Chapter 2 Time-Domain Representations of Linear Time-Invariant Systems

2-1 The Representation of Signals in Terms of Impulses

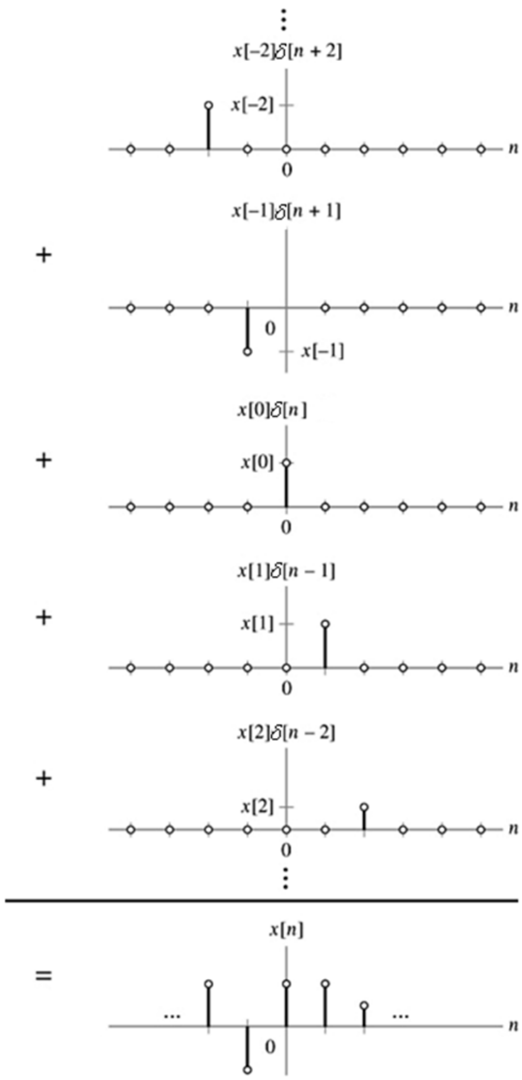
1. Discrete-time case

$$x[n]\delta[n]=x[0]\delta[n]\underset{\text{generalized}}{\Rightarrow}x[n]\delta[n-k]=x[k]\delta[n-k] \tag{1}$$

where $x[k]$ represents a specific value of the signal $x[n]$ at time k. Therefore, $x[n]$ can be expressed as the following weighted sum of time-shifted impulses:

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k] \tag{2}$$

***Example* 1**:



■ **Figure 1**. Graphical example illustrating the representation of a signal $x[n]$ as a weighted sum of time-shifted impulses.

***Example* 2**: $u[n]=\sum_{k=0}^{\infty}\delta[n-k]$

2. Continuous-time case

Define

$$\delta_\Delta(t)=\begin{cases}1/\Delta\ , & 0<t<\Delta\\ 0\ , & \text{otherwise}\end{cases},\ \Delta\delta_\Delta(t)=1 \tag{3}$$

$$\hat{x}(t)=\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \tag{4}$$

$$\begin{aligned}x(t)&=\lim_{\Delta\to 0}\hat{x}(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_\Delta(t-k\Delta)\Delta\\&=\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau\end{aligned} \tag{5}$$

- $x(t)$ equals the limit as $\Delta\to 0$ of the area under $x(\tau)\delta_\Delta(t-\tau)$.

$$\delta_\Delta(t-\tau)\xrightarrow{\Delta\to 0}\delta(t-\tau) \tag{6}$$

***Example* 3**:

$$u(t)=\int_{-\infty}^{\infty}u(\tau)\delta(t-\tau)d\tau=\int_{0}^{\infty}\delta(t-\tau)d\tau \qquad \blacksquare$$

- $\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau=\int_{-\infty}^{\infty}x(t)\delta(t-\tau)d\tau=x(t)\int_{-\infty}^{\infty}\delta(t-\tau)d\tau=x(t)$

$$(\because x(\tau)\delta(t-\tau)=x(t)\delta(t-\tau)) \tag{7}$$

2-2 Discrete-time LTI Systems

1. $x[n]=\sum_{-\infty}^{\infty}x[k]\delta[n-k]\Rightarrow y[n]$

Let $h[n]=H\{\delta[n]\}$ be the impulse response of the LTI system and $h[n-k]=H\{\delta[n-k]\}$ denote the response of the system to the shifted unit sample $\delta[n-k]$. Then

$$\begin{aligned}y[n]&=H\{x[n]\}=H\left\{\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]\right\}\\&=\sum_{k=-\infty}^{\infty}H\{x[k]\delta[n-k]\}\quad(\because \text{linearity property})\\&=\sum_{k=-\infty}^{\infty}x[k]\cdot H\{\delta[n-k]\}\end{aligned} \tag{8}$$

