Chapter 3 Fourier Representations of Signals and Linear Time-Invariant Systems

3-1 Eigenfunctions and Eigenvalues of Continuous-Time LTI Systems

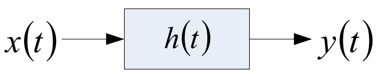
1. A signal for which the system output is just a (possibly complex) constant times the input is referred to as an eigenfunction of the system, and the amplitude factor is referred to as the eigenvalue.

$$x(t) \to y(t) = \underbrace{H}_{eigenvalue} \cdot \underbrace{x(t)}_{eigenfunction} \tag{3.1}$$

Note: $x(t)$ is called an eigenfunction of the system if $T\{x(t)\} = H \cdot x(t)$, where H is called the eigenvalue corresponding to the eigenfunction.

2. Complex exponentials e^{st} are eigenfunctions of continuous time LTI systems

$x(t) = e^{st}$



$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau) \cdot e^{s(t-\tau)} d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \\ &= e^{st} \cdot H(s) = H(s) \cdot x(t) \end{aligned} \tag{3.2}$$

$H(s)$ is a complex constant whose value depends on "s".

e^{st} ································· eigenfunction

$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$ ················ eigenvalue

3. $x(t) = \sum_k a_k e^{s_k t} \to y(t) = \sum_k a_k H(s_k) e^{s_k t}$

Note:

- $T\{x(t)\} = \sum_k a_k T\{e^{s_k t}\} = \sum_k a_k H(s_k) e^{s_k t}$
- $s \to s_k$: Fourier series
- $s = j\omega$: Fourier transform
- $s = \sigma + j\omega$: Laplace transform

3-2 Fourier Series Representation of Periodic Continuous-Time Signals: The Continuous-Time Fourier Series

1. Fourier series representation

Harmonically related complex exponentials:

$$\phi_k(t) = e^{jk\omega_0 t} \quad , k = 0, \ \pm 1, \ \pm 2, \ .... \tag{3.3}$$

Note:

- Each of these exponentials is periodic with period $T_0 = 2\pi/\omega_0$.
- Any linear combination of these exponentials is also periodic with period T_0.

Let $x(t)$ be a periodic continuous-time signal with fundamental period T_0.

Then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{3.4}$$

is referred to as the Fourier series representation of $x(t)$.

Note:

- The terms for $k = \pm 1$: the fundamental components or the lst harmonic components.
  The terms for $k = \pm 2$: the 2nd harmonic components.
  ⋮
  The terms for $k = \pm N$: the *N*th harmonic components.
- Alternative forms for the Fourier series of real periodic signals

$$\begin{cases} x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \\ x(t) = a_0 + 2\sum_{k=1}^{\infty} \left[b_k \cos(k\omega_0 t) - c_k \sin(k\omega_0 t) \right] \end{cases} \tag{3.5}$$

$\because x(t)$ is real $\quad \therefore x^*(t) = x(t)$.

Let $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$. Then

$$x^*(t) = x(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} \underset{k \text{ replaced with} -k}{=} \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} \tag{3.6}$$

$\Rightarrow a_k = a_{-k}^*$ or $a_k^* = a_{-k}$

$$\begin{aligned}\Rightarrow x(t) &= a_0 + \sum_{k=-\infty}^{-1} a_k e^{jk\omega_0 t} + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} \\ &= a_0 + \sum_{k=1}^{\infty}\left[a_{-k}e^{-jk\omega_0 t} + a_k e^{jk\omega_0 t}\right] \\ &= a_0 + \sum_{k=1}^{\infty}\left[\left(a_k e^{jk\omega_0 t}\right)^* + a_k e^{jk\omega_0 t}\right] \\ &= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\} \\ &= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{A_k e^{j(k\omega_0 t+\theta_k)}\right\} \qquad \left(a_k = A_k e^{j\theta_k}\right) \\ &= a_0 + 2\sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \theta_k\right)\end{aligned} \tag{3.7}$$

Let $a_k = b_k + jc_k$. Then

$$\begin{aligned}x(t) &= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\} \\ &= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\left(b_k + jc_k\right)\left(\cos\left(k\omega_0 t\right) + j\sin\left(k\omega_0 t\right)\right)\right\} \\ &= a_0 + 2\sum_{k=1}^{\infty}\left[b_k \cos\left(k\omega_0 t\right) - c_k \sin\left(k\omega_0 t\right)\right]\end{aligned} \tag{3.8}$$

- $x(t) = \sum_k a_k e^{jk\omega_0 t} \rightarrow y(t) = \sum_k a_k \cdot H\left(jk\omega_0\right) e^{jk\omega_0 t}$

$$H\left(jk\omega_0\right) = \int_{-\infty}^{\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau \tag{3.9}$$

2. Determination of the Fourier series coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{3.10}$$

$$x(t)\cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \tag{3.11}$$

$$\begin{aligned}\int_0^{T_0} x(t)\cdot e^{-jn\omega_0 t} dt &= \int_0^{T_0} \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \int_0^{T_0} e^{j(k-n)\omega_0 t} dt\end{aligned} \tag{3.12}$$

$$\int_0^{T_0} e^{j(k-n)\omega_0 t} dt = \int_0^{T_0} \cos\left(k-n\right)\omega_0 t \; dt + j\int_0^{T_0} \sin\left(k-n\right)\omega_0 t \; dt$$

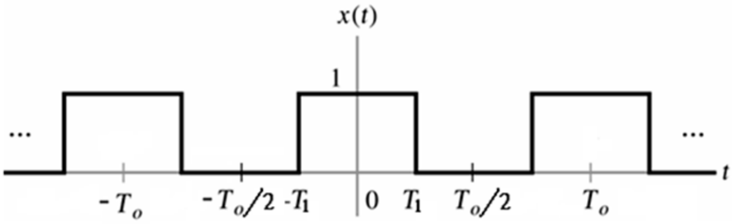
$$= \begin{cases} 0 \;,\; \text{for } k \neq n \rightarrow \begin{pmatrix} \sin\left(k-n\right)\omega_0 t \text{ and } \cos\left(k-n\right)\omega_0 t \\ \text{are periodic with fundamental period } T_0/\left|k-n\right| \end{pmatrix} \\ T_0 \;,\; \text{for } k = n \end{cases}$$

$$\Rightarrow a_n = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jn\omega_0 t} dt = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jn\omega_0 t} dt \tag{3.13}$$

where $\int_{T_0}$ denotes integral over any interval of length T_0.

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k &= \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jk\omega_0 t}dt \end{aligned}} \quad (3.14)$$

***Example* 3.1**: Derive the Fourier series of $x(t)$.



$$x(t) = \begin{cases} 1 \ , & |t| < T_1 \\ 0 \ , & T_1 < |t| < T_0/2 \end{cases}$$ for one period. Fundamental period $= T_0$.

The Fourier series representation of $x(t)$ is

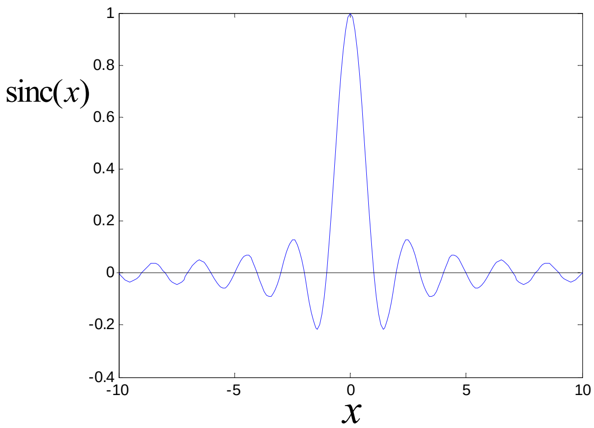
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ , \quad \omega_0 = 2\pi/T_0$$

$$\begin{aligned} a_k &= \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt \\ &= \begin{cases} 2T_1/T_0 & , \text{ for } k = 0 \\ \dfrac{\sin(k\omega_0 T_1)}{k\pi} & , \text{ for } k \neq 0 \end{cases} . \end{aligned} \quad (3.15)$$

Note:

- $\because \text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

$$\therefore \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\omega_0 T_1}{\pi}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1} = \frac{\omega_0 T_1}{\pi}\text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$



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3-3 Approximation of Periodic Signals Using Finite Fourier Series and the Convergence of Fourier Series

