the computation. (Fast Fourier Transform algorithms) For example, a power of 2 ($2^m=N$) is often chosen as a transform length.

4-5 Properties of the Discrete-Time Fourier Transform

1. Periodicity

   The discrete-time Fourier transform is always periodic in Ω with period 2π.

$$\begin{cases} X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n} \\ x[n]=\dfrac{1}{2\pi}\int_{2\pi}X(e^{j\Omega})e^{j\Omega n}d\Omega \end{cases} \tag{4.44}$$

2. Linearity

$$x_1[n]\xleftrightarrow{\mathcal{F}}X_1(e^{j\Omega}) \tag{4.45}$$

$$x_2[n]\xleftrightarrow{\mathcal{F}}X_2(e^{j\Omega}) \tag{4.46}$$

$$a_1x_1[n]+b_2x_2[n]\xleftrightarrow{\mathcal{F}}a_1X_1(e^{j\Omega})+b_2X_2(e^{j\Omega}) \tag{4.47}$$

3. Symmetry properties

If $x[n]$ is a real-valued sequence, then

(1) $X\left(e^{j\Omega}\right)=X^{*}\left(e^{j(-\Omega)}\right)$ (4.48)

(2) $\mathrm{Re}\left\{X\left(e^{j\Omega}\right)\right\}=\mathrm{Re}\left\{X\left(e^{j(-\Omega)}\right)\right\}$: even function (4.49)

(3) $\mathrm{Im}\left\{X\left(e^{j\Omega}\right)\right\}=-\mathrm{Im}\left\{X\left(e^{j(-\Omega)}\right)\right\}$: odd function (4.50)

(4) $\left|X\left(e^{j\Omega}\right)\right|=\left|X\left(e^{j(-\Omega)}\right)\right|$ (4.51)

(5) $\angle X\left(e^{j\Omega}\right)=-\angle X\left(e^{j(-\Omega)}\right)$ (4.52)

(6) $x_e[n]\overset{\mathcal{F}}{\longleftrightarrow}\mathrm{Re}\left\{X\left(e^{j\Omega}\right)\right\}$ (4.53)

(7) $x_o[n]\overset{\mathcal{F}}{\longleftrightarrow}j\,\mathrm{Im}\left\{X\left(e^{j\Omega}\right)\right\}$ (4.54)

4. Time shifting and frequency shifting

If $x[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j\Omega}\right)$, then

$$x[n-n_0]\overset{\mathcal{F}}{\longleftrightarrow}e^{-j\Omega n_0}X\left(e^{j\Omega}\right) \tag{4.55}$$

$$e^{j\Omega_0 n}x[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j(\Omega-\Omega_0)}\right) \tag{4.56}$$

$$\begin{aligned}\frac{1}{2\pi}\int_{2\pi}X\left(e^{j(\Omega-\Omega_0)}\right)e^{j\Omega n}d\Omega&=\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\Omega'}\right)e^{j(\Omega'+\Omega_0)n}d\Omega'\\&=e^{j\Omega_0 n}\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\Omega'}\right)e^{j\Omega' n}d\Omega'=e^{j\Omega_0 n}x[n]\end{aligned} \tag{4.57}$$

5. Differencing and Summation

$$x[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j\Omega}\right) \tag{4.58}$$

(1) $x[n]-x[n-1]\overset{\mathcal{F}}{\longleftrightarrow}\left(1-e^{-j\Omega}\right)X\left(e^{j\Omega}\right)$ (4.59)

(2) $y[n]=\sum_{m=-\infty}^{n}x[m]=x[n]*u[n]$:

$$y[n]+c-y[n-1]-c=x[n]\Rightarrow Y\left(e^{j\Omega}\right)\left(1-e^{-j\Omega}\right)=X\left(e^{j\Omega}\right) \tag{4.60}$$

$$\Rightarrow\ Y\left(e^{j\Omega}\right)=\frac{1}{1-e^{-j\Omega}}X\left(e^{j\Omega}\right).$$ This is partly correct!

$$\sum_{m=-\infty}^{n}x[m]\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{1-e^{-j\Omega}}X\left(e^{j\Omega}\right)+\underbrace{\pi X\left(e^{j\cdot 0}\right)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)}_{\text{This term reflects the dc or average value that can result from summation.}}$$

$$\left(1\overset{\mathcal{F}}{\longleftrightarrow}2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)\right)$$

(4.61)

Note:

- Average value (or dc value) is $\frac{1}{2}X\left(e^{j\cdot 0}\right)=\frac{1}{2}\sum_{m=-\infty}^{n}x[m]$.

***Example* 4.8**:

$$x[n]=\delta[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j\Omega}\right)=1$$

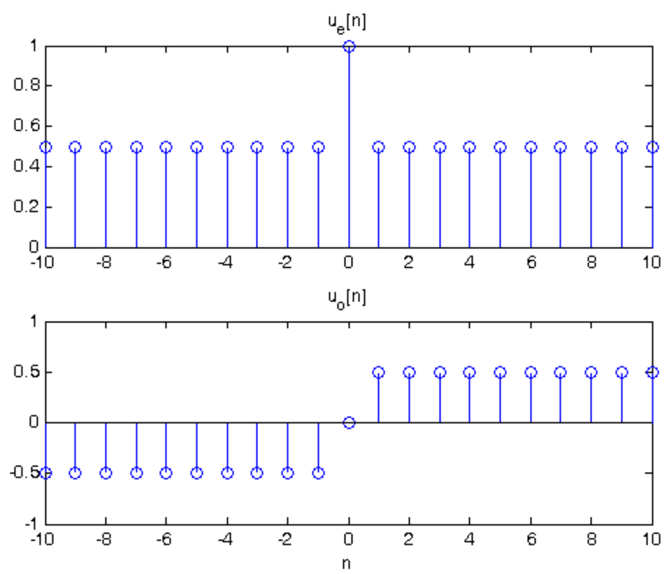
$$u[n]=\sum_{m=-\infty}^{n}\delta[m]\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{1-e^{-j\Omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$$

$$\because u[n]+c-u[n-1]-c=\delta[n]$$

$$\therefore \mathcal{F}\{u[n]\}=\frac{1}{1-e^{-j\Omega}}+g\left(e^{j\Omega}\right)$$

where $g\left(e^{j\Omega}\right)$ accounts for the dc value of $u[n]$.

$$u[n]=\underbrace{\left(u[n]-\frac{1}{2}-\frac{1}{2}\delta[n]\right)}_{\text{odd part, }u_o[n]}+\underbrace{\frac{1}{2}+\frac{1}{2}\delta[n]}_{\text{even part, }u_e[n]}$$



$$\mathcal{F}\{u_o[n]\} = \mathcal{F}\{u[n]\} - \frac{1}{2}2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k) - \frac{1}{2}$$
$$= \frac{1}{1-e^{-j\Omega}} + g\left(e^{j\Omega}\right) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k) - \frac{1}{2}$$
$$= \left(\frac{1}{1-\cos\Omega + j\sin\Omega} - \frac{1}{2}\right) + g\left(e^{j\Omega}\right) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k)$$
$$= \left(\frac{1-\cos\Omega - j\sin\Omega}{2-2\cos\Omega} - \frac{1}{2}\right) + g\left(e^{j\Omega}\right) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k)$$
$$= \frac{-j\sin\Omega}{2-2\cos\Omega} + g\left(e^{j\Omega}\right) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k)$$

$\because \mathcal{F}\{u_o[n]\}$ is purely imaginary.

$\therefore g\left(e^{j\Omega}\right) = \pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k)$

$\because \sum_{m=-\infty}^{n} x[m] = x[n] * u[n]$

$$\mathcal{F}\left\{\sum_{m=-\infty}^{n} x[m]\right\} = \mathcal{F}\{x[n]\}\cdot\mathcal{F}\{u[n]\} \quad \text{(convolution property)}$$

$$\therefore \quad = X\left(e^{j\Omega}\right)\left[\frac{1}{1-e^{-j\Omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k)\right]$$
$$= \frac{1}{1-e^{-j\Omega}}X\left(e^{j\Omega}\right) + \pi X\left(e^{j\cdot 0}\right)\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k)$$

($X(e^{j\Omega})$ is periodic with period 2π.) ■

6. Time and frequency scaling

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right) \tag{4.62}$$

(1) $x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j(-\Omega)}\right)$

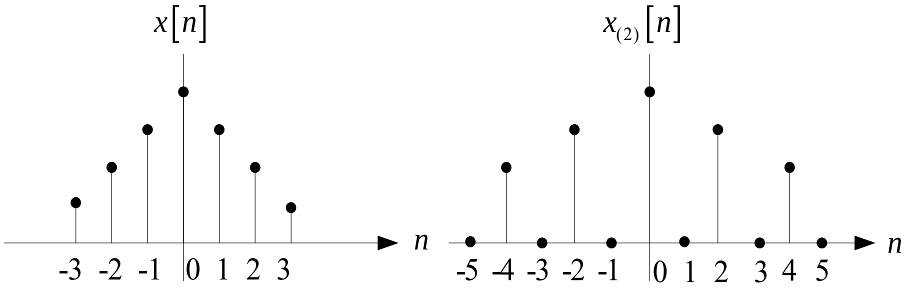
$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[-n]e^{-j\Omega n} &= \sum_{m=-\infty}^{\infty} x[m]e^{j\Omega m} \qquad (m=-n)\\ &= \sum_{m=-\infty}^{\infty} x[m]e^{-j(-\Omega)m}\end{aligned} \tag{4.63}$$

(2) $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(j\frac{\omega}{a}\right)$: continuous-time case

In the discrete-time case, the corresponding property is quite different. If a is an integer, $x[an]$ consists only of part of $x[n]$. What happens if a is not an integer?