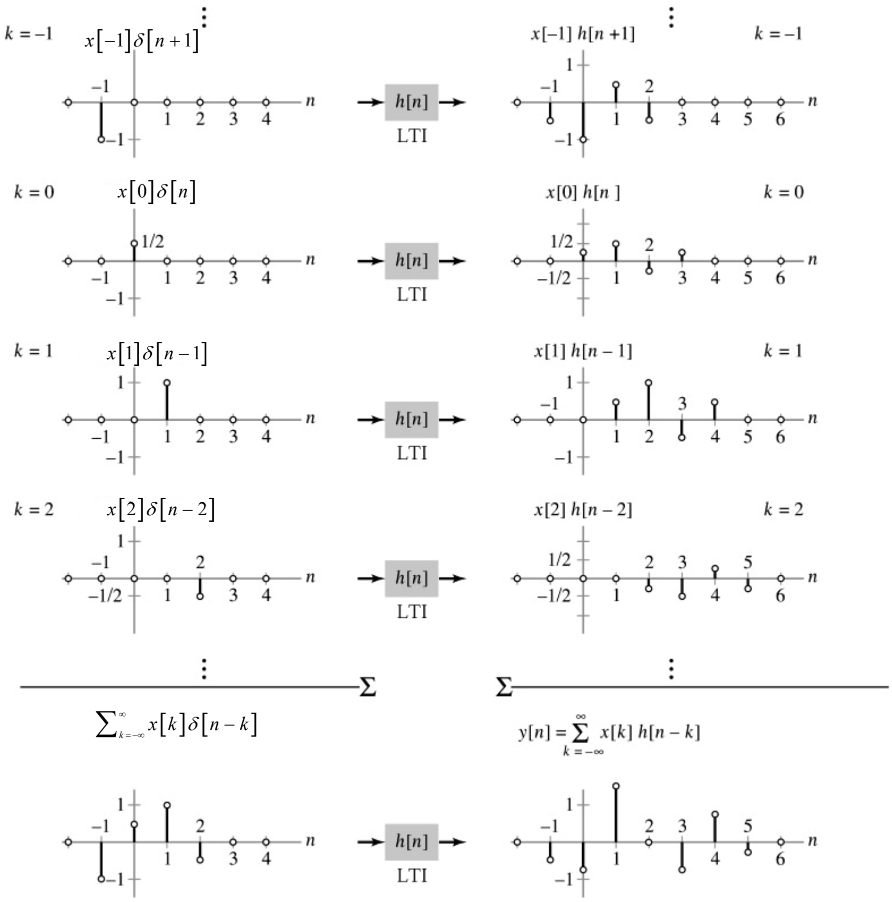
$$=\sum_{k=-\infty}^{\infty}x[k]\cdot h[n-k]\quad(\because \text{time-invariance property}) \tag{9}$$

$$\Rightarrow y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k] \tag{10}$$

This result is referred to as the convolution sum or superposition sum. We will represent symbolically as

$$y[n] = x[n] * h[n] \tag{11}$$

Interpretation of the convolution of two sequences



■ **Figure 2**. Illustration of the convolution sum.

Example 4:

$x[n] = \alpha^n u[n];\ h[n] = u[n]$

$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$x[k]h[n-k] = \begin{cases} \alpha^k\ , & 0 \le k \le n \\ 0\ , & \text{otherwise} \end{cases} \quad (\because n-k \ge 0, k \ge 0)$

(1) For $n \ge 0$, $y[n] = \sum_{k=0}^{n} \alpha^k = \dfrac{1-\alpha^{n+1}}{1-\alpha}$.

(2) For $n < 0$, $y[n] = 0$.

$\Rightarrow y[n] = \dfrac{1-\alpha^{n+1}}{1-\alpha} u[n]$ ■

***Example* 5**: Multipath communication channel

$$y[n] = x[n] + \frac{1}{2}x[n-1]$$

$$x[n] = \delta[n] \Rightarrow h[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

Determine the output of this system in response to the input

$$x[n] = 2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$$

$$\Rightarrow y[n] = 2h[n] + 4h[n-1] - 2h[n-2] = 2\delta[n] + 5\delta[n-1] - \delta[n-3] \quad \blacksquare$$

2. When the input is of long duration, the procedure can be cumbersome. So we need to use an alternative approach to evaluate the convolution sum.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} w_n[k] \qquad (12)$$

where $w_n[k] = x[k]h[n-k]$ is called the intermediate signal.

In this definition, k is the independent variable and $h[n-k] = h[-(k-n)]$ is a reflected and time-shifted version of $h[k]$.

$$\begin{cases} n < 0, \ h[n-k] = \text{time shift } h[-k] \text{ to the left.} \\ n > 0, \ h[n-k] = \text{time shift } h[-k] \text{ to the right.} \end{cases}$$

***Example* 6**: Convolution sum evaluation by using an intermediate signal

$$h[n] = (3/4)^n u[n]$$

Using the intermediate signal to determine the output of the system at time $n = -5$, 5, and 10 when the input is $x[n] = u[n]$.