1. Finite Fourier series

$$\begin{cases} x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} & \cdots\cdots \text{ infinite series} \\ x_N(t)=\sum_{k=-N}^{N} a'_k e^{jk\omega_0 t} & \cdots\cdots \text{ finite series} \end{cases} \tag{3.16}$$

Let $e_N(t)$ denote the approximation error.

$$e_N(t)=x(t)-x_N(t)=x(t)-\sum_{k=-N}^{N} a'_k e^{jk\omega_0 t} \tag{3.17}$$

Quantitative measure of the size of the approximation error:
"Mean Squared-Error" (MSE)

$$E_N=\int_{T_0}\left|e_N(t)\right|^2 dt=\int_{T_0} e_N(t)e_N^*(t)\,dt \tag{3.18}$$

$E=\int_a^b \left|z(t)\right|^2 dt$ $\cdots\cdots$energy in $z(t)$ over the time interval $a \le t \le b$

E_N : Error energy in the approximation over one period

Determining the Fourier series coefficients for x_N such that E_N is minimized

$$E_N=\int_{T_0}\left|x(t)-\sum_{k=-N}^{N} a'_k e^{jk\omega_0 t}\right|^2 dt \tag{3.19}$$

$$\frac{\partial E_N}{\partial a'_k}=0 \Rightarrow \boxed{a'_k=\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt} \tag{3.20}$$

$\Rightarrow$ The best approximation is obtained by truncating the Fourier series to the desired number of terms.

$\Rightarrow$ If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.

Any one of the following conditions is sufficient to ensure the MSE convergence of the Fourier series for $x(t)$:

Condition 1: If the periodic signal $x(t)$ is a continuous function of t, the Fourier series converges. (Actually, in the case, the convergence is uniform, which is a stronger criterion than MSE convergence.)

Condition 2: If $x(t)$ is *square-integrable* over a period *T*, that is, if

$$\int_T |x(t)|^2 \, dt < \infty,$$

the Fourier series converges in the MSE sense. Since this condition is equivalent to the requirement that the average power in $x(t)$ be finite, it clearly applies to all periodic signals encountered in the laboratory, as well as to most theoretical signals of interest. In particular, note that any bounded signal satisfies the condition because, if $|x(t)| < B$ for finite *B* and all *t*, then

$$\int_T |x(t)|^2 \, dt < TB^2.$$

Condition 3 (Dirichlet):

(1) Over any period, $x(t)$ must be absolutely integrable

$$\int_{T_0} |x(t)| dt < \infty \tag{3.21}$$

(2) In any finite interval of time, $x(t)$ is of bounded variation, i.e., there are no more than a finite number of maxima and minima during any single period of the signal.
(The number of maxima and minima points must be countable during any period.)

(3) In a finite number of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities must be finite.

Note:

- These conditions guarantee that:
  - (i) $x(t)$ has a Fourier series representation.
  - (ii) $x(t)$ is equal to its Fourier series representation for all t except at isolated values of t for which $x(t)$ is discontinuous. At the isolated points of discontinuity, the series converges to the average value of the discontinuity.

$\Rightarrow$ The integrals of both signals ($x(t)$ and its Fourier series) over any interval are identical.

$\Rightarrow$ The two signals behave identically under convolution and consequently are identical from the standpoint of the analysis of LTI systems.

- For a periodic signal that varies continuously, the Fourier series representation converges and equals the original signal at any t.

***Example* 3.2**:

(a) $x(t) = 1/t$, $0 < t \leq 1$, period $= 1$

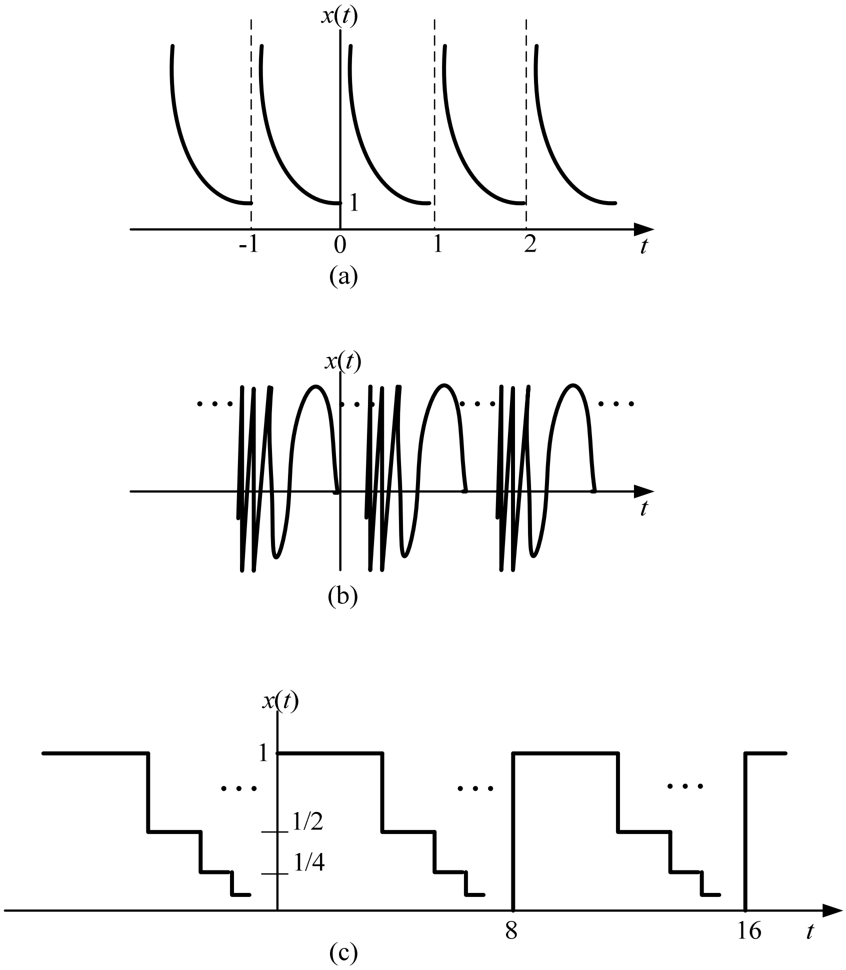
(cannot be represented by a Fourier series)
This signal violates the first Dirichlet Condition.
(See Fig. 3.1(a))

(b) $x(t) = \sin(2\pi/t)$, $0 < t \leq 1$, period $= 1$

(Converge in the MSE sense)
This signal meets the 1st Dirichlet condition but not the 2nd one.

$\int_0^1 \left|\sin(2\pi/t)\right| dt < 1$

Having an infinite number of maxima and minima in the interval
(See Fig. 3.1(b))
(c) This signal violates the 3rd Dirichlet condition.
Having an infinite number of discontinuities (See Fig. 3.1(c))



■**Figure 3.1.** Signals that violate the Dirichlet conditions: (a) the signal $x(t)$, periodic with period 1, with $x(t) = 1/t$ for $0 < t \leq 1$ (this signal violates the first Dirichlet condition); (b) the periodic signal $x(t) = \sin(2\pi/t)$ which violates the second Dirichlet condition; (c) a signal, periodic with period 8, that violates the third Dirichlet condition [for $0 \leq t < 8$ the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

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Note:

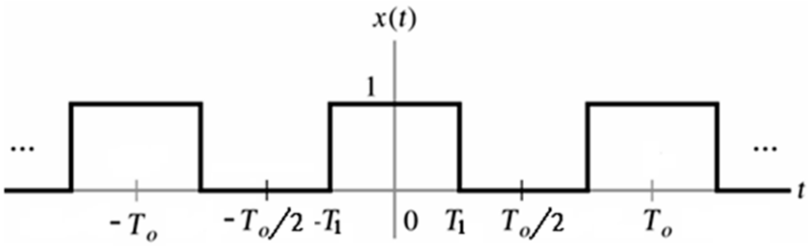
- The signals that do not satisfy the Dirichlet conditions are generally pathological in nature and thus are not particularly important in the study of signals and systems.

3-4 The Fourier Transform of Aperiodic Continuous-Time Signals: The Continuous-Time Fourier Transform

1. Development of the Fourier Transform

   Periodic signals $\rightarrow$ Fourier series

   Aperiodic signals $\rightarrow$ Fourier transform



■ **Figure 3.2.** Periodic square wave.