Let k be a positive integer, and define

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{, if } n \text{ is a multiple of } k \\ 0 & \text{, if } n \text{ is not a multiple of } k \end{cases} \tag{4.64}$$



■ **Figure 4.8** The signal $x_{(2)}[n]$ obtained from $x[n]$ by inserting one zero between successive values of the original signal.

$$\begin{aligned} X_{(k)}\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\Omega n} \\ &= \sum_{r=-\infty}^{\infty} x_{(k)}[rk] e^{-j\Omega rk} \qquad \left(x_{(k)}[n] \neq 0 \text{ when } n = rk\right) \\ &= \sum_{r=-\infty}^{\infty} x[r] e^{-j(k\Omega)r} = X\left(e^{j(k\Omega)}\right) \end{aligned} \tag{4.65}$$

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{X\left(e^{j(k\Omega)}\right)}_{\text{periodic with period } 2\pi/k} \tag{4.66}$$

7. Differentiation in frequency

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \tag{4.67}$$

$$\frac{dX\left(e^{j\Omega}\right)}{d\Omega} = -\sum_{n=-\infty}^{\infty} jnx[n] e^{-j\Omega n} \Rightarrow j\frac{dX\left(e^{j\Omega}\right)}{d\Omega} = \sum_{n=-\infty}^{\infty} nx[n] e^{-j\Omega n} \tag{4.68}$$

$$\Rightarrow nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX\left(e^{j\Omega}\right)}{d\Omega} \tag{4.69}$$

8. Parseval's relation

For aperiodic signal:

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right) \tag{4.70}$$

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X\left(e^{j\Omega}\right)\right|^2 d\Omega \tag{4.71}$$

For periodic signals:

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{4.72}$$

$$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2 \tag{4.73}$$

Proof:

(1) $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \sum_{n=-\infty}^{\infty} x[n]\frac{1}{2\pi}\int_{2\pi} X^*(e^{j\Omega})e^{-j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\Omega})\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\right)d\Omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\Omega})X(e^{j\Omega})d\Omega = \frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\Omega})\right|^2 d\Omega \tag{4.74}$$

(2) $$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \frac{1}{N}\sum_{n=<N>} x[n]x^*[n]$$

$$= \frac{1}{N}\sum_{n=<N>} x[n]\sum_{k=<N>} a_k^* e^{-jk\frac{2\pi}{N}n}$$

$$= \sum_{k=<N>} a_k^*\left(\frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n}\right) = \sum_{k=<N>} a_k a_k^* \tag{4.75}$$

$$= \sum_{k=<N>} |a_k|^2$$

■

9. Convolution property

If $y[n] = x[n] * h[n]$, then

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) \tag{4.76}$$

where $X(e^{j\Omega}) = \mathcal{F}\{x[n]\}$, $H(e^{j\Omega}) = \mathcal{F}\{h[n]\}$, and $Y(e^{j\Omega}) = \mathcal{F}\{y[n]\}$.

Proof:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \tag{4.77}$$

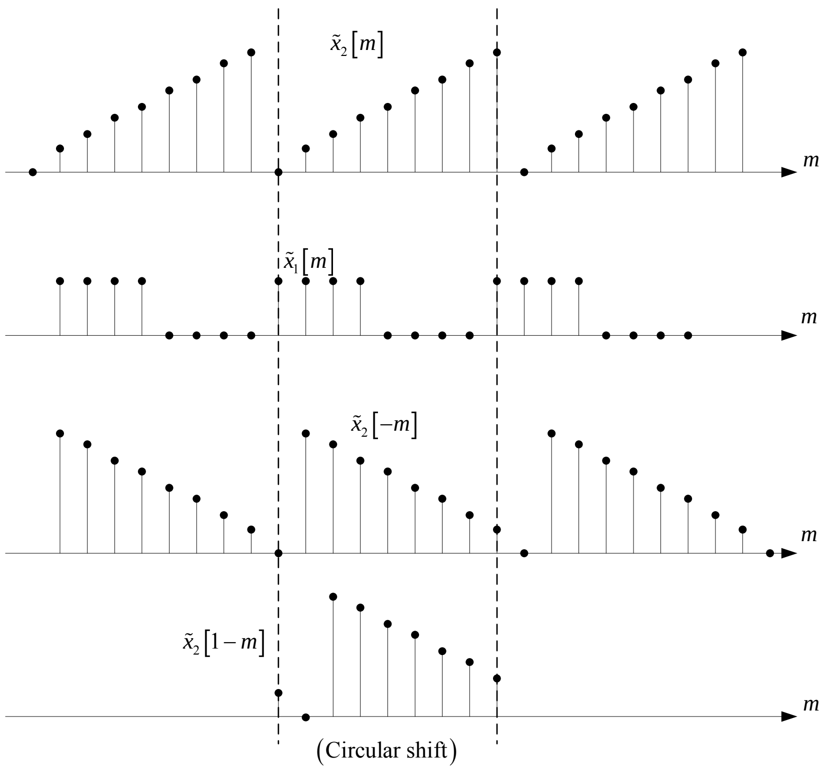
$$\begin{aligned} Y(e^{j\Omega}) = \mathcal{F}\{y[n]\} &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[m]h[n-m]e^{-j\Omega n} \\ &= \sum_{m=-\infty}^{\infty} x[m]\sum_{n=-\infty}^{\infty} h[n-m]e^{-j\Omega n} \\ &= \sum_{m=-\infty}^{\infty} x[m]H(e^{j\Omega})e^{-j\Omega m} = H(e^{j\Omega})\sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m} \\ &= H(e^{j\Omega})X(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) \end{aligned} \tag{4.78}$$

(1) Periodic convolution

Consider the periodic convolution of two sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ which are periodic with the same period *N*. The periodic convolution of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ is defined as

$$\begin{aligned}\tilde{y}[n] &= \tilde{x}_1[n] \circledast \tilde{x}_2[n] \\ &= \sum_{m=<N>} \tilde{x}_1[m]\tilde{x}_2[n-m]\end{aligned} \tag{4.79}$$

where $\tilde{y}[n]$ is also periodic with period *N*.



■ **Figure 4.9** Procedure in forming the periodic convolution of two periodic sequences.

For periodic convolution, the counterpart of the convolution property can be expressed in terms of the Fourier series coefficients.

Let

$$\tilde{x}_1[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} \qquad (\Omega_0 = 2\pi/N) \tag{4.80}$$

$$\tilde{x}_2[n] = \sum_{k=<N>} b_k e^{jk\Omega_0 n} \tag{4.81}$$

$$\tilde{y}[n] = \sum_{k=<N>} c_k e^{jk\Omega_0 n} \tag{4.82}$$

Then

$$c_k = Na_k b_k \tag{4.83}$$

Proof:

$$\tilde{y}[n] = \sum_{m=<N>} \tilde{x}_1[m]\tilde{x}_2[n-m] \tag{4.84}$$

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{n=<N>} \tilde{y}[n]e^{-jk\Omega_0 n} \\ &= \frac{1}{N}\sum_{n=<N>}\sum_{m=<N>} \tilde{x}_1[m]\tilde{x}_2[n-m]e^{-jk\Omega_0 n} \\ &= \sum_{m=<N>} \tilde{x}_1[m]\frac{1}{N}\sum_{n=<N>} \tilde{x}_2[n-m]e^{-jk\Omega_0 n} \\ &= \sum_{m=<N>} \tilde{x}_1[m]\frac{1}{N}\sum_{n'=<N>} \tilde{x}_2[n']e^{-jk\Omega_0 (n'+m)} \\ &= \sum_{m=<N>} \tilde{x}_1[m]e^{-jk\Omega_0 m}b_k = Na_k b_k \end{aligned} \tag{4.85}$$

■

(2) Let $x_1[n]$ and $x_2[n]$ be two finite-duration sequences, and suppose that

$$x_1[n] = 0 \text{, outside the interval } 0 \le n \le N_1 - 1 \tag{4.86}$$

$$x_2[n] = 0 \text{, outside the interval } 0 \le n \le N_2 - 1 \tag{4.87}$$

Let $y[n] = x_1[n] * x_2[n]$ (aperiodic convolution). Then we can find

$$y[n] = 0 \text{, outside the interval } 0 \le n \le N_1 + N_2 - 2 \tag{4.88}$$

Choose $N \ge N_1 + N_2 - 1$ and define signals $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ that are periodic with period *N* and such that

$$\tilde{x}_1[n] = x_1[n], \ \ 0 \le n \le N-1 \tag{4.89}$$

$$\tilde{x}_2[n] = x_2[n], \ \ 0 \le n \le N-1 \tag{4.90}$$

Let $\tilde{y}[n] = \tilde{x}_1[n] \circledast \tilde{x}_2[n]$ (periodic convolution), then we obtain $y[n] = \tilde{y}[n], \ 0 \le n \le N-1$.