$$h[n-k] = (3/4)^{n-k} u[n-k]$$

$$n = -5, \ w_{-5} = x[k]h[-5-k] = u[k](3/4)^{-5-k} u[-5-k] = 0$$

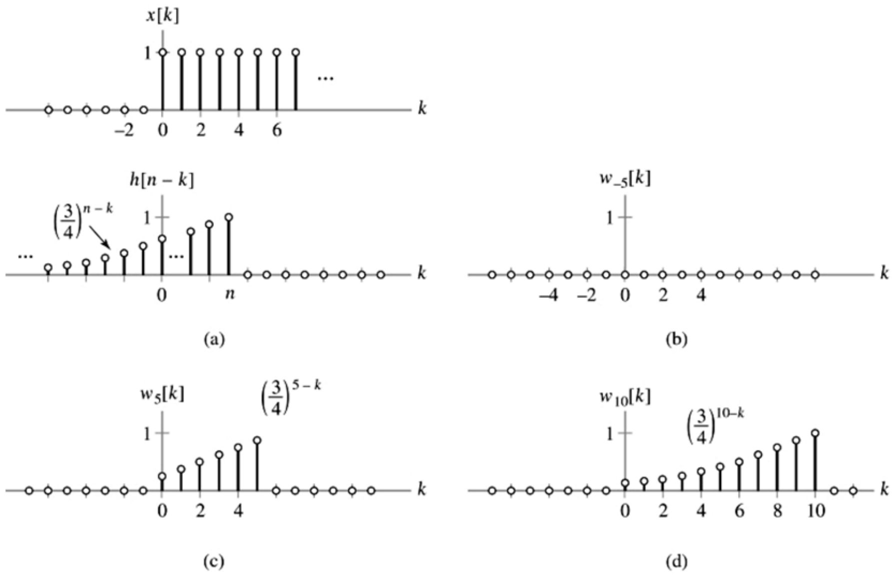
$$\therefore y[-5] = 0$$

$$n = 5, \ w_5 = x[k]h[5-k] = \begin{cases} (3/4)^{5-k} & ,0 \le k \le 5 \\ 0 & , \text{otherwise} \end{cases}$$

$$\therefore y[5] = \sum_{k=0}^{5} (3/4)^{5-k} = \frac{1-(3/4)^6}{1-3/4} = 3.288$$

$$n = 10, \ w_{10} = x[k]h[10-k] = \begin{cases} (3/4)^{10-k} & ,0 \le k \le 10 \\ 0 & , \text{otherwise} \end{cases}$$

$$\therefore y[10] = \sum_{k=0}^{10} (3/4)^{10-k} = \frac{1-(3/4)^{11}}{1-3/4} = 3.831$$



■ **Figure 3**. (a) The input signal $x[k]$ above the reflected and shifted response $h[n-k]$, depicted as a function of k. (b) The product signal $w_{-5}[k]$ used to evaluate $y[-5]$. (c) The product signal $w_5[k]$ used to evaluate $y[5]$. (d) The product signal $w_{10}[k]$ used to evaluate $y[10]$.

***Example* 7**: MA systems

$$y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k] \Rightarrow x[n] = \delta[n] \Rightarrow h[n] = \frac{1}{4}\left(u[n] - u[n-4]\right)$$

Determine the output of the system when the input is defined as

$$x[n] = u[n] - u[n-10]$$

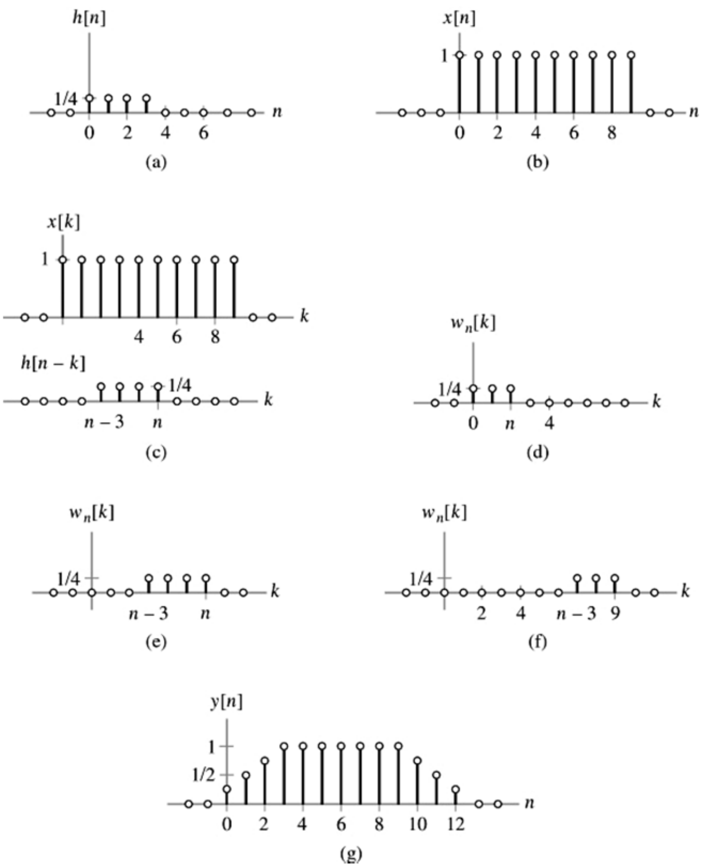
$$n < 0,\ w_n[k] = 0,\ \text{for all } k \Rightarrow y[n] = 0$$

$$0 \le n \le 3,\ w_n[k] = \begin{cases} 1/4\ ,0 \le k \le n \\ 0\ ,\ \text{otherwise} \end{cases} \Rightarrow y[n] = \sum_{k=0}^{n} 1/4 = (n+1)/4$$

$$3 < n \le 9,\ w_n[k] = \begin{cases} 1/4, n-3 \le k \le n \\ 0\ ,\ \text{otherwise} \end{cases} \Rightarrow y[n] = \sum_{k=n-3}^{n} 1/4 = 1$$

$$9 < n \le 12,\ w_n[k] = \begin{cases} 1/4, n-3 \le k \le 9 \\ 0\ ,\ \text{otherwise} \end{cases} \Rightarrow y[n] = \sum_{k=n-3}^{9} 1/4 = (13-n)/4$$

$$12 < n,\ w_n[k] = 0,\ \text{for all } k \Rightarrow y[n] = 0$$



■ **Figure 4**. (a) The system impulse response. (b) The input signal. (c) The input above the reflected and time-shifted impulse response $h[n-k]$, depicted as a function of k. (d) The product signal $w_n[k]$ for the interval of shifts $0 \le n \le 3$. (e) The product signal $w_n[k]$ for the interval of shifts $3 < n \le 9$. (f) The product signal $w_n[k]$ for the interval of shifts $9 < n \le 12$. (g) The output $y[n]$.