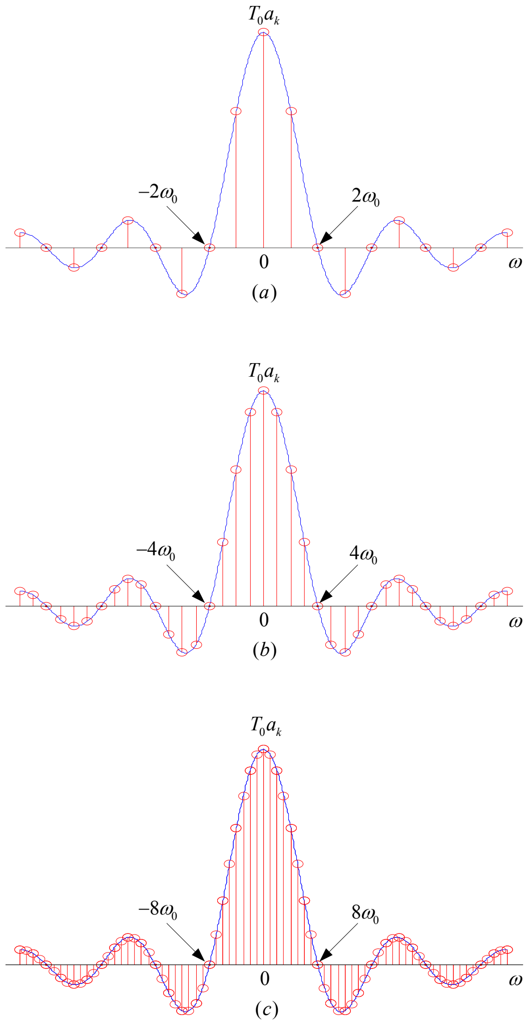
Fourier series representation

$$x(t)=\sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t} \qquad \omega_0 = 2\pi/T_0$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T_0} \tag{3.22}$$

$$T_0 a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega=k\omega_0} = 2T_1 \cdot \left.\frac{\sin\omega T_1}{\omega T_1}\right|_{\omega=k\omega_0} = 2T_1 \cdot \left.\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)\right|_{\omega=k\omega_0}$$

$\Rightarrow$ The function $\left(2\sin(\omega T_1)/\omega\right)$ represents the envelope of $T_0 a_k$ (See Fig. 3.3).



■ **Figure 3.3.** Fourier coefficient and their envelope for the periodic square wave: (a) $T_0 = 4T_1$; (b) $T_0 = 8T_1$; (c) $T_0 = 16T_1$.

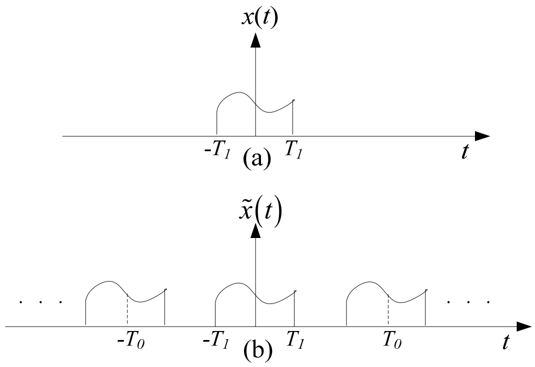
i.e., $T_0 a_k$ is a sampled value of $\left(2\sin(\omega T_1)/\omega\right)$

$$\begin{cases} \because \text{The sampling interval is } \omega_0 \\ \therefore T_0 \uparrow \Rightarrow \omega_0 \downarrow \Rightarrow \text{sampling spacing} \downarrow \Rightarrow \text{Fourier series coefficients approach the envelope function} \\ T_0 \to \infty \Rightarrow x(t) \text{ is a rectangular pulse (aperiodic)} \end{cases}$$

Note:

- We can think of an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large.

  Consider a general aperiodic signal $x(t)$ that is of finite duration. From this aperiodic signal, we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is of one period (See Fig. 3.4).



■ **Figure 3.4.** (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

As $T_0 \to \infty$, $\tilde{x}(t) \to x(t)$

Fourier series representation of $\tilde{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \tilde{x}(t)\cdot e^{-jk\omega_0 t}dt \tag{3.23}$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cdot e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\cdot e^{-jk\omega_0 t}dt$$

Defining the envelope of $T_0 a_k$ as $X(j\omega)$, we have

$$T_0 a_k = X(j\omega)\Big|_{\omega=k\omega_0} = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}dt\Big|_{\omega=k\omega_0} \tag{3.24}$$

$$a_k = \frac{1}{T_0} X(j\omega)\Big|_{\omega=k\omega_0} \tag{3.25}$$

$$\Rightarrow \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(jk\omega_0) e^{jk\omega_0 t} \tag{3.26}$$

$$\because T_0 = \frac{2\pi}{\omega_0} \qquad \therefore \frac{1}{T_0} = \frac{1}{2\pi}\omega_0 \tag{3.27}$$

$$\Rightarrow \tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t}\omega_0 \tag{3.28}$$

As $T_0 \to \infty$, $\tilde{x}(t) \to x(t)$, and the above equation becomes a representation of $x(t)$.

As $T_0 \to \infty$, $\omega_0 \to 0$. $\left(\sum \to \int\right)$ $\omega_0 \to d\omega$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

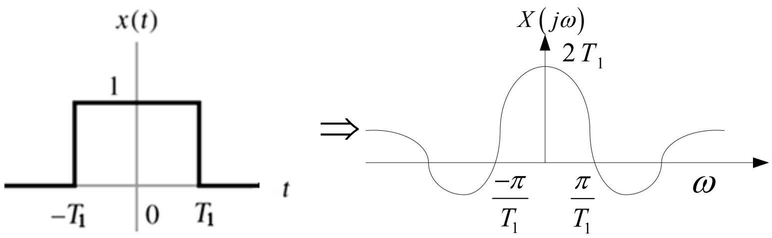
Fourier transform pair: $X(j\omega)$ is referred to as the Fourier transform or Fourier integral of $x(t)$.

2. Examples of Continuous-Time Fourier Transform

***Example* 3.3**:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \qquad \text{rectangular pulse}$$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = 2\cdot\frac{\sin\omega T_1}{\omega} = 2T_1\cdot\frac{\sin\omega T_1}{\omega T_1} = 2T_1\text{sinc}\left(\frac{\omega T_1}{\pi}\right) \tag{3.29}$$



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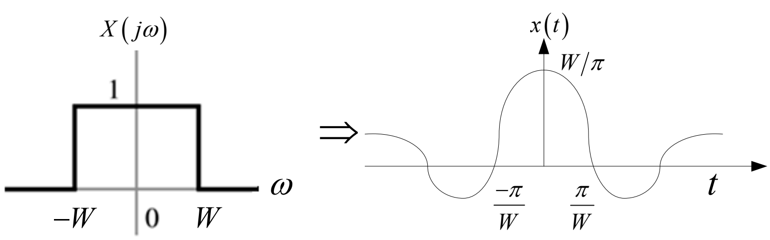
***Example* 3.4**:

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin Wt}{\omega\pi} = \frac{W}{\pi}\frac{\sin Wt}{Wt} = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right) \tag{3.30}$$



Note:

- Broader in time domain $\rightarrow$ narrower in frequency domain

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3-5 Periodic Signals and the Continuous-Time Fourier Transform

1. Fourier series coefficients as samples of the Fourier transform of one period

Fourier series representation

$$\tilde{x}(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{fundamental period} = T_0$$

$$a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)\cdot e^{-jk\omega_0 t}dt$$

$$x(t)=\begin{cases}\tilde{x}(t) & , \quad -\frac{T_0}{2}\le t\le \frac{T_0}{2}\\ 0 & , \quad \text{otherwise}\end{cases} \tag{3.31}$$

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\Rightarrow a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \tilde{x}(t)\cdot e^{-jk\omega_0 t}dt$$

$$=\frac{1}{T_0}X(jk\omega_0)=\frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\cdot e^{-jk\omega_0 t}dt$$

General statement:

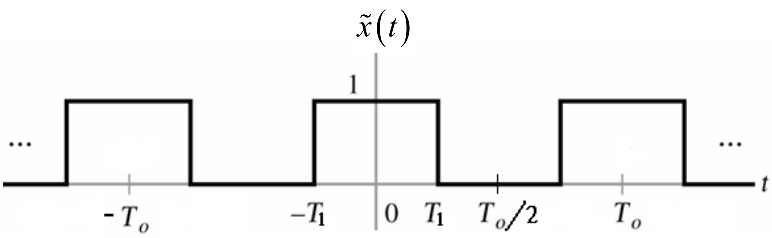
Let $x(t)=\begin{cases}\tilde{x}(t) & , \quad s\le t\le s+T_0\\ 0 & , \quad \text{otherwise}\end{cases}$

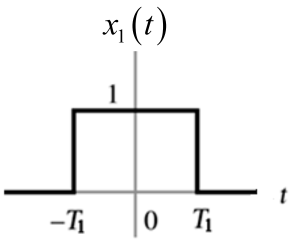
Then the Fourier series coefficients of $\tilde{x}(t)$ are given by

$$a_k=\frac{1}{T_0}X(jk\omega_0)$$

where $X(j\omega)$ is the Fourier transform of $x(t)$.