$\Rightarrow$ The periodic convolution $\tilde{y}[n]$ equals the aperiodic convolution $y[n]$ over one period.

An algorithm for the calculation of the aperiodic convolution of $x_1[n]$ and $x_2[n]$:

(a) Calculate the DFTs $\tilde{X}_1(k)$ and $\tilde{X}_2(k)$ of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

(b) Multiply these DFTs together to obtain the DFT of $y[n]$:

$$\tilde{Y}(k) = \tilde{X}_1(k) \cdot \tilde{X}_2(k) \tag{4.91}$$

(c) Calculate the inverse DFT of $\tilde{Y}(k)$. The result is the desired convolution $\tilde{y}[n]$.

$$\begin{cases} \tilde{X}_1(k) = Na_k = \sum_{n=0}^{N-1} x_1[n] e^{-jk\frac{2\pi}{N}n},\ k = 0,1,2,\dots,N-1 & (4.92) \\ \tilde{X}_2(k) = Nb_k = \sum_{n=0}^{N-1} x_2[n] e^{-jk\frac{2\pi}{N}n},\ k = 0,1,2,\dots,N-1 & (4.93) \\ \tilde{Y}(k) = Nc_k = N^2 a_k b_k = \tilde{X}_1(k) \cdot \tilde{X}_2(k),\ k = 0,1,2,\dots,N-1 & (4.94) \\ \tilde{y}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{Y}(k) e^{jk\frac{2\pi}{N}n},\ n = 0,1,2,\dots,N-1 & (4.95) \end{cases}$$

***Example* 4.9**:

$$h[n] = \alpha^n u[n] \xleftrightarrow{\mathcal{F}} H\left(e^{j\Omega}\right) = \frac{1}{1-\alpha e^{-j\Omega}}$$

$$x[n] = \beta^n u[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j\Omega}\right) = \frac{1}{1-\beta e^{-j\Omega}}$$

$$Y\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right) X\left(e^{j\Omega}\right) = \frac{1}{\left(1-\alpha e^{-j\Omega}\right)\left(1-\beta e^{-j\Omega}\right)}$$

If $\alpha \neq \beta$,

$$Y\left(e^{j\Omega}\right) = \frac{A}{1-\alpha e^{-j\Omega}} + \frac{B}{1-\beta e^{-j\Omega}}$$

$$A = \frac{\alpha}{\alpha-\beta},\ B = \frac{\beta}{\alpha-\beta}$$

$$\Rightarrow y[n] = \frac{\alpha}{\alpha-\beta} \alpha^n u[n] + \frac{\beta}{\alpha-\beta} \beta^n u[n]$$

If $\alpha = \beta$,

$$Y\left(e^{j\Omega}\right) = \left(\frac{1}{1-\alpha e^{-j\Omega}}\right)^2 = \frac{j}{\alpha} e^{j\Omega} \frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\Omega}}$$

$$n\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$(n+1)\alpha^{n+1} u[n+1] \overset{\mathcal{F}}{\longleftrightarrow} je^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$\left(\text{time shifting property, } x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\Omega n_0} X\left(e^{j\Omega}\right)\right)$$

$$\begin{aligned} y[n] &= \frac{1}{\alpha}(n+1)\alpha^{n+1}u[n+1] \\ &= (n+1)\alpha^n u[n+1] \\ &= (n+1)\alpha^n u[n] \quad (\because n=-1,\ n+1=0) \end{aligned}$$

■

***Example* 4.10**:

Let $x_1[n] = x_2[n] = \begin{cases} 1, 0 \le n \le N-1 \\ 0, \ \text{otherwise} \end{cases}$.

(i) Find $\tilde{y}_1[n] = \tilde{x}_1[n] \circledast \tilde{x}_2[n]$ via DFT: $\tilde{x}_1[n] = \tilde{x}_2[n]$ is periodic with period *N*. $\tilde{x}_1[n]$ is equal to $\tilde{x}_2[n]$ for $0 \le n \le N-1$.

$$\tilde{X}_1(k) = \tilde{X}_2(k) = \sum_{n=0}^{N-1} e^{-jk\frac{2\pi}{N}n} = \begin{cases} N, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{Y}_1(k) = \tilde{X}_1(k)\tilde{X}_2(k) = \begin{cases} N^2, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{y}_1[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{Y}_1(k) e^{jk\frac{2\pi}{N}n} = N,\ 0 \le n \le N-1$$

(ii) Find $y_2[n] = x_1[n] * x_2[n]$ via DFT:

Since $2N > (N+N-1)$, we use 2*N*-point DFT and IDFT for calculating $y_2[n]$.

$$\tilde{X}_1(k) = \tilde{X}_2(k) = \sum_{n=0}^{2N-1} e^{-jk\frac{2\pi}{2N}n},\ k = 0,1,2,\ldots,2N-1$$

$$\tilde{Y}_2(k) = \tilde{X}_1(k)\tilde{X}_2(k),\ k = 0,1,2,\ldots,2N-1$$

$$y_2[n] = \frac{1}{2N}\sum_{k=0}^{2N-1} \tilde{Y}_2(k) e^{jk\frac{2\pi}{2N}n},\ 0 \le n \le 2N-1$$

■

10. Modulation property

$$y[n] = x_1[n]x_2[n] \tag{4.96}$$

$$x_1[n] \xleftrightarrow{\mathcal{F}} X_1\left(e^{j\Omega}\right) \tag{4.97}$$

$$x_2[n] \xleftrightarrow{\mathcal{F}} X_2\left(e^{j\Omega}\right) \tag{4.98}$$

$$y[n] \xleftrightarrow{\mathcal{F}} Y\left(e^{j\Omega}\right) \tag{4.99}$$

$$\Rightarrow Y\left(e^{j\Omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right) X_2\left(e^{j(\Omega-\theta)}\right) d\theta = \frac{1}{2\pi} X_1\left(e^{j\Omega}\right) \circledast X_2\left(e^{j\Omega}\right) \tag{4.100}$$

(periodic convolution)

Proof:

$$Y\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\Omega n} \tag{4.101}$$

$$\because x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)e^{j\theta n}d\theta \tag{4.102}$$

$$\begin{aligned}\therefore Y\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)e^{j\theta n}d\theta\right\}e^{-j\Omega n} \\ &= \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)\left(\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\Omega-\theta)n}\right)d\theta \\ &= \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)X_2\left(e^{j(\Omega-\theta)}\right)d\theta\end{aligned} \tag{4.103}$$

■

4-6 Duality

1. Discrete-time Fourier series

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{4.104}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n} \tag{4.105}$$

$$f[m] = \frac{1}{N}\sum_{r=<N>} g[r]e^{-jm\frac{2\pi}{N}r} \tag{4.106}$$

(1) Let $m = k$ and $r = n$, the sequence $f[k]$ corresponds to the Fourier series coefficients of the signal $g[n]$, i.e.,

$$g[n] \xleftrightarrow{\mathcal{F}} f[k] = \frac{1}{N}\sum_{n=<N>} g[n]e^{-jk\frac{2\pi}{N}n} \tag{4.107}$$

(2) Let $m = n$ and $r = -k$, the sequence $f[m]$ becomes

$$f[n] = \frac{1}{N}\sum_{k=<N>} g[-k]e^{jk\frac{2\pi}{N}n} \quad (4.108)$$

$$f[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{N}g[-k] \quad (4.109)$$

($\frac{1}{N}g[-k]$ corresponds to the Fourier series coefficients of $f[n]$.)

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} a_k \ (\text{also periodic}) \quad (4.110)$$

There are some notes about it

Note:

- The duality property implies that the Fourier series coefficients for the periodic sequence a_k are the values $\frac{1}{N}x[-n]$ (i.e., are proportional to the original reversed in time).
- The duality property implies that every property of the discrete-time Fourier series has a dual.

***Example* 4.11**:

$$\begin{cases} x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} a_k e^{-jk\frac{2\pi}{N}n_0} \\ e^{jM\frac{2\pi}{N}n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} a_{k-M} \end{cases}$$

$$\begin{cases} \sum_{r=<N>} x[r]y[n-r] \overset{\mathcal{F}}{\longleftrightarrow} Na_k b_k \\ x[n]y[n] \overset{\mathcal{F}}{\longleftrightarrow} \sum_{l=<N>} a_l b_{k-l} \end{cases}$$

■

2. Discrete-time Fourier transform and continuous-time Fourier series

$$\begin{cases} x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega \\ X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \end{cases} \quad \text{Discrete-time Fourier transform} \quad (4.111)$$

$$\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt \end{cases} \quad \text{Continuous-time Fourier series} \quad (4.112)$$

Let $f(u)$ represent a periodic function of a continuous variable with period 2π, and let $g[m]$ be a discrete sequence related to $f(u)$ by

$$f(u)=\sum_{m=-\infty}^{\infty} g[m]e^{-jum} \quad (4.113)$$

(1) $u=\Omega$ and $m = n$: $f(e^{j\Omega})$ is the discrete-time Fourier transform of $g[n]$, i.e.,

$$g[n] \overset{\mathcal{F}}{\longleftrightarrow} f(e^{j\Omega}) \quad (4.114)$$

(2) $u=t$ and $m=-k$: $g[-k]$ is the Fourier series coefficients of $f(t)$, i.e.,

$$f(t) \overset{\mathcal{FS}}{\longleftrightarrow} g[-k] \quad (\omega_0 = 2\pi/T_0 = 1) \quad (4.115)$$

Note:

- Since $X(e^{j\Omega})$ is a periodic function of a continuous variable, we can expand it in a Fourier series with $\omega_0 = 1$ $(T_0 = 2\pi)$ and Ω, rather than *t*, as the continuous variable.
  $\Rightarrow$ From the duality relationship, we can conclude that the Fourier series coefficients of $X(e^{j\Omega})$ will be the original sequence $x[n]$ reversed in order.