***Example* 8**:

$$x[n] = \begin{cases} 1\ , & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} \alpha^n\ , & 0 \le n \le 6 \\ 0\ , & \text{otherwise} \end{cases}$$

$$\Rightarrow h[n-k] = \begin{cases} \alpha^{n-k}, & n-6 \le k \le n \\ 0 & , \text{otherwise} \end{cases}$$

Interval 1: $n < 0$

$$w_n[k] = x[k]h[n-k] = 0 \Rightarrow y[n] = 0$$

Interval 2: $0 \le n \le 4$

$$w_n[k] = \begin{cases} \alpha^{n-k} & ,0 \le k \le n \\ 0 & , \text{otherwise} \end{cases} \Rightarrow y[n] = \sum_{k=0}^{n} \alpha^{n-k} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Interval 3: $4 < n \le 6$

$$w_n[k] = \begin{cases} \alpha^{n-k} & ,0 \le k \le 4 \\ 0 & , \text{otherwise} \end{cases} \Rightarrow y[n] = \sum_{k=0}^{4} \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}$$

Interval 4: $6 < n \le 10$

$$w_n[k] = \begin{cases} \alpha^{n-k} & ,n-6 \le k \le 4 \\ 0 & , \text{otherwise} \end{cases} \Rightarrow y[n] = \sum_{k=n-6}^{4} \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^{7}}{1-\alpha}$$

Interval 5: $10 < n,\ \ w_n[k] = 0 \Rightarrow y[n] = 0$ ■

3. Basic properties of convolution

(1) Commutative property

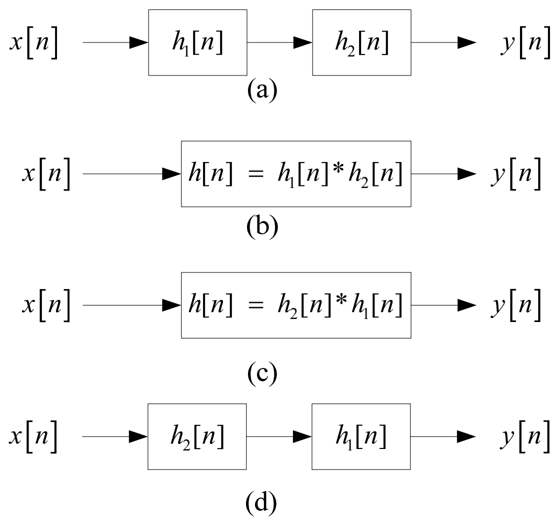
$$x[n] * h[n] = h[n] * x[n] \tag{13}$$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n] \tag{14}$$

(2) Associative property

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n] \tag{15}$$

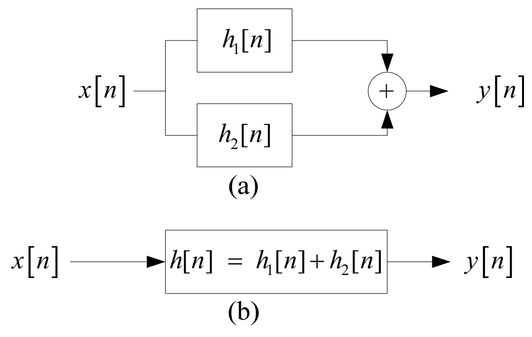
Interpretation of the associative property



■ **Figure 5**. Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI system.

(3) Distributive property

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n] \quad (16)$$



■ **Figure 6**. Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

Note:

- Convolution sum formula $\overset{\text{imply}}{\Rightarrow}$ unit impulse response

  It completely characterizes the behavior of an LTI system.
- The unit impulse response of a nonlinear system does not completely characterize the behavior of the system.

***Example* 9**:

$$h[n] = \begin{cases} 1 \ , \ n = 0,1 \\ 0, \text{ otherwise} \end{cases}$$

LTI system:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$
$$= h[0]x[n]+h[1]x[n-1]$$

"There is exactly one LTI system with $h[n]$ as its impulse response."