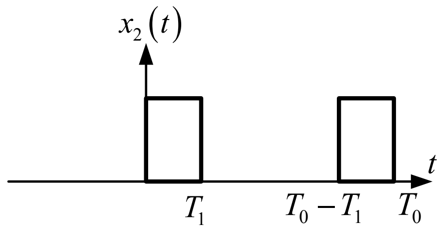
***Example* 3.5**: Compare the Fourier series coefficients of $x_1(t)$ and $x_2(t)$.





$$\Rightarrow X_1(j\omega)\underset{\text{sampling period}=\omega_o=2\pi/T_0}{\Rightarrow} \text{Fourier series coefficients of } \tilde{x}(t)$$

$$X_1(j\omega) = \frac{2\sin\omega T_1}{\omega} \tag{3.32}$$



$\Rightarrow X_2(j\omega) \underset{\text{sampling period}=\omega_o=2\pi/T_0}{\Rightarrow}$ Fourier series coefficients of $\tilde{x}(t)$

$$\begin{aligned} X_2(j\omega) &= \int_0^{T_1} e^{-j\omega t}dt + \int_{T_0-T_1}^{T_0} e^{-j\omega t}dt \\ &= \frac{2}{\omega}\sin\left(\frac{\omega T_1}{2}\right)\left[e^{-j\omega T_1/2} + e^{-j\omega(T_0-T_1)/2}\right] \end{aligned} \tag{3.33}$$

$X_1(j\omega) \neq X_2(j\omega)$, but $X_1(jk\omega_0) = X_2(jk\omega_0) = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0}$ (only some sample points are the same)

The Fourier coefficients of a periodic signal can be obtained from samples of the Fourier transform of an aperiodic signal that equals the original periodic signal over any arbitrary interval of length T_0 and that is zero outside this interval.

■

2. The Fourier Transform for Periodic Signals

Consider a signal $x(t)$ with Fourier transform $X(j\omega) = 2\pi\delta(\omega - \omega_0)$.

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}dt = e^{j\omega_0 t}$$

If $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$, then

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ …… corresponding to the Fourier series representation of a periodic signal.

Note:

- $e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$ (3.34)

$$e^{j\cdot 0\cdot t} = 1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega) \tag{3.35}$$

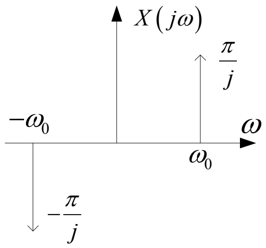
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \tag{3.36}$$

Periodic signal $\rightarrow$ Fourier series representation
$\rightarrow$ Fourier transform

***Example* 3.6**: Represent the periodic signal $x(t)$ with Fourier transform.

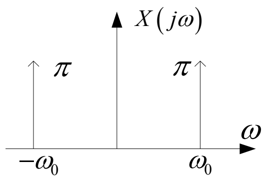
$$x(t) = \sin\omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \tag{3.37}$$

$$\begin{aligned}\Rightarrow X(j\omega) &= \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right] \\ &= \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)\end{aligned} \tag{3.38}$$



$$x(t) = \cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \tag{3.39}$$

$$\Rightarrow X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0) \tag{3.40}$$



■

***Example* 3.7**: Represent the periodic signal $x(t)$ with Fourier series representation.

$$x(t) = \sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T}$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\omega_0 t}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty}\frac{1}{T}2\pi\delta(\omega-k\omega_0) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right) \quad , \quad \left(\omega_0 = \frac{2\pi}{T}\right) \tag{3.41}$$

Impulse train in time domain $\xleftrightarrow{\mathcal{F}}$ Impulse train in frequency domain

■

3-6 Properties of the Continuous-Time Fourier Transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$
$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt \qquad (3.42)$$

Notation:

$$x(t)\stackrel{\mathcal{F}}{\longleftrightarrow}X(j\omega)$$

$$X(j\omega)=\mathcal{F}\{x(t)\}$$

$$x(t)=\mathcal{F}^{-1}\{X(j\omega)\}$$

1. Linearity

$$x_1(t)\stackrel{\mathcal{F}}{\longleftrightarrow}X_1(j\omega)$$
$$x_2(t)\stackrel{\mathcal{F}}{\longleftrightarrow}X_2(j\omega)$$

$$\Rightarrow ax_1(t)+bx_2(t)\stackrel{\mathcal{F}}{\longleftrightarrow}aX_1(j\omega)+bX_2(j\omega) \qquad (3.43)$$

2. Symmetry Properties

If $x(t)$ is a real-valued function, then

$$X(-j\omega)=X^*(j\omega) \qquad * \text{ : complex conjugate} \qquad (3.44)$$

Proof:

$$X^*(j\omega)=\left[\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right]^*=\int_{-\infty}^{\infty}x^*(t)e^{j\omega t}dt$$
$$=\int_{-\infty}^{\infty}x(t)e^{j\omega t}dt=X(-j\omega)$$

Note:

- $X(j\omega)=\text{Re}\{X(j\omega)\}+j\,\text{Im}\{X(j\omega)\}$

  If $x(t)$ is real, then

  $$\text{Re}\{X(j\omega)\}=\text{Re}\{X(-j\omega)\} \cdots\cdots \text{even function}$$
  $$\text{Im}\{X(j\omega)\}=-\text{Im}\{X(-j\omega)\} \cdots\cdots \text{odd function}$$

- $X(j\omega) = |X(j\omega)| e^{j\theta(j\omega)}$ ……polar form

  If $x(t)$ is real, then

$$\begin{aligned} |X(j\omega)| &= |X(-j\omega)| \quad \text{……even function} \\ \theta(j\omega) &= -\theta(-j\omega) \quad \text{……odd function} \end{aligned} \tag{3.45}$$

- If $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even.

$$\begin{aligned} X(-j\omega) &= \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(-\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \qquad (\because x(t) = x(-t)) \\ &= X(j\omega) \qquad \text{…… even} \end{aligned} \tag{3.46}$$

  symmetry property $\Rightarrow X^*(j\omega) = X(-j\omega) = X(j\omega)$ …… real

- If $x(t)$ is both real and odd, then $X(j\omega)$ is pure imaginary and odd.
- A real function $x(t)$ can always be expressed as

$$x(t) = \underset{\text{even part}}{x_e(t)} + \underset{\text{odd part}}{x_o(t)} \tag{3.47}$$

$$\mathcal{F}\{x(t)\} = \underbrace{\mathcal{F}\{x_e(t)\}}_{\text{Re}\{X(j\omega)\}} + \underbrace{\mathcal{F}\{x_o(t)\}}_{j\,\text{Im}\{X(j\omega)\}} \tag{3.48}$$

3. Time Shifting

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega) \tag{3.49}$$

Proof:

$$\begin{aligned} \mathcal{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(\sigma) e^{-j\omega(\sigma+t_0)} d\sigma \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\sigma) e^{-j\omega\sigma} d\sigma \\ &= e^{-j\omega t_0} X(j\omega) \end{aligned}$$

Time shifting only introduces a phase shift but leaves the magnitude unchanged.