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\Omega}) \overset{\mathcal{FS}}{\longleftrightarrow} x[-k] \quad (4.116)$$

- $$\frac{1}{2\pi}\int_{2\pi} x_1(\tau)x_2(t-\tau)d\tau \overset{\mathcal{FS}}{\longleftrightarrow} a_k b_k \quad (4.117)$$

$$x[n]y[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\Omega-\theta)})d\theta \quad (4.118)$$

- Summary of Fourier series and transform expressions (See Table 4.1)

Table 4.1 Summary of Fourier series and transform expressions

	Continuous-time	
	Time domain	**Frequency domain**
Fourier Series	$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k=\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$ discrete frequency aperiodic in frequency
Fourier Transform	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time ⟷ duality aperiodic in time	$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency

	Discrete-time	
	Time domain	**Frequency domain**
Fourier Series	$x[n]=\sum_{k=<N>} a_k e^{jk(2\pi/N)n}$ discrete time ⟷ duality periodic in time	$a_k=\frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x[n]=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\Omega})e^{j\Omega t}d\Omega$ discrete time aperiodic in time	$X(e^{j\Omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ continuous frequency periodic in frequency

* : duality

4-7 The Polar Representation of Discrete-Time Fourier Transforms

$$x[n]\overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right) \tag{4.119}$$

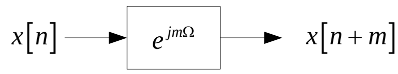
$$X\left(e^{j\Omega}\right)=\left|X\left(e^{j\Omega}\right)\right|e^{j\angle X\left(e^{j\Omega}\right)} \tag{4.120}$$

where $\left|X\left(e^{j\Omega}\right)\right|$ and $\angle X\left(e^{j\Omega}\right)$ are the magnitude and phase of $X\left(e^{j\Omega}\right)$.

1. Both $\left|X\left(e^{j\Omega}\right)\right|$ and $e^{j\angle X\left(e^{j\Omega}\right)}$ are periodic with period 2π.
2. $\left|X\left(e^{j\Omega}\right)\right|$ contains the information about the relative magnitudes of the complex exponentials that make up $x[n]$.
3. $\angle X\left(e^{j\Omega}\right)$ provides a description of the relative phases of the different complex exponentials in the Fourier transform $x[n]$. A change in the phase function of $X\left(e^{j\Omega}\right)$ may lead to a distortion of the signal $x(t)$.

(1) Linear phase: the phase shift at frequency Ω is a linear function of Ω.

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right)$$
$$\Rightarrow X\left(e^{j\Omega}\right)e^{jm\Omega} = \left|X\left(e^{j\Omega}\right)\right|e^{j\left(\angle X(\Omega)+m\Omega\right)} \overset{\mathcal{F}}{\longleftrightarrow} x[n+m] \tag{4.121}$$



■ **Figure 4.10** Illustration of the linear phase system.

"No distortion occurs." The output is simply a shifted version of the input.

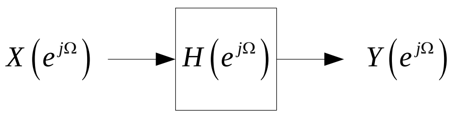
(2) Nonlinear phase: the phase shift at frequency Ω is a nonlinear function of Ω.

***Example* 4.12**: $x[n] = \underbrace{e^{j\Omega_1 n}}_{x_1[n]} + \underbrace{e^{j\Omega_2 n}}_{x_2[n]}$

$$x'[n] = \underbrace{e^{j\Omega_1 n}e^{j\Omega_1}}_{x_1[n+1]} + \underbrace{e^{j\Omega_2 n}e^{j2\Omega_2}}_{x_2[n+2]}$$

"Distortion occurs." The delays of different frequency elements may be different. ■

4. LTI systems



■ **Figure 4.11** The representation of an LTI system in frequency domain.

$$Y\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right)X\left(e^{j\Omega}\right) \tag{4.122}$$

where $H\left(e^{j\Omega}\right)$ is the frequency response.

$$\left|Y\left(e^{j\Omega}\right)\right| = \left|H\left(e^{j\Omega}\right)\right|\left|X\left(e^{j\Omega}\right)\right| \tag{4.123}$$
$$\angle Y\left(e^{j\Omega}\right) = \angle H\left(e^{j\Omega}\right) + \angle X\left(e^{j\Omega}\right) \tag{4.124}$$

Note: The magnitude of the frequency response of an LTI system is sometimes referred to as the gain of the system.

5. Graphical representation of the discrete-time Fourier transform

Plotting $\angle H\left(e^{j\Omega}\right)$ in radians for $-\pi \le \Omega \le \pi$

Plotting $\left|H\left(e^{j\Omega}\right)\right|$ in decibels ($20\log_{10}\left|H\left(e^{j\Omega}\right)\right|$) for $-\pi \le \Omega \le \pi$

If the signal (or function) $h[n]$ is real, we actually need plot $H\left(e^{j\Omega}\right)$ only for $0 \le \Omega \le \pi$. For $-\pi \le \Omega \le 0$, we can calculate $H\left(e^{j\Omega}\right)$ using the relations

$$\left|H\left(e^{j(-\Omega)}\right)\right| = \left|H\left(e^{j\Omega}\right)\right| \tag{4.125}$$

$$\angle H\left(e^{j(-\Omega)}\right) = -\angle H\left(e^{j\Omega}\right) \tag{4.126}$$

4-8 The Frequency Response of Systems Characterized by Linear Constant–Coefficient Difference Equations

1. Calculation of the frequency and impulse responses

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{4.127}$$

Assume that the Fourier transforms of $x[n]$, $y[n]$, and the system impulse response $h[n]$ all exist.

$$x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j\Omega}\right) \tag{4.128}$$

$$y[n] \xleftrightarrow{\mathcal{F}} Y\left(e^{j\Omega}\right) \tag{4.129}$$

$$h[n] \xleftrightarrow{\mathcal{F}} H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} \tag{4.130}$$

$$\Rightarrow \sum_{k=0}^{N} a_k e^{-jk\Omega} Y\left(e^{j\Omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\Omega} X\left(e^{j\Omega}\right) \tag{4.131}$$

$$\Rightarrow H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\Omega}}{\sum_{k=0}^{N} a_k e^{-jk\Omega}} \tag{4.132}$$

***Example* 4.13**: $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$

$$Y\left(e^{j\Omega}\right) - \frac{3}{4}e^{-j\Omega}Y\left(e^{j\Omega}\right) + \frac{1}{8}e^{-j2\Omega}Y\left(e^{j\Omega}\right) = 2X\left(e^{j\Omega}\right)$$

$$Y\left(e^{j\Omega}\right)\left(1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}\right) = 2X\left(e^{j\Omega}\right)$$

$$\Rightarrow H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}}$$

$$= \frac{2}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)} = \frac{4}{1-\frac{1}{2}e^{-j\Omega}} - \frac{2}{1-\frac{1}{4}e^{-j\Omega}}$$

$$\Rightarrow h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Note:

- $H\left(e^{j\Omega}\right) = \frac{1}{1-ae^{-j\Omega}} = 1 + ae^{-j\Omega} + a^2 e^{-j2\Omega} + \cdots,\ |a|<1$

  $h[n] = \delta[n] + a\delta[n-1] + a^2\delta[n-2] + \cdots = a^n u[n]$

- $a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-ae^{-j\Omega}},\ |a|<1$