Nonlinear system:

$$\left.\begin{aligned} y[n] &= (x[n]+x[n-1])^2 \\ y[n] &= \max(x[n],x[n-1]) \end{aligned}\right\} \text{with the same impulse response}$$

Let $x[n]=\delta[n]$, then $y[n]=h[n]$

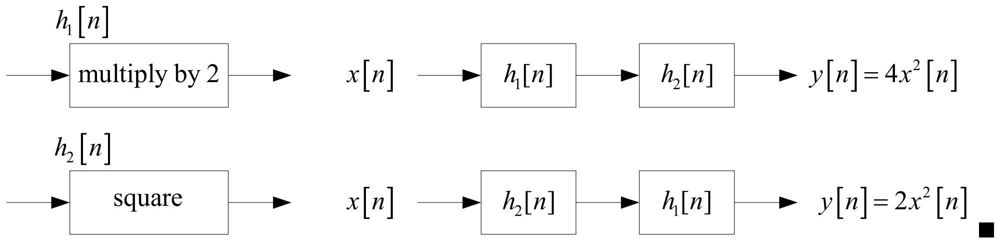
$$y[-1] = \max(x[-1],x[-2]) = 0$$
$$y[0] = \max(x[0],x[-1]) = 1$$
$$y[1] = \max(x[1],x[0]) = 1$$
$$y[2] = \max(x[2],x[1]) = 0$$

$$y[3] = \max\left(x[3], x[2]\right) = 0$$
$$\vdots$$

"There are many nonlinear systems with $h[n]$ as its unit impulse response."

Note: It is not true in general that the order in which nonlinear systems are cascaded can be changed without changing the overall response. ■

***Example* 10**:



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2-3 Continuous-Time LTI Systems

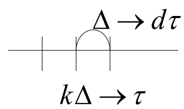
1. $x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$ (17)

   where $\delta_\Delta(t - k\Delta)$ is a rectangular pulse with unity amplitude and width Δ.

$$\Rightarrow \begin{aligned} y(t) &= \lim_{\Delta \to 0} H\left\{\sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta\right\} \\ &= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) H\left\{\delta_\Delta(t - k\Delta)\right\}\Delta \ (\because \text{linear property}) \end{aligned} \quad (18)$$

   where $H\{\delta_\Delta(t - k\Delta)\}$ is defined as the response of an LTI system to the input $\delta_\Delta(t - k\Delta)$.

   As Δ→0, $H\{\delta_\Delta(t - k\Delta)\} \to H\{\delta(t - k\Delta)\}$.



$$\Rightarrow y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) H\{\delta(t - k\Delta)\}\Delta \quad (19)$$

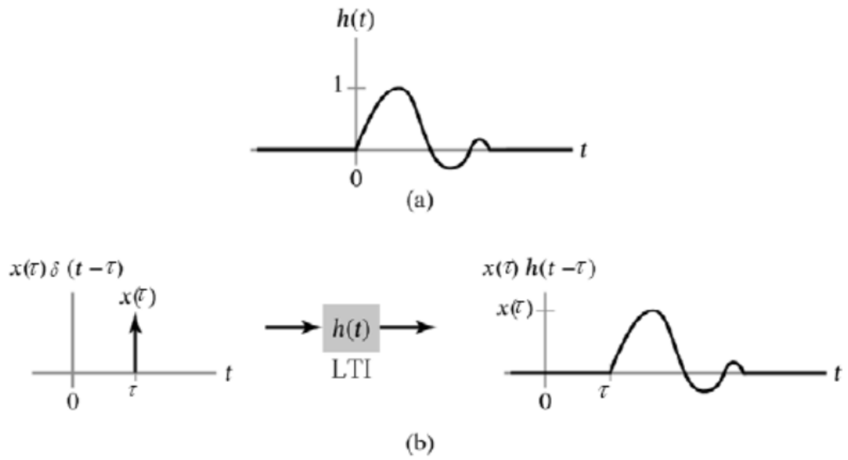
$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) H\{\delta(t - \tau)\} d\tau \quad (20)$$

   ∵ The system is time invariant.

$$\therefore \ H\{\delta(t - \tau)\} = h(t - \tau),\ H\{\delta(t)\} = h(t) \quad (21)$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = \int_{-\infty}^{\infty} w_t(\tau) d\tau \quad (22)$$

where $w_t(\tau) = x(\tau)h(t-\tau)$ is the intermediate signal.



■ **Figure 7**. (a) Impulse response of an LTI system *H*. (b) The output of an LTI system to a time-shifted and amplitude-scaled impulse is a time-shifted and amplitude-scaled impulse response.

The convolution of two signals $x(t)$ and $h(t)$ will be represented symbolically as

$$y(t) = x(t) * h(t) \tag{23}$$

2. Properties of the continuous-time convolution
   (1) Commutativity

$$x(t) * h(t) = h(t) * x(t) \tag{24}$$

The roles of input signal and impulse response are interchangeable.

   (2) Associativity

$$x(t) * \left[h_1(t) * h_2(t)\right] = \left[x(t) * h_1(t)\right] * h_2(t) \tag{25}$$

A cascade combination of LTI systems can be condensed into a single system whose impulse response is the convolution of the individual impulse responses.

   (3) Distributivity

$$x(t) * \left[h_1(t) + h_2(t)\right] = \left[x(t) * h_1(t)\right] + \left[x(t) * h_2(t)\right] \tag{26}$$

A parallel combination of LTI systems is equivalent to a single system whose impulse response is the sum of the individual impulse response in the parallel configuration.

Note:

- The overall impulse response of a cascade of two nonlinear systems (or even linear but time-varying system) does depend upon the order in which the systems are cascaded.
- A nonlinear continuous-time system is not completely described by its unit impulse response.