4. Differentiation and Integration

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

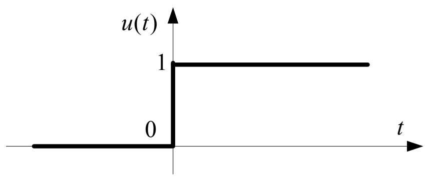
$$\frac{dx(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) \tag{3.50}$$

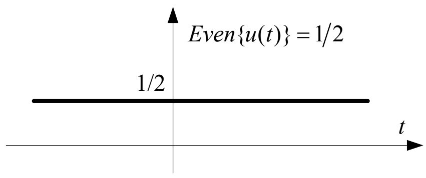
$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega) \tag{3.51}$$

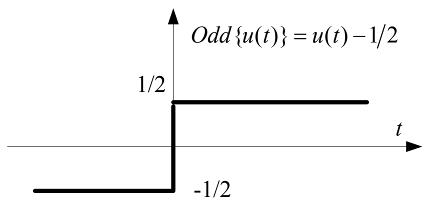
$X(0)$: reflects the dc or average value that can result from the integration

$$\begin{cases} x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \\ \dfrac{dx(t)}{dt} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega t} d\omega \end{cases} \tag{3.52}$$

***Example* 3.8:** Determine the Fourier transform of step function $u(t)$.







$$u(t) = \frac{1}{2} + \underbrace{\left[u(t) - \frac{1}{2}\right]}_{v(t)} \tag{3.53}$$

$$\because v'(t) = u'(t) = \delta(t)$$

$$\therefore \mathcal{F}\{\delta(t)\} = \mathcal{F}\left\{\frac{dv(t)}{dt}\right\} = j\omega V(j\omega)$$

$$\because \mathcal{F}\{\delta(t)\} = 1 \qquad \therefore V(j\omega) = \frac{1}{j\omega}$$

$$Even\{u(t)\} = \frac{1}{2} \Rightarrow \mathcal{F}\left\{\frac{1}{2}\right\} = \pi\delta(\omega)$$

$$\Rightarrow \boxed{\mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega)} \text{ "agrees with the integration property"} \quad (3.54)$$

Note:

- $\delta(t) = \dfrac{du(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega\left[\dfrac{1}{j\omega} + \pi\delta(\omega)\right] = 1 \qquad (\because \omega\pi\delta(\omega) = 0)$

5. Time and Frequency Scaling

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right) \qquad (3.55)$$

Proof:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \qquad (\tau = at)$$

$$= \begin{cases} \dfrac{1}{-a}\displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau & (a < 0) \\ \dfrac{1}{a}\displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau & (a > 0) \end{cases}$$

6. Duality

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} f(j\omega) \qquad \text{where } f(u) = \int_{-\infty}^{\infty} g(v) e^{-juv} dv$$

$$f(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi g(-j\omega) \qquad g(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u) e^{juv} du \quad (3.56)$$

Proof:

$u = j\omega$ and $v = t$

$$\begin{cases} f(j\omega) = \displaystyle\int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \\ f(j\omega) = \mathcal{F}\{g(t)\} \end{cases}$$

$u = t$ and $v = -j\omega$

$$\begin{cases} f(t) = \int_{\infty}^{-\infty} g(-j\omega) e^{j\omega t} d\omega \\ \quad = \int_{\infty}^{-\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right] e^{j\omega t} d\omega \\ \quad = \int_{\infty}^{-\infty} \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right] e^{j\omega t} d\omega \\ \quad = \int_{\infty}^{-\infty} \frac{1}{2\pi} \left[\mathcal{F}\{f(t)\} \right] e^{j\omega t} d\omega \end{cases}$$

$$\Rightarrow f(t) \xleftrightarrow{\mathcal{F}} 2\pi g(-j\omega)$$

■

***Example* 3.9**: Compare the relationship between rectangular and sinc function with the duality property.

rectangular in time domain→ sinc in frequency domain

sinc in time domain→ rectangular in frequency domain

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \xleftrightarrow{\mathcal{F}} X_1(j\omega) = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi} \right) \tag{3.57}$$

$$x_2(t) = \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi} \right) \xleftrightarrow{\mathcal{F}} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} \tag{3.58}$$

(Let $W = T_1$, and we can find that they satisfy the duality property.)

■

***Example* 3.10**: Calculate the Fourier transform of $x(t)$ with duality property.

$$x(t) = \frac{2}{t^2 + 1} \tag{3.59}$$

$$\text{Let } f(u) = \frac{2}{u^2 + 1} \tag{3.60}$$

$$g(t) \xleftrightarrow{\mathcal{F}} f(j\omega) = \frac{2}{\omega^2 + 1} \tag{3.61}$$

$$g(t) = e^{-|t|} \xleftrightarrow{\mathcal{F}} f(j\omega) = \frac{2}{\omega^2 + 1} \tag{3.62}$$

$$x(t) = f(t) \xleftrightarrow{\mathcal{F}} 2\pi g(-j\omega) = 2\pi e^{-|\omega|} \tag{3.63}$$

$$\therefore \mathcal{F}\{x(t)\} = 2\pi e^{-|\omega|} \tag{3.64}$$

Note:

- $$-jtx(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega} \qquad X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{3.65}$$

  $$\frac{dx(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) \qquad \frac{dX(j\omega)}{dt}=\int_{-\infty}^{\infty} -jtx(t)e^{-j\omega t}dt \tag{3.66}$$

- $$e^{j\omega_0 t}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega-\omega_0))$$

  $$x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega) \tag{3.67}$$

- $$-\frac{1}{jt}x(t)+\pi x(0)\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} X(\eta)d\eta$$

  $$\int_{-\infty}^{t} x(t)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega) \tag{3.68}$$

7. Parseval's Relation

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \tag{3.69}$$

Proof:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\underbrace{\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]}_{X(j\omega)}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

■

Note:

- The total energy in the signal $x(t)$ may be determined either by computing the energy per unit time and integrating over all time or by computing the energy per unit frequency and integrating over all frequencies.
- For periodic signals,

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{3.70}$$

8. Convolution Property

$$y(t)=h(t)*x(t)\xleftrightarrow{\mathcal{F}}Y(j\omega)=H(j\omega)X(j\omega) \tag{3.71}$$

Proof:

$$\begin{aligned}
y(t)&=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau \\
Y(j\omega)&=\mathcal{F}\{y(t)\}=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt \\
&=\int_{-\infty}^{\infty}x(\tau)\left[\int_{-\infty}^{\infty}h(t-\tau)e^{-j\omega t}dt\right]d\tau \\
&=\int_{-\infty}^{\infty}x(\tau)\cdot e^{-j\omega\tau}H(j\omega)d\tau \\
&=H(j\omega)\int_{-\infty}^{\infty}x(\tau)e^{-j\omega\tau}d\tau \\
&=H(j\omega)X(j\omega)
\end{aligned}$$

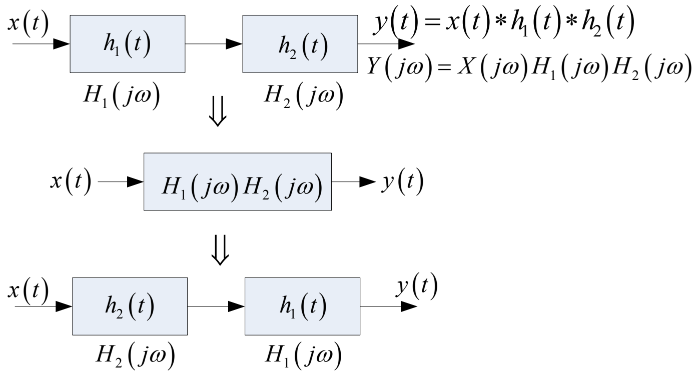
■

- $$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(jk\omega_0)\underbrace{e^{jk\omega_0 t}}_{\text{eigenfunction}}\omega_0 \tag{3.72}$$

$$\underbrace{H(jk\omega_0)}_{\text{eigenvalue}}=\int_{-\infty}^{\infty}h(t)e^{-jk\omega_0 t}dt$$

$$\begin{aligned}
\Rightarrow y(t)&=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}H(jk\omega_0)X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)X(j\omega)e^{j\omega t}d\omega \\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j\omega)e^{j\omega t}d\omega
\end{aligned}$$

- $H(j\omega)$: the Fourier transform of the system impulse response or the frequency response of the system.