■

***Example* 4.14**: $H\left(e^{j\Omega}\right) = \frac{2}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\Omega}\right) = \frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

$$Y\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right) X\left(e^{j\Omega}\right) = \frac{2}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2\left(1-\frac{1}{2}e^{-j\Omega}\right)}$$

$$= \frac{B_{11}}{\left(1-\frac{1}{4}e^{-j\Omega}\right)} + \frac{B_{12}}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2} + \frac{B_{21}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)}$$

$$Y(v) = \frac{2}{\left(1-\frac{1}{4}v\right)^2\left(1-\frac{1}{2}v\right)},\ v_1 = 4,\ v_2 = 2$$

$$B_{12} = \left(1-\frac{1}{4}v\right)^2 Y(v)\Big|_{v=v_1=4} = \frac{2}{1-\frac{1}{2}v}\Bigg|_{v=v_1=4} = -2$$

$$B_{11} = -v_1 \frac{d}{dv}\left[\left(1-\frac{1}{4}v\right)^2 Y(v)\right]\Bigg|_{v=v_1=4} = -4\frac{d}{dv}\left(\frac{2}{1-\frac{1}{2}v}\right)\Bigg|_{v=4} = 4\frac{2\cdot\frac{1}{2}}{\left(1-\frac{1}{2}v\right)^2}\Bigg|_{v=4} = -4$$

$$\left(\left(1-\frac{1}{4}v\right)^2 Y(v) = B_{11}\left(1-\frac{1}{4}v\right) \xrightarrow{\frac{d}{dv}} \frac{d}{dv}\left(1-\frac{1}{4}v\right)^2 Y(v) = -\frac{1}{4}B_{11} = -\frac{1}{v_1}B_{11}\right)$$

$$B_{21} = \left(1-\frac{1}{2}v\right)Y(v)\Big|_{v=v_2=2} = \frac{2}{\left(1-\frac{1}{4}v\right)^2}\Bigg|_{v=2} = 8$$

$$\Rightarrow Y\left(e^{j\Omega}\right) = \frac{-4}{\left(1-\frac{1}{4}e^{-j\Omega}\right)} + \frac{-2}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2} + \frac{8}{\left(1-\frac{1}{2}e^{-j\Omega}\right)}$$

$$\Rightarrow y[n] = -4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n] + 8\left(\frac{1}{2}\right)^n u[n]$$

- $$X\left(e^{j\Omega}\right) = \frac{1}{\left(1-\alpha e^{-j\Omega}\right)^2} = \frac{j}{\alpha}e^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$\alpha^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\alpha e^{-j\Omega}}$$

$$n\alpha^n u[n] \xleftrightarrow{\mathcal{F}} j\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$(n+1)\alpha^{(n+1)}u[n+1] \xleftrightarrow{\mathcal{F}} je^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right)$$

$$(n+1)\alpha^n u[n+1] \xleftrightarrow{\mathcal{F}} \frac{j}{\alpha}e^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right) = \frac{1}{\left(1-\alpha e^{-j\Omega}\right)^2}$$

$$\Rightarrow (n+1)\alpha^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{\left(1-\alpha e^{-j\Omega}\right)^2}$$

$$\left(\because (n+1)\alpha^n u[n+1] = 0 \text{ when } n=-1\right)$$

■

2. Cascade- and parallel-form structures

(1) Cascade form

$$H\left(e^{j\Omega}\right) = \frac{b_0\prod_{k=1}^{N}\left(1+\mu_k e^{-j\Omega}\right)}{a_0\prod_{k=1}^{M}\left(1+\eta_k e^{-j\Omega}\right)} \tag{4.133}$$

where μ_k and η_k may be complex, but they then appear in complex-conjugate pairs.

Let $M = N$. Multiplying out $\left(1+\mu_k e^{-j\Omega}\right)\left(1+\mu_k^* e^{-j\Omega}\right)$ and $\left(1+\eta_k e^{-j\Omega}\right)\left(1+\eta_k^* e^{-j\Omega}\right)$, we obtain

$$1+\left(\mu_k+\mu_k^*\right)e^{-j\Omega}+\left|\mu_k\right|^2 e^{-j2\Omega}=1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-j2\Omega} \quad (4.134)$$

and

$$1+\left(\eta_k+\eta_k^*\right)e^{-j\Omega}+\left|\eta_k\right|^2 e^{-j2\Omega}=1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega} \quad (4.135)$$

where $\left(\mu_k+\mu_k^*\right)$, $\left|\mu_k\right|^2$, $\left(\eta_k+\eta_k^*\right)$, and $\left|\eta_k\right|^2$ are all real.

$$H\left(e^{j\Omega}\right)=\frac{b_0}{a_0}\cdot\frac{\prod_{k=1}^{P}\left(1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-j2\Omega}\right)\prod_{k=1}^{N-2P}\left(1+\mu_k e^{-j\Omega}\right)}{\prod_{k=1}^{Q}\left(1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}\right)\prod_{k=1}^{N-2Q}\left(1+\eta_k e^{-j\Omega}\right)} \quad (4.136)$$

where the coefficients are all real.

Note:

- The frequency response of any LTI system described by a linear constant coefficient difference equation can be written as the product of first- and second-order terms.
- The LTI system can be realized as the cascade of first- and second-order LTI systems.

(a) Realization of a second-order LTI system

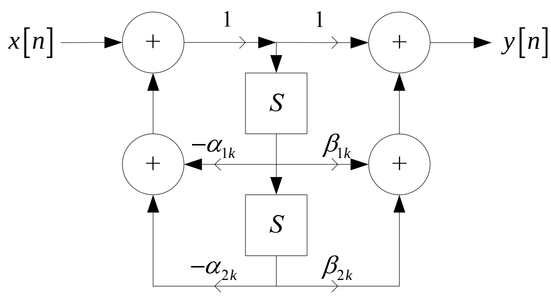
$$H\left(e^{j\Omega}\right)=\frac{1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-j2\Omega}}{1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}}=\frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} \quad (4.137)$$

$$Y\left(e^{j\Omega}\right)\left[1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}\right]=X\left(e^{j\Omega}\right)\left[1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-j2\Omega}\right] \quad (4.138)$$

$$y[n]+\alpha_{1k}y[n-1]+\alpha_{2k}y[n-2]=x[n]+\beta_{1k}x[n-1]+\beta_{2k}x[n-2] \quad (4.139)$$

$$y[n]=-\alpha_{1k}y[n-1]-\alpha_{2k}y[n-2]+\underbrace{x[n]+\beta_{1k}x[n-1]+\beta_{2k}x[n-2]}_{w[n]}$$

(4.140)



■ **Figure 4.12** Realization of a second-order LTI system with direct form II for cascade structure.

(b) The first-order terms can also be realized using the second-order structure with β_{2k} and α_{2k} equal to zero.

(2) Parallel form

$$H\left(e^{j\Omega}\right)=\frac{b_N}{a_N}+\sum_{k=1}^{N}\frac{A_k}{1+\eta_k e^{-j\Omega}} \quad (4.141)$$

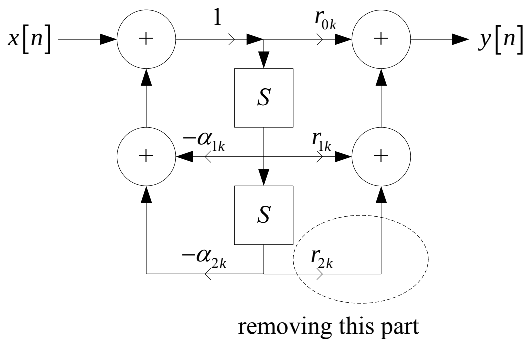
Adding the pairs involving complex conjugate η_k 's, we obtain

$$H\left(e^{j\Omega}\right)=\frac{b_N}{a_N}+\sum_{k=1}^{Q}\frac{r_{0k}+r_{1k}e^{-j\Omega}}{1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}}+\sum_{k=1}^{N-2Q}\frac{A_k}{1+\eta_k e^{-j\Omega}} \quad (4.142)$$

where all the coefficients are real.

We can realize the LTI system using a parallel interconnection of first- and second-order LTI systems.

Realization of $H\left(e^{j\Omega}\right)=\dfrac{r_{0k}+r_{1k}e^{-j\Omega}}{1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-j2\Omega}}$



■ **Figure 4.13** Realization of a second-order LTI system with direct form II for parallel structure.

4-9 First-Order and Second-Order Systems

1. First-order systems

Consider the first-order causal LTI system described by the difference equation

$$y[n]-ay[n-1]=x[n],\ |a|<1 \quad (4.143)$$

$$H\left(e^{j\Omega}\right)=\frac{1}{1-ae^{-j\Omega}} \quad (4.144)$$

$$h[n]=a^n u[n]\cdots\cdots \text{impulse response} \quad (4.145)$$

$$s[n]=h[n]*u[n]=\sum_{k=0}^{n}a^k=\frac{1-a^{n+1}}{1-a}u[n]\cdots\cdots \text{step response} \quad (4.146)$$

(1) The magnitude of “a” plays a role similar to that of the time constant τ of a continuous-time first-order system. (See Fig. 4.14)

(2) Unlike its continuous-time analog, the first-order discrete-time system can play oscillatory behavior when $a<0$.