***Example* 11**:

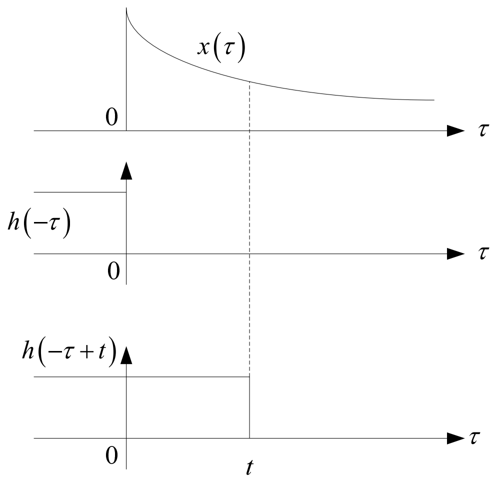
Let $x(t)=e^{-at}u(t)$ and $h(t)=u(t)$ where $a>0$.

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau \Rightarrow w_t(\tau)=x(\tau)h(t-\tau)=\begin{cases} e^{-a\tau} &, 0<\tau<t \\ 0 &, \text{otherwise}\end{cases}$$

For $t>0$, $y(t)=\int_0^t e^{-a\tau}d\tau=\frac{1}{a}\left(1-e^{-at}\right)$

For $t<0$, $y(t)=0$

$$y(t)=\frac{1}{a}\left(1-e^{-at}\right)u(t)$$



■ **Figure 8**. Calculation of the convolution integral for Example 11.

***Example* 12**:

$$x(t)=\begin{cases}1 &, 0<t<T \\ 0, & \text{otherwise}\end{cases} \quad \text{and} \quad h(t)=\begin{cases}t, & 0<t<2T \\ 0, & \text{otherwise}\end{cases}$$

$t < 0,\ y(t) = 0$

$0 < t < T,\ 0 < \tau < t$

$$w_t(\tau) = x(\tau)h(t-\tau) = t - \tau \Rightarrow y(t) = \int_0^t (t-\tau)\,d\tau = t^2 - \frac{1}{2}t^2 = \frac{1}{2}t^2$$

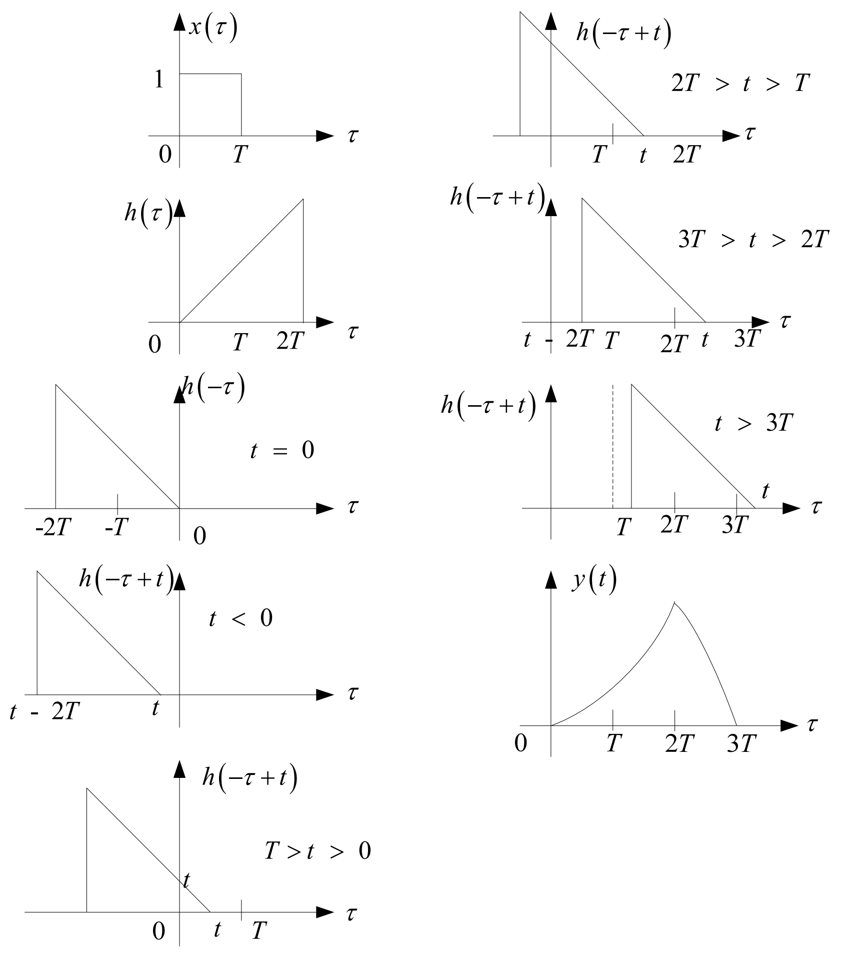
$T < t < 2T,\ 0 < \tau < T$

$$y(t) = \int_0^T (t-\tau)\,d\tau = Tt - \frac{1}{2}T^2$$

$2T < t < 3T,\ t - 2T < \tau < T$

$$y(t) = \int_{t-2T}^{T} (t-\tau)\,d\tau = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2$$

$3T < t,\ \ y(t) = 0$



■ **Figure 9**. Signal $x(\tau)$, $h(t-\tau)$, and $y(t)$ for different values of *t* for Example 12.