- Periodic Convolution: [(periodic signal)* (periodic signal)]
  Consider two periodic signals $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ with common period T_0, the periodic convolution of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ is defined as

  $$\tilde{y}(t)=\int_{T_0}\tilde{x}_1(\tau)\tilde{x}_2(t-\tau)d\tau$$

  $$\tilde{x}_1(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \rightarrow T_0 a_k = X_1(jk\omega_0),\ \underbrace{X_1(j\omega)}_{\text{envelope}}$$

  Let $\tilde{x}_2(t)=\sum_{k=-\infty}^{\infty}b_k e^{jk\omega_0 t} \rightarrow T_0 b_k = X_2(jk\omega_0),\ \underbrace{X_2(j\omega)}_{\text{envelope}}$

  $$\tilde{y}(t)=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t} \rightarrow T_0 c_k = Y(jk\omega_0),\ \underbrace{Y(j\omega)}_{\text{envelope}}$$

  then $c_k = T_0 a_k b_k$

***Example* 3.11**:

$h(t)=e^{-at}u(t),\ a>0$

$x(t)=e^{-bt}u(t),\ b>0$ $\qquad\qquad$ $y(t)=h(t)*x(t)=?$

$H(j\omega)=\dfrac{1}{a+j\omega}$ $\qquad\qquad$ $X(j\omega)=\dfrac{1}{b+j\omega}$

(i) If $a\neq b$

$$Y(j\omega)=H(j\omega)X(j\omega)=\frac{1}{a+j\omega}\cdot\frac{1}{b+j\omega}=\frac{A}{a+j\omega}+\frac{B}{b+j\omega}$$

$$\Rightarrow A(b+j\omega)+B(a+j\omega)=1$$

$$\Rightarrow \left.\begin{array}{r} A+B=0 \\ Ab+Ba=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=1/(b-a) \\ B=-1/(b-a) \end{array}$$

$$\Rightarrow Y(j\omega)=\frac{1}{b-a}\left[\frac{1}{a+j\omega}-\frac{1}{b+j\omega}\right]$$

$$\Rightarrow y(t)=\frac{1}{b-a}\left[e^{-at}u(t)-e^{-bt}u(t)\right]$$

(ii) If $a=b$

$$\left.\begin{array}{l} Y(j\omega)=\dfrac{1}{(a+j\omega)^2}=j\dfrac{d}{d\omega}\left[\dfrac{1}{a+j\omega}\right] \\ \because e^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}\dfrac{1}{a+j\omega} \\ \Rightarrow te^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}j\dfrac{d}{d\omega}\left[\dfrac{1}{a+j\omega}\right]=\dfrac{1}{(a+j\omega)^2} \end{array}\right\}\Rightarrow y(t)=te^{-at}u(t)$$

***Example* 3.12**:

$$h(t) = e^{-t}u(t)$$
$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t} \qquad \mathrm{y}(t) = h(t) * x(t) = ?$$

$$H(j\omega) = \frac{1}{1+j\omega}$$
$$X(j\omega) = \sum_{k=-3}^{3} a_k 2\pi\delta(\omega - 2\pi k)$$
$$\Rightarrow Y(j\omega) = \sum_{k=-3}^{3} 2\pi a_k \cdot \frac{1}{1+j\omega}\delta(\omega - 2\pi k)$$
$$= \sum_{k=-3}^{3} \frac{2\pi a_k}{1+j2\pi k}\delta(\omega - 2\pi k)$$
$$\Rightarrow y(t) = \sum_{k=-3}^{3} \frac{a_k}{1+j2\pi k} e^{j2\pi kt}$$

■

***Example* 3.13**:

$$x(t) = e^{-t}u(t) - e^{-1}e^{-(t-1)}u(t-1)$$
$$y(t) = x(t) * h(t) = e^{-t}u(t), \qquad h(t) = ?$$

$$X(j\omega) = \frac{1}{1+j\omega} - \frac{e^{-1}e^{-j\omega}}{1+j\omega} = \frac{1-e^{-(1+j\omega)}}{1+j\omega}$$
$$Y(j\omega) = \frac{1}{1+j\omega}$$
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1-e^{-(1+j\omega)}}$$

$$\because \left|e^{-(1+j\omega)}\right| < 1$$
$$\therefore \frac{1}{1-e^{-(1+j\omega)}} = 1 + e^{-1}e^{-j\omega} + e^{-2}e^{-2j\omega} + e^{-3}e^{-3j\omega} + \cdots\cdots$$
$$\Rightarrow h(t) = \delta(t) + e^{-1}\delta(t-1) + e^{-2}\delta(t-2) + e^{-3}\delta(t-3) + \cdots\cdots$$

※ $\delta(t) \xleftrightarrow{\mathcal{F}} 1$

$\delta(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}$

■

9. Modulation Property

$$s(t) \stackrel{\mathcal{F}}{\longleftrightarrow} S(j\omega)$$
$$p(t) \stackrel{\mathcal{F}}{\longleftrightarrow} P(j\omega)$$

$$r(t) = s(t)p(t) \stackrel{\mathcal{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right] \tag{3.73}$$

Note:

- Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other.
  ⇒The multiplication of two signals is often referred to as amplitude modulation.
  Proof:

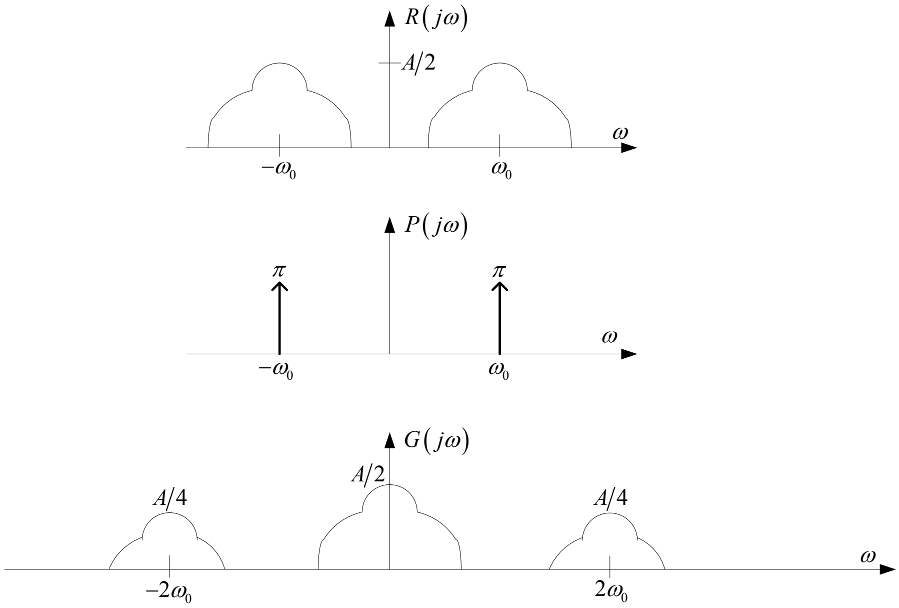
$$r(t) = s(t)p(t)$$
$$R(j\omega) = \int_{-\infty}^{\infty} s(t)p(t)e^{-j\omega t}dt$$
$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(jv)e^{jvt}dv$$
$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(t)\left[\int_{-\infty}^{\infty} P(jv)e^{jvt}dv\right]e^{-j\omega t}dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(jv)\underbrace{\left[\int_{-\infty}^{\infty} s(t)e^{-j(\omega-v)t}dt\right]}_{S(j(\omega-v))}dv$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(jv)S(j(\omega-v))dv$$
$$= \frac{1}{2\pi}P(j\omega) * S(j\omega)$$

■

***Example* 3.14**: Given $R(j\omega)$ and $p(t)$, calculate $G(j\omega) = \frac{1}{2\pi}\left[R(j\omega) * P(j\omega)\right]$.

$$g(t) = r(t)p(t) \stackrel{\mathcal{F}}{\longleftrightarrow} G(\omega) = \frac{1}{2\pi}\left[R(j\omega) * P(j\omega)\right]$$

$$p(t) = \cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$
$$\Rightarrow P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0) \tag{3.74}$$



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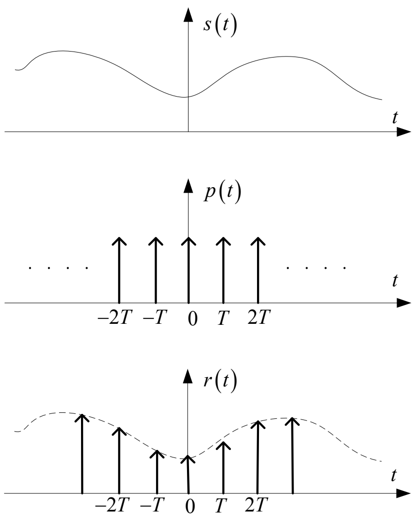
***Example* 3.15**: Given *s*(*t*) and *p*(*t*), calculate $R(j\omega)=\frac{1}{2\pi}\left[S(j\omega)*P(j\omega)\right]$.

$$r(t)=s(t)\,p(t)$$

$$p(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)\Rightarrow P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right) \tag{3.75}$$