Note:

- For $a>0$, the system amplifies low frequencies and attenuates high frequencies. (See Fig. 4.16)
  For $a<0$, the system amplifies high frequencies and attenuates low frequencies. (See Fig. 4.17)

  $$\begin{cases} \text{low frequencies: } \Omega \text{ near } 0. \\ \text{high frequencies: } \Omega \text{ near } \pm\pi. \end{cases}$$

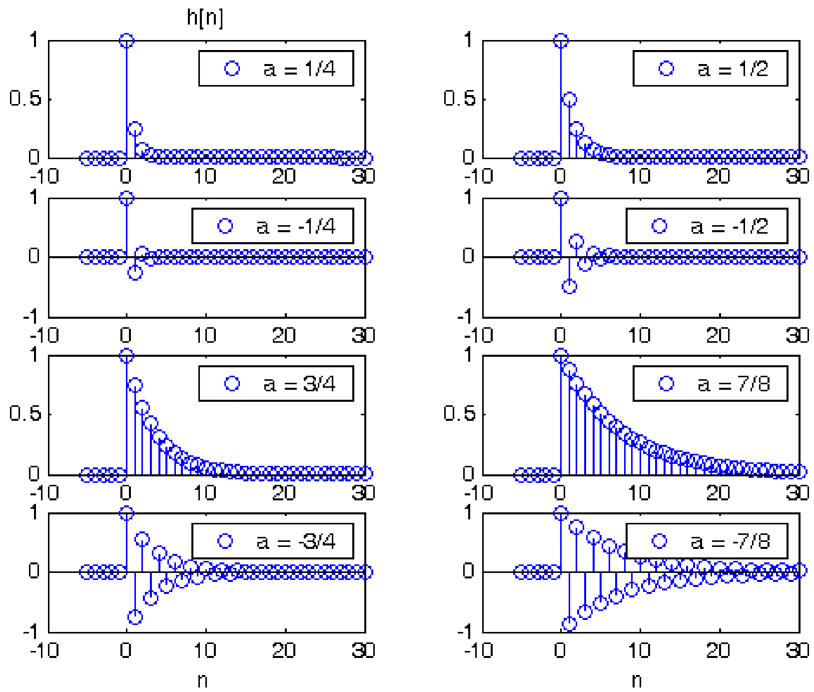
- $$H\left(e^{j\Omega}\right)=\frac{1}{1-ae^{-j\Omega}} \quad \begin{cases} \max\left|H\left(e^{j\Omega}\right)\right| = 1/\left(1-|a|\right) \\ \min\left|H\left(e^{j\Omega}\right)\right| = 1/\left(1+|a|\right) \end{cases} \tag{4.147}$$

  For $|a|$ small, $1/\left(1+|a|\right)$ and $1/\left(1-|a|\right)$ are close.

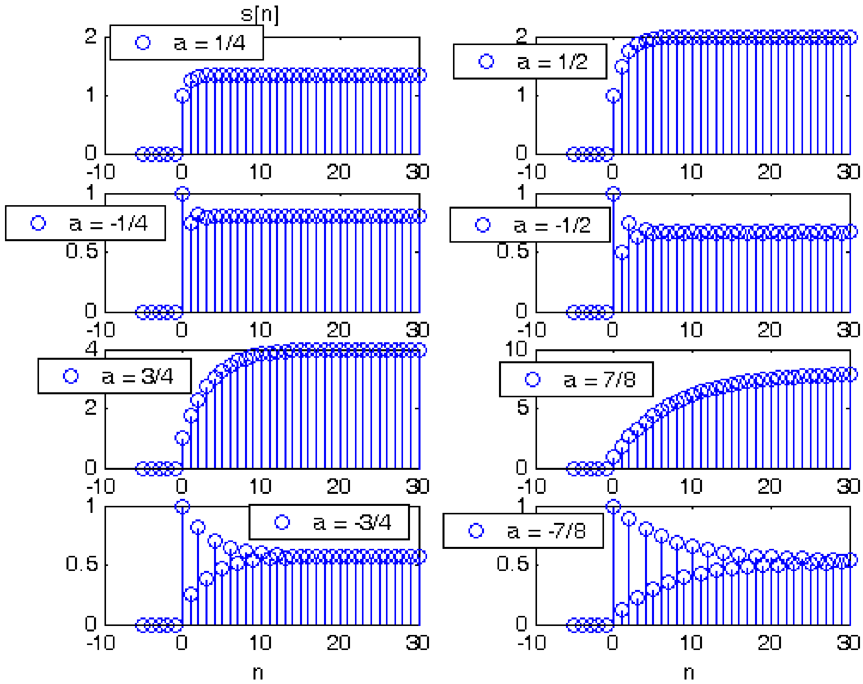
  $\Rightarrow$ The graph of $\left|H\left(e^{j\Omega}\right)\right|$ is relatively flat. (See Fig. 4.16)

  For $|a|$ near 1, $1/\left(1+|a|\right)$ and $1/\left(1-|a|\right)$ differ significantly.

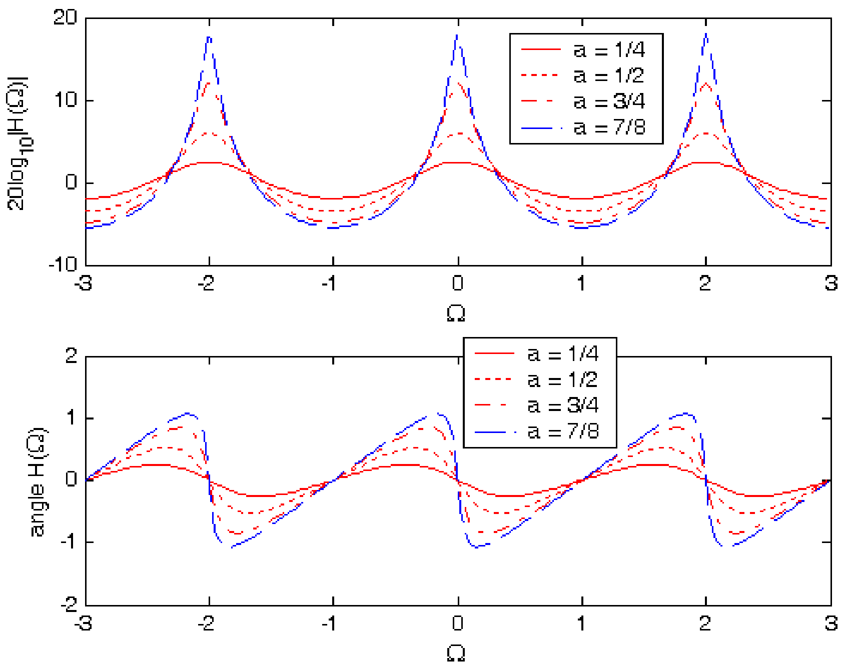
  $\Rightarrow$ The graph of $\left|H\left(e^{j\Omega}\right)\right|$ is more sharply peaked. (See Fig. 4.16)



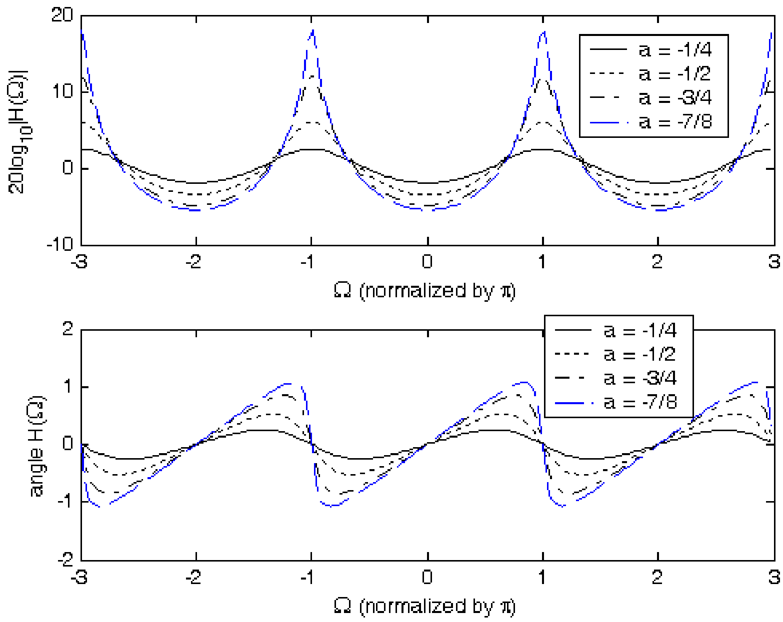
■ **Figure 4.14** Impulse response $h[n]=a^n u[n]$ of a first-order system.



■ **Figure 4.15** Step response $s[n]$ of a first-order system.



■ **Figure 4.16** Magnitude and phase of the frequency response of Eq. (4.144) for a first-order system. ($a > 0$)



■ **Figure 4.17** Magnitude and phase of the frequency response of Eq. (4.144) for a first-order system. ($a < 0$)

2. Second-order systems

Consider the second-order causal LTI system described by

$$y[n]-2r\cos\theta\, y[n-1]+r^2y[n-2]=x[n] \qquad (4.148)$$

with $0<r<1$ and $0\le\theta\le\pi$,

$$H\left(e^{j\Omega}\right)=\frac{1}{1-2r\cos\theta e^{-j\Omega}+r^2e^{-j2\Omega}}=\frac{1}{\left[1-\left(re^{j\theta}\right)e^{-j\Omega}\right]\left[1-\left(re^{-j\theta}\right)e^{-j\Omega}\right]} \qquad (4.149)$$

(1) For $\theta\ne 0$ or π, $re^{j\theta}\ne re^{-j\theta}$ and

$$H\left(e^{j\Omega}\right)=\frac{A}{1-\left(re^{j\theta}\right)e^{-j\Omega}}+\frac{B}{1-\left(re^{-j\theta}\right)e^{-j\Omega}} \qquad (4.150)$$

where $A=\dfrac{e^{j\theta}}{2j\sin\theta}$ and $B=\dfrac{e^{-j\theta}}{2j\sin\theta}$.

$$h[n]=\left[A\left(re^{j\theta}\right)^n+B\left(re^{-j\theta}\right)^n\right]u[n]=r^n\frac{\sin\left[(n+1)\theta\right]}{\sin\theta}u[n] \qquad (4.151)$$