***Example* 13**:

$x(t) = u(t-1) - u(t-3)$ and $h(t) = u(t) - u(t-2)$

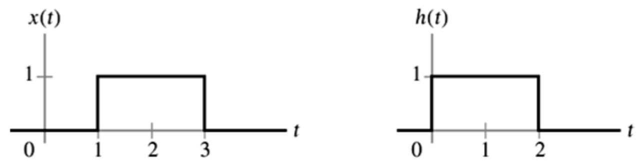
Evaluate the convolution integral for a system with $x(t)$ and $h(t)$.

$t<1,\ w_t(\tau)=0 \Rightarrow y(t)=0$

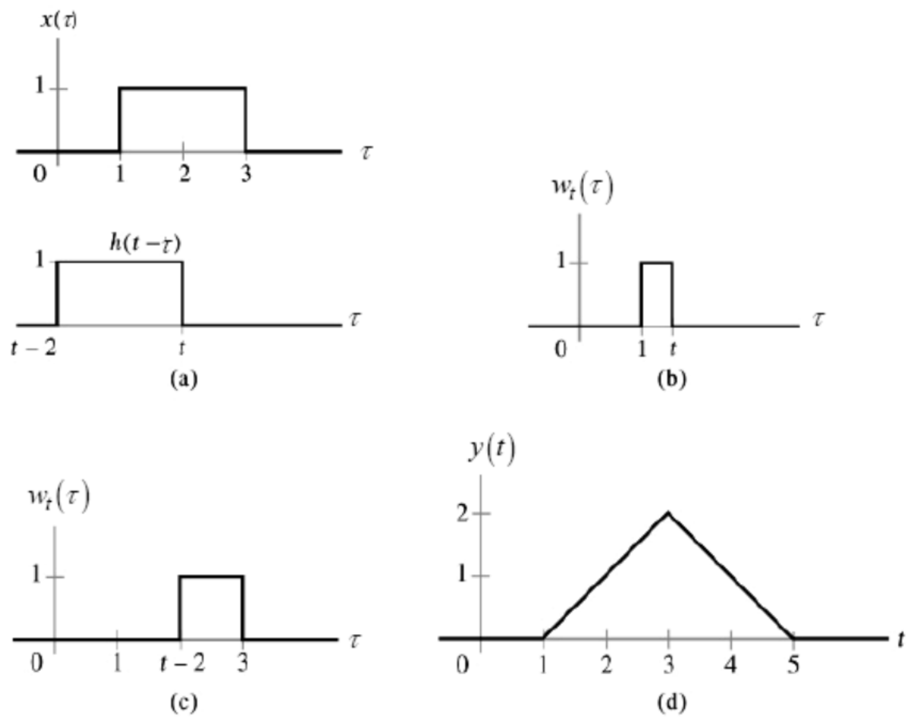
$1 \le t < 3,\ w_t(\tau)=\begin{cases}1\ ,\ 1<\tau<t \\ 0,\ \text{otherwise}\end{cases} \Rightarrow y(t)=t-1$

$3 \le t < 5,\ w_t(\tau)=\begin{cases}1,\ t-2<\tau<3 \\ 0\ \ ,\ \text{otherwise}\end{cases} \Rightarrow y(t)=5-t$

$5 \le t,\ w_t(\tau)=0 \Rightarrow y(t)=0$



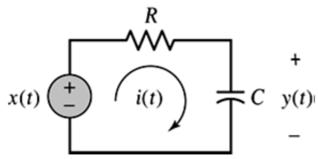
■ **Figure 10**. Input signal and LTI system impulse response for Example 13.



■ **Figure 11**. Evaluation of the convolution of integral for Example 13. (a) The input $x(\tau)$ depicted above the reflected and time-shifted impulse response $h(t-\tau)$, depicted as a function of τ. (b) The product signal $w_t(\tau)$ for $1 \le t < 3$. (c) The product signal $w_t(\tau)$ for $3 \le t < 5$. (d) The system output $y(t)$.

***Example* 14**: RC circuit output

Assume RC = 1 sec. $h(t)=e^{-t}u(t)$ and $x(t)=u(t)-u(t-2)$.



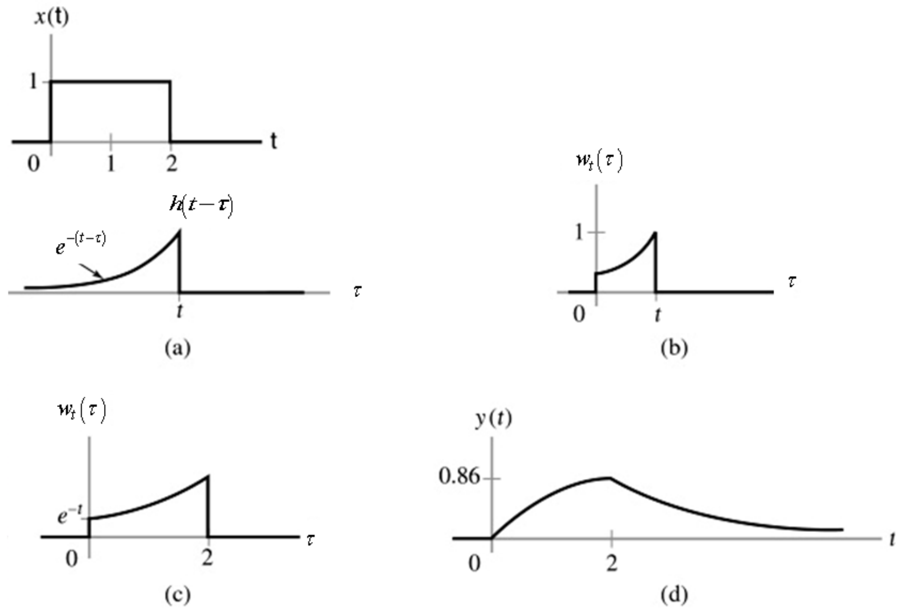
■ **Figure 12**. RC circuit system with the voltage source $x(t)$ as input and the voltage measured across the capacitor, $y(t)$, as output.