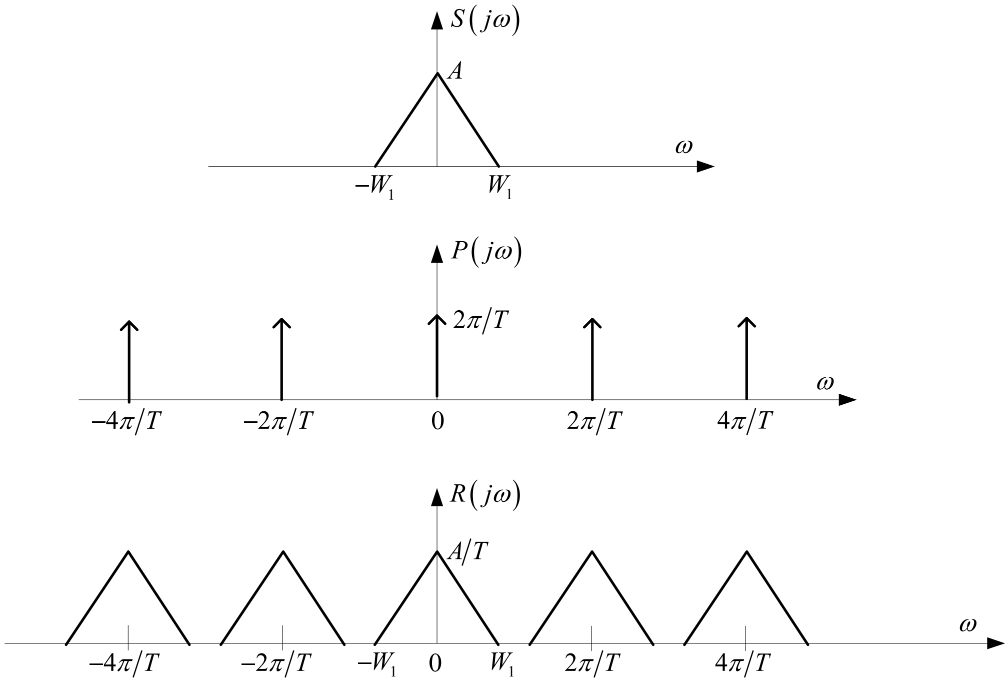


$$\begin{aligned}R(j\omega)&=\frac{1}{2\pi}\left[S(j\omega)*P(j\omega)\right]=\frac{1}{T}\sum_{k=-\infty}^{\infty}S(j\omega)*\delta\left(\omega-\frac{2\pi k}{T}\right)\\&=\frac{1}{T}\sum_{k=-\infty}^{\infty}S\left(j\left(\omega-\frac{2\pi k}{T}\right)\right)\end{aligned} \tag{3.76}$$

■

***Example* 3.16**: (Sampling Theorem)

$$r(t) = s(t)\,p(t)$$



■

- $\frac{2\pi}{T} \geq 2W_1$ i.e., $\omega_0 \geq 2W_1$

  (Sampling frequency ≥ 2 (Bandwidth of the signal))

  $\Rightarrow$ no aliasing in $R(j\omega)$

  $\Rightarrow$ $s(t)$ can be reconstructed from $r(t)$.

3-7 The Frequency Response of Systems Characterized by Linear Constant-Coefficient Differential Equations

1. Calculation of Frequency and Impulse Response

$$\sum_{k=0}^{N} a_k \cdot \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \cdot \frac{d^k x(t)}{dt^k}$$

$$\updownarrow \mathcal{F} \qquad\qquad \updownarrow \mathcal{F}$$

$$\sum_{k=0}^{N} a_k \cdot (j\omega)^k \cdot Y(j\omega) = \sum_{k=0}^{M} b_k \cdot (j\omega)^k \cdot X(j\omega)$$

$$\Rightarrow Y(j\omega)\left[\sum_{k=0}^{N} a_k \cdot (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k \cdot (j\omega)^k\right]$$

$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{\sum_{k=0}^{M} b_k \cdot (j\omega)^k}{\sum_{k=0}^{N} a_k \cdot (j\omega)^k} \quad \cdots\cdots \text{ frequency response}$$

***Example* 3.17**: Find out $h(t)$ for $y(t)=x(t)*h(t)$.

$$\frac{dy(t)}{dt}+ay(t)=x(t) \qquad (a>0)$$

$$j\omega Y(j\omega)+aY(j\omega)=X(j\omega)\Rightarrow Y(j\omega)[a+j\omega]=X(j\omega)$$

$$\Rightarrow H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{a+j\omega}\Rightarrow h(t)=e^{-at}u(t)$$

***Example* 3.18**: Find out $h(t)$ for $y(t)=x(t)*h(t)$.

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$$

$$(j\omega)^2 Y(j\omega)+4j\omega Y(j\omega)+3Y(j\omega)=j\omega X(j\omega)+2X(j\omega)$$

$$\Rightarrow Y(j\omega)\left[(j\omega)^2+4j\omega+3\right]=X(j\omega)[j\omega+2]$$

$$\Rightarrow H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{j\omega+2}{(j\omega)^2+4j\omega+3}$$

$$=\frac{j\omega+2}{(j\omega+1)(j\omega+3)}$$

$$=\frac{A}{j\omega+1}+\frac{B}{j\omega+3}$$

$$j\omega+2=A(j\omega+3)+B(j\omega+1)$$

$$=(A+B)j\omega+(3A+B)$$

$$\left.\begin{array}{l}A+B=1\\3A+B=2\end{array}\right\}\Rightarrow A=B=\frac{1}{2}$$

$$\Rightarrow H(j\omega)=\frac{1/2}{j\omega+1}+\frac{1/2}{j\omega+3}$$

$$\Rightarrow h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$$

■

***Example* 3.19**:

$$x(t)=e^{-t}u(t)$$

$$h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$$

$$y(t)=x(t)*h(t)=?$$

$$
\begin{aligned}
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= \frac{1}{j\omega+1}\cdot\frac{j\omega+2}{(j\omega+1)(j\omega+3)} \\
&= \frac{A}{j\omega+1}+\frac{B}{(j\omega+1)^2}+\frac{C}{j\omega+3}
\end{aligned}
$$

$j\omega+2 = A(j\omega+1)(j\omega+3)+B(j\omega+3)+C(j\omega+1)^2$

Let $s = j\omega$

$\Rightarrow s+2 = A(s+1)(s+3)+B(s+3)+C(s+1)^2$

Set $s=-1$

$\Rightarrow 1 = B\cdot 2 \Rightarrow B = 1/2$

Set $s=-3$

$\Rightarrow -1 = 4C \Rightarrow C = -1/4$

$A+C=0 \qquad (\text{the } s^2 \text{ term})$

$\Rightarrow A = -C = 1/4 \Rightarrow y(t) = \left[\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{1}{4}e^{-3t}\right]u(t)$

■

2. Cascade and Parallel-Form Structures for the Implementation of LTI Systems

(i) Cascade-form structure

$$H(j\omega) = \frac{b_M\prod_{k=1}^{M}(\lambda_k+j\omega)}{a_N\prod_{k=1}^{N}(v_k+j\omega)} \tag{3.77}$$

where λ_k and v_k may be complex.

By multiplying together the two first-order terms involving complex conjugate λ_k's or v_k's, we obtain second-order terms with real coefficients. For example,

$(\lambda+j\omega)(\lambda^*+j\omega) = |\lambda|^2+2\,\mathrm{Re}\{\lambda\}j\omega+(j\omega)^2$

$$\Rightarrow H(j\omega) = \frac{b_M}{a_N}\frac{\prod_{k=1}^{P}\left[\beta_{0k}+\beta_{1k}(j\omega)+(j\omega)^2\right]\prod_{k=1}^{M-2P}(\lambda_k+j\omega)}{\prod_{k=1}^{Q}\left[\alpha_{0k}+\alpha_{1k}(j\omega)+(j\omega)^2\right]\prod_{k=1}^{N-2Q}(v_k+j\omega)} \tag{3.78}$$

where the coefficients are all real.

$\Rightarrow$ The system can be implemented using a cascade (let *P*=*Q*) of *P* second-order systems and (*N-2P*) first-order systems.