(2) For $\theta=0$, $re^{j\theta}=re^{-j\theta}=r$ and

$$H\left(e^{j\Omega}\right)=\frac{1}{\left(1-re^{-j\Omega}\right)^2} \qquad (4.152)$$

$$h[n]=(n+1)r^nu[n] \qquad (4.153)$$

(3) For $\theta = \pi$, $re^{j\theta} = re^{-j\theta} = -r$

$$H\left(e^{j\Omega}\right) = \frac{1}{\left(1 + re^{-j\Omega}\right)^2} \tag{4.154}$$

$$h[n] = (n+1)(-r)^n u[n] \tag{4.155}$$

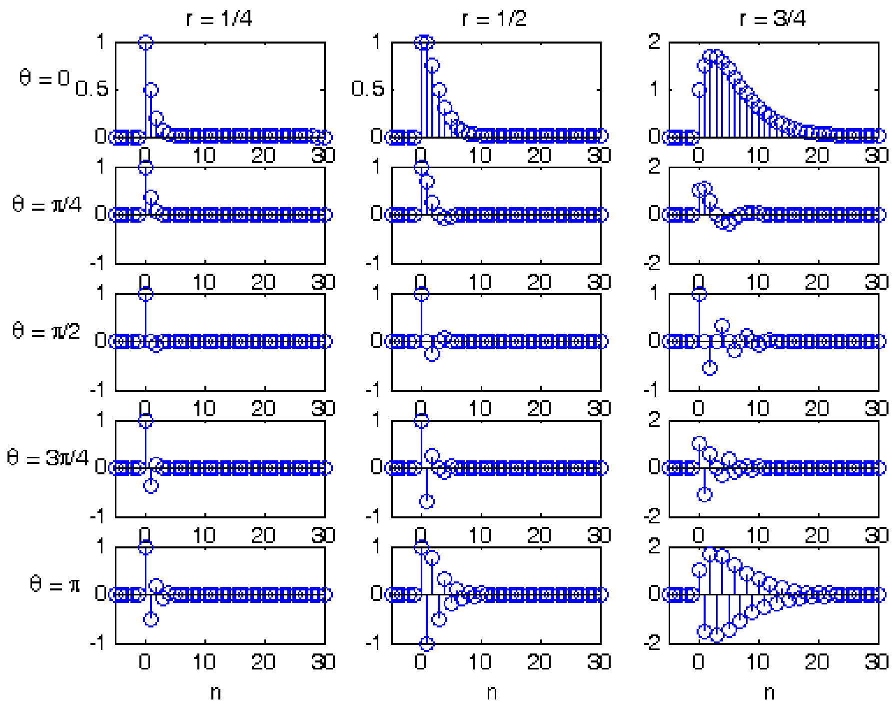
(The impulse response for second-order systems are plotted in Fig. 4.18 for a range of values of r and θ.)

Note:

- The rate of decay of $h[n]$ is controlled by r. The closer r is to 1, and the slower the decay in $h[n]$.
- The value of θ determines the frequency of oscillation.

$\underset{\text{(low frequency)}}{\theta = 0}$ $\Rightarrow$ No oscillation.

$\underset{\text{(high frequency)}}{\theta = \pi}$ $\Rightarrow$ Oscillations are rapid.



■ **Figure 4.18** Impulse response of the second-order system of Eq. (4.148) for a range of values of r and θ.

- The effect of different values of *r* and θ can also be seen by examining the step response.

$$s[n]=h[n]*u[n]=\left[A\left(\frac{1-\left(re^{j\theta}\right)^{n+1}}{1-re^{j\theta}}\right)+B\left(\frac{1-\left(re^{-j\theta}\right)^{n+1}}{1-re^{-j\theta}}\right)\right]u[n] \quad (4.156)$$

For $\theta=0$,

$$s[n]=\left[\frac{1}{(r-1)^2}-\frac{r}{(r-1)^2}r^n+\frac{r}{r-1}(n+1)r^n\right]u[n] \quad (4.157)$$

For $\theta=\pi$,

$$s[n]=\left[\frac{1}{(r+1)^2}-\frac{r}{(r+1)^2}(-r)^n+\frac{r}{r+1}(n+1)(-r)^n\right]u[n] \quad (4.158)$$

The step response for a range of *r* and θ is plotted in Fig. 4.19.

- For any value of θ other than zero, the impulse response has a damped oscillatory behavior, and the step response exhibits ringing and overshot.
- The frequency response of the system is depicted in Fig. 4.20.
  θ essentially controls the location of band that is amplified.
  r determines how sharply peaked the frequency response is within the band that is amplified.
- Consider $H\left(e^{j\Omega}\right)$ of the form

$$H\left(e^{j\Omega}\right)=\frac{1}{\left(1-d_1e^{-j\Omega}\right)\left(1-d_2e^{-j\Omega}\right)}=\frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} \quad (4.159)$$

where d_1 and d_2 are both real with $|d_1|,\ |d_2|<1$.

$$y[n]-(d_1+d_2)y[n-1]+d_1d_2y[n-2]=x[n] \quad (4.160)$$

Using the partial function expansion technique, $H\left(e^{j\Omega}\right)$ can be expressed as

$$H\left(e^{j\Omega}\right)=\frac{A}{1-d_1e^{-j\Omega}}+\frac{B}{1-d_2e^{-j\Omega}} \quad (4.161)$$

where $A=\frac{d_1}{d_1-d_2}$ and $B=\frac{d_2}{d_2-d_1}$.

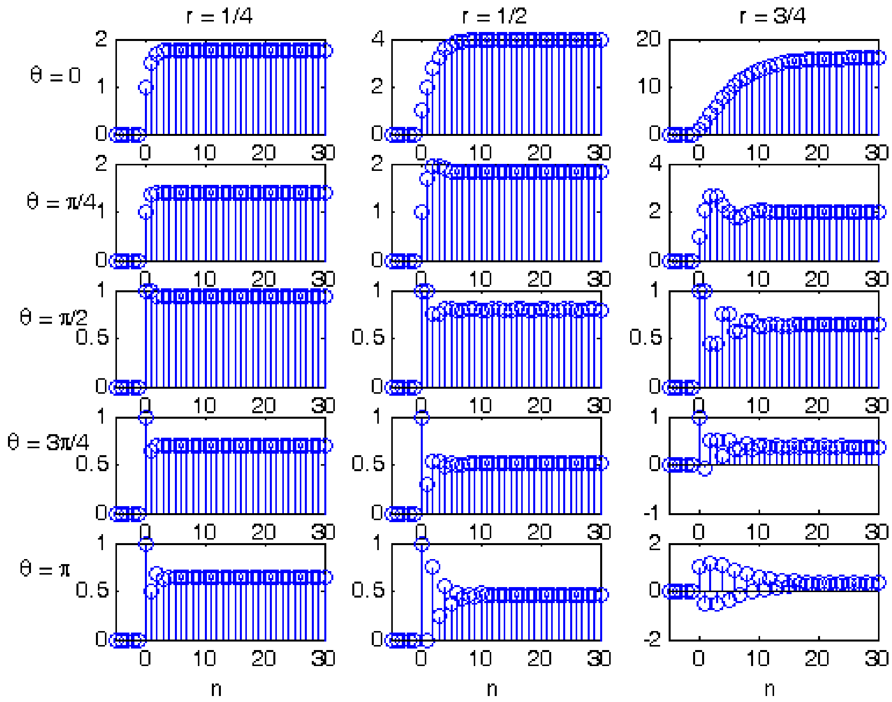
$$h[n]=\left(Ad_1^n+Bd_2^n\right)u[n] \quad (4.162)$$

$$s[n] = \left(A \frac{1-d_1^{n+1}}{1-d_1} + B \frac{1-d_2^{n+1}}{1-d_2} \right) u[n] \quad (4.163)$$

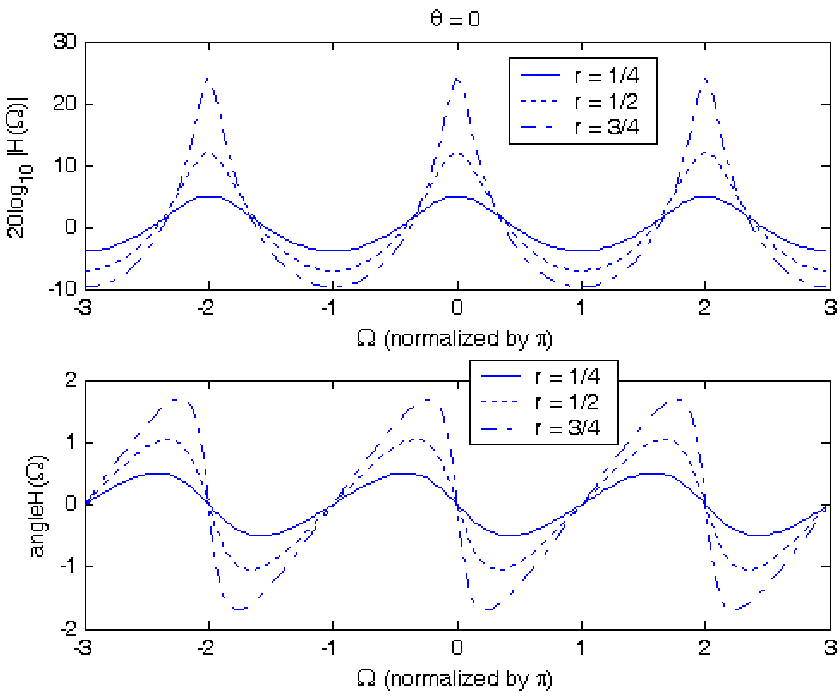
The system corresponds to a parallel interconnection of two first-order systems. We can deduce most of its properties from our understanding of the first-order systems.