- Realization of a second-order system

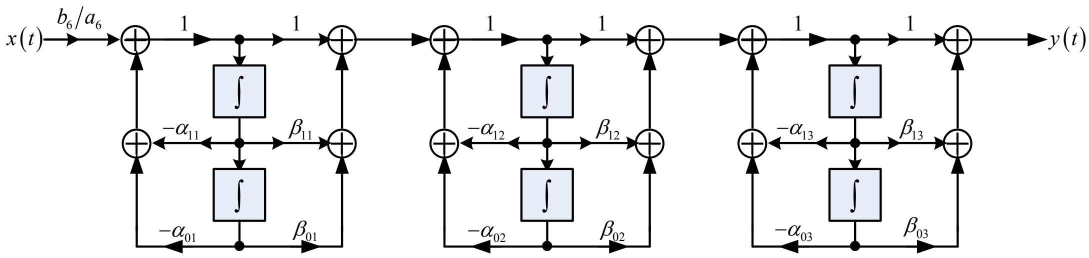
$$H_k(j\omega)=\frac{\beta_{0k}+j\omega\beta_{1k}+(j\omega)^2}{\alpha_{0k}+j\omega\alpha_{1k}+(j\omega)^2}=\frac{Y_k(j\omega)}{X_k(j\omega)}$$

$$\Rightarrow \alpha_{0k}Y_k(j\omega)+j\omega\alpha_{1k}Y_k(j\omega)+(j\omega)^2Y_k(j\omega)$$
$$=\beta_{0k}X_k(j\omega)+j\omega\beta_{1k}X_k(j\omega)+(j\omega)^2X_k(j\omega)$$

$$\Rightarrow \alpha_{0k}y_k(t)+\alpha_{1k}\frac{dy_k(t)}{dt}+\frac{d^2y_k(t)}{dt^2}$$
$$=\beta_{0k}x_k(t)+\beta_{1k}\frac{dx_k(t)}{dt}+\frac{d^2x_k(t)}{dt^2}$$

For convenience, we only consider the second-order terms for realization of a cascade system. (See Fig. 3.5.)

$$H(j\omega)=\frac{b_M}{a_N}\frac{\prod_{k=1}^{P}\left[\beta_{0k}+j\omega\beta_{1k}+(j\omega)^2\right]}{\prod_{k=1}^{Q}\left[\alpha_{0k}+j\omega\alpha_{1k}+(j\omega)^2\right]}$$



■ **Figure 3.5.** Cascade structure of each second-order subsystem with N=M=6 and P=Q=3.

(ii) Parallel-form structure

$$H(j\omega)=\frac{b_N\prod_{k=1}^{N}(\lambda_k+j\omega)}{a_N\prod_{k=1}^{N}(v_k+j\omega)} \tag{3.79}$$

If all of the v_k are distinct, then

$H(j\omega)$ can be expressed as

$$H(j\omega)=\left(\frac{b_N}{a_N}\right)+\sum_{k=1}^{N}\frac{A_k}{v_k+j\omega} \tag{3.80}$$

Adding together the pairs involving complex conjugate v_k's , we obtain

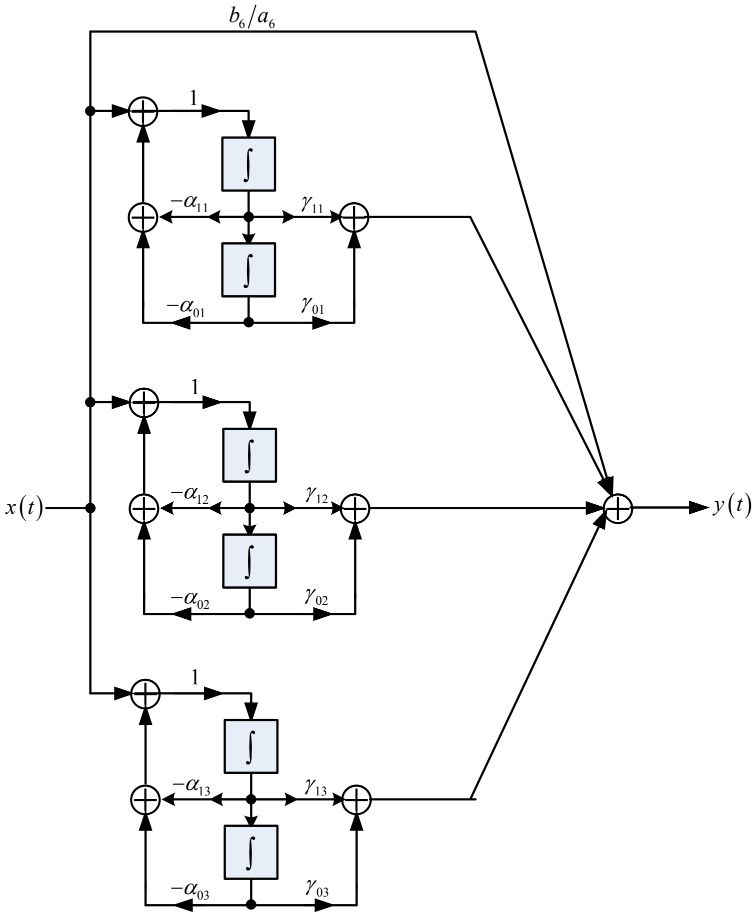
$$H(j\omega)=\left(\frac{b_N}{a_N}\right)+\sum_{k=1}^{Q}\frac{\gamma_{0k}+j\omega\gamma_{1k}}{\alpha_{0k}+j\omega\alpha_{1k}+(j\omega)^2} + \sum_{l=1}^{N-2Q}\frac{A_l}{v_l+j\omega} \tag{3.81}$$

(All the coefficients are real.)

$\Rightarrow$ We can implement the system by using a parallel interconnection of Q second-order systems and $(N\text{-}2Q)$ first-order systems.

For convenience, we only consider the second-order terms for realization of a parallel system. (See Fig. 3.6.)

$$H(j\omega)=\left(\frac{b_N}{a_N}\right)+\sum_{k=1}^{Q}\frac{\gamma_{0k}+j\omega\gamma_{1k}}{\alpha_{0k}+j\omega\alpha_{1k}+(j\omega)^2}$$



■ **Figure 3.6.** Parallel-form realization for each second-order subsystem with N=6 and Q=3.

Appendix

- $$G(v)=\frac{b_2v^2+b_1v+b_0}{(v-\rho_1)^2(v-\rho_2)}$$

$$G(v)=\frac{A_{11}}{v-\rho_1}+\frac{A_{12}}{(v-\rho_1)^2}+\frac{A_{21}}{v-\rho_2}$$

(i) $$(v-\rho_1)^2G(v)=A_{11}(v-\rho_1)+A_{12}+\frac{A_{21}(v-\rho_1)^2}{v-\rho_2}$$

$$\Rightarrow\left[(v-\rho_1)^2G(v)\right]\Big|_{v=\rho_1}=A_{12}$$

(ii) $$\frac{d\left[(v-\rho_1)^2G(v)\right]}{dv}=A_{11}+A_{21}\left[\frac{2(v-\rho_1)}{v-\rho_2}-\frac{(v-\rho_1)^2}{(v-\rho_2)^2}\right]$$

$$\Rightarrow\left.\frac{d\left[(v-\rho_1)^2G(v)\right]}{dv}\right|_{v=\rho_1}=A_{11}$$

(iii) $$(v-\rho_2)G(v)=\frac{A_{11}}{(v-\rho_1)}(v-\rho_2)+\frac{A_{12}}{(v-\rho_1)^2}(v-\rho_2)+A_{21}$$

$$\Rightarrow\left[(v-\rho_2)G(v)\right]\Big|_{v=\rho_2}=A_{21}$$

- $$G(v)=\frac{b_{n-1}v^{n-1}+\cdots+b_1v+b_0}{(v-\rho_1)^{\sigma_1}(v-\rho_2)^{\sigma_2}\cdots(v-\rho_r)^{\sigma_r}}$$

$$=\frac{A_{11}}{v-\rho_1}+\frac{A_{12}}{(v-\rho_1)^2}+\cdots+\frac{A_{1\sigma_1}}{(v-\rho_1)^{\sigma_1}}$$

$$+\frac{A_{21}}{v-\rho_2}+\frac{A_{22}}{(v-\rho_2)^2}+\cdots+\frac{A_{2\sigma_2}}{(v-\rho_2)^{\sigma_2}}$$

$$\vdots$$

$$+\frac{A_{r1}}{v-\rho_r}+\frac{A_{r2}}{(v-\rho_r)^2}+\cdots+\frac{A_{r\sigma_r}}{(v-\rho_r)^{\sigma_r}}$$

$$=\sum_{i=1}^{r}\sum_{k=1}^{\sigma_i}\frac{A_{ik}}{(v-\rho_i)^k}$$

$$\boxed{A_{ik}=\frac{1}{(\sigma_i-k)!}\left\{\frac{d^{\sigma_i-k}}{dv^{\sigma_i-k}}\left[(v-\rho_i)^{\sigma_i}G(v)\right]\right\}\Big|_{v=\rho_i}}$$