- We have only examined those first-order and second-order systems that are stable and consequently have frequency response.

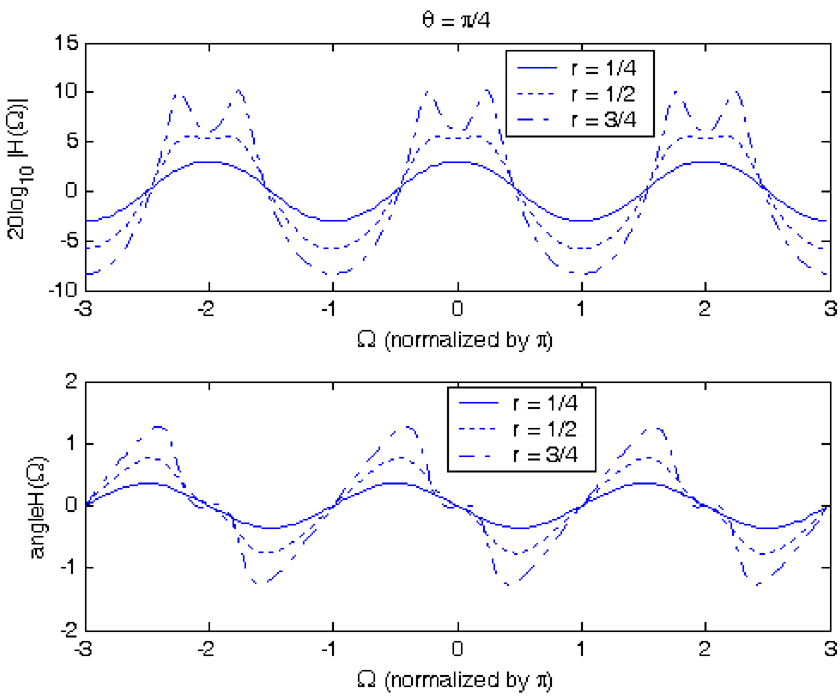
$$\underbrace{|a|<1}_{\text{first-order case}} \;,\; \underbrace{r<1,\ |d_1|<1,\ |d_2|<1}_{\text{second-order case}}$$



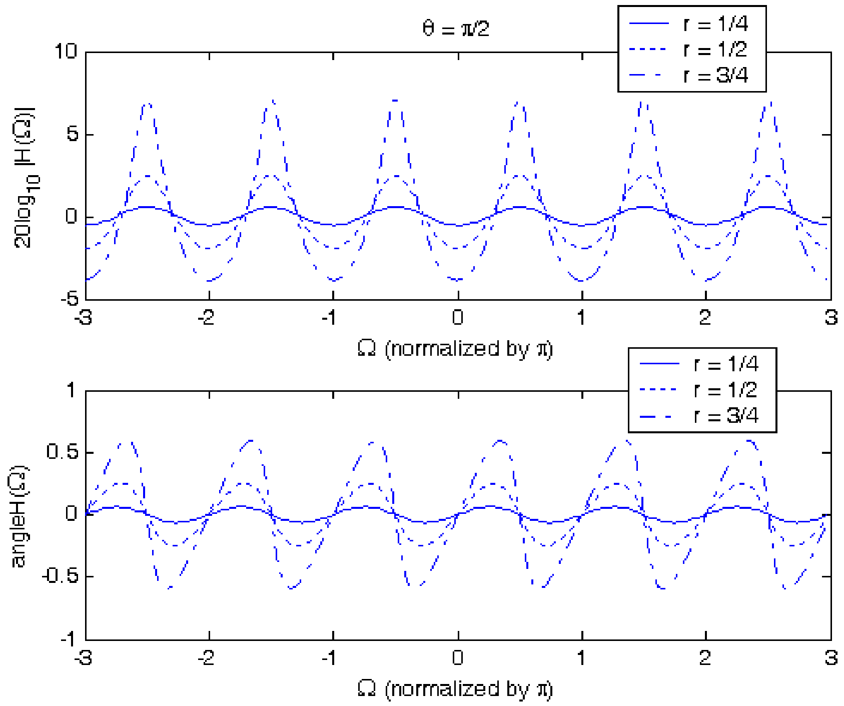
■ **Figure 4.19** Step response of the second-order system of Eq. (4.148) for a range of values of r and θ.



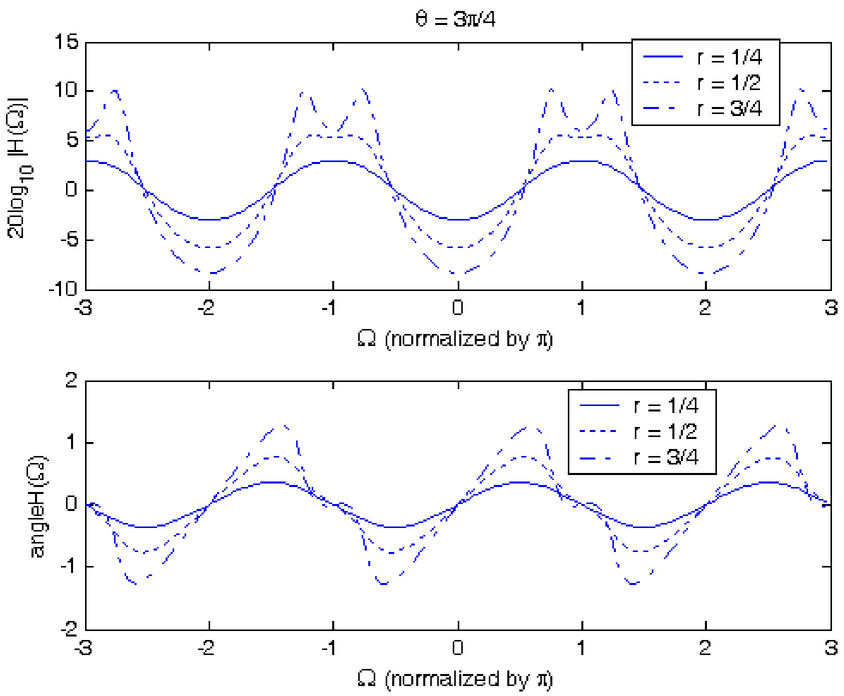
■ **Figure 4.20** Magnitude and phase of the frequency response of the second-order system of Eq. (4.148). ($\theta = 0$)



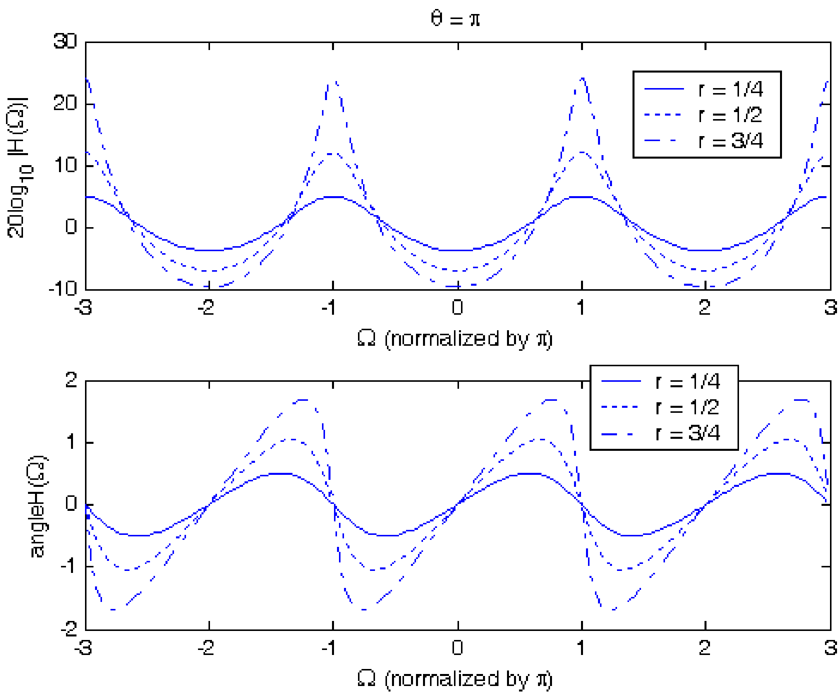
■ **Figure 4.20** (contd.) ($\theta = \pi/4$)



■ **Figure 4.20** (contd.) ($\theta = \pi/2$)



■ **Figure 4.20** (contd.) ($\theta = 3\pi/4$)



■ **Figure 4.20** (contd.) ($\theta = \pi$)

References:

[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, *Signals and Systems*, 2nd Ed., Prentice-Hall, 1997.