$$h(t-\tau)=e^{-(t-\tau)}u(t-\tau)$$

$$t<0,\ w_t(\tau)=0 \Rightarrow y(t)=0$$

$$0\le t<2,\ w_t(\tau)=\begin{cases} e^{-(t-\tau)}, & 0<\tau<t \\ 0, & \text{otherwise} \end{cases} \Rightarrow y(t)=\int_0^t e^{-(t-\tau)}d\tau = 1-e^{-t}$$

$$2\le t,\ w_t(\tau)=\begin{cases} e^{-(t-\tau)}, & 0<\tau<2 \\ 0, & \text{otherwise} \end{cases} \Rightarrow y(t)=\int_0^2 e^{-(t-\tau)}d\tau = \left(e^2-1\right)e^{-t}$$



■ **Figure 13**. Evaluate of the convolution integral for Example 14. (a) The input $x(\tau)$ depicted above the reflected and time-shifted impulse response $h(t-\tau)$, depicted as a function of τ. (b) The product signal $w_t(\tau)$ for $0\le t<2$. (c) The product signal $w_t(\tau)$ for $2\le t$. (d) The system output $y(t)$.

***Example* 15**: Radar range measurement: propagation model

$$x(t)=\begin{cases} \sin(\omega_c t), & 0\le t\le T_0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h(t)=a\delta(t-\beta)$$

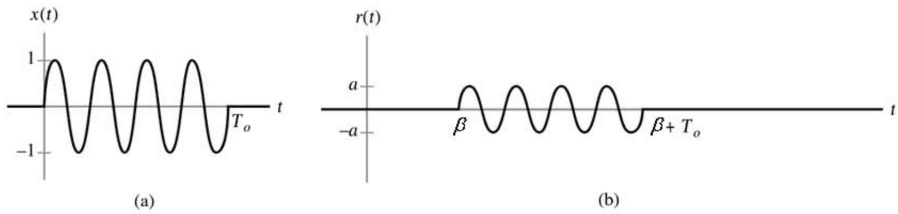
where a represents the attenuation factor and β the round-trip time delay.

$$h(\tau)=a\delta(\tau-\beta)\Rightarrow h(-\tau)=a\delta(-\tau-\beta)=a\delta\big(-(\tau+\beta)\big)=a\delta(\tau+\beta)$$

$(\because \text{even symmetry})$

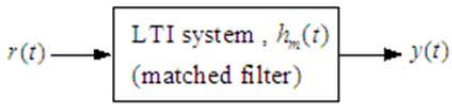
$$h(t-\tau)=a\delta\big(\tau-(t-\beta)\big)$$

$$r(t)=\int_{-\infty}^{\infty}x(\tau)a\delta\big(\tau-(t-\beta)\big)d\tau=ax(t-\beta)$$



■ **Figure 14**. Radar range measurement. (a) Transmitted RF pulse. (b) The received echo is an attenuated and delayed version of the transmitted pulse.

Example **16**: Radar range measurement : the matched filter



$$h_m(t)=x(-t)=\begin{cases}-\sin(\omega_c t), & -T_0\le t\le 0\\ 0 & \text{, otherwise}\end{cases}$$

$$w_t(\tau)=r(\tau)h_m(t-\tau)=r(\tau)x(\tau-t)$$

$$t+T_0<\beta\Rightarrow t<\beta-T_0,\ w_t(\tau)=0\Rightarrow y(t)=0$$

$$\beta\le t+T_0<\beta+T_0\Rightarrow \beta-T_0<t\le\beta,$$

$$w_t(\tau)=\begin{cases}a\sin\big(\omega_c(\tau-\beta)\big)\sin\big(\omega_c(\tau-t)\big), & \beta<\tau<t+T_0\\ 0 & \text{, otherwise}\end{cases}$$

$$\Rightarrow\ y(t)=\int_{\beta}^{t+T_0}a\sin\big(\omega_c(\tau-\beta)\big)\sin\big(\omega_c(\tau-t)\big)d\tau$$

$$=\frac{a}{2}\int_{\beta}^{t+T_0}\Big[\cos\big(\omega_c(t-\beta)\big)-\cos\big(\omega_c(2\tau-\beta-t)\big)\Big]d\tau$$

$$=\frac{a}{2}\cos\big(\omega_c(t-\beta)\big)[t+T_0-\beta]+\frac{a}{4\omega_c}\sin\big(\omega_c(2\tau-\beta-t)\big)\Big|_{\beta}^{t+T_0}$$

$$=\frac{a}{2}\cos\big(\omega_c(t-\beta)\big)[t+T_0-\beta]$$

$$+\frac{a}{4\omega_c}\Big[\sin\big(\omega_c(t+2T_0-\beta)\big)-\sin\big(\omega_c(\beta-t)\big)\Big]$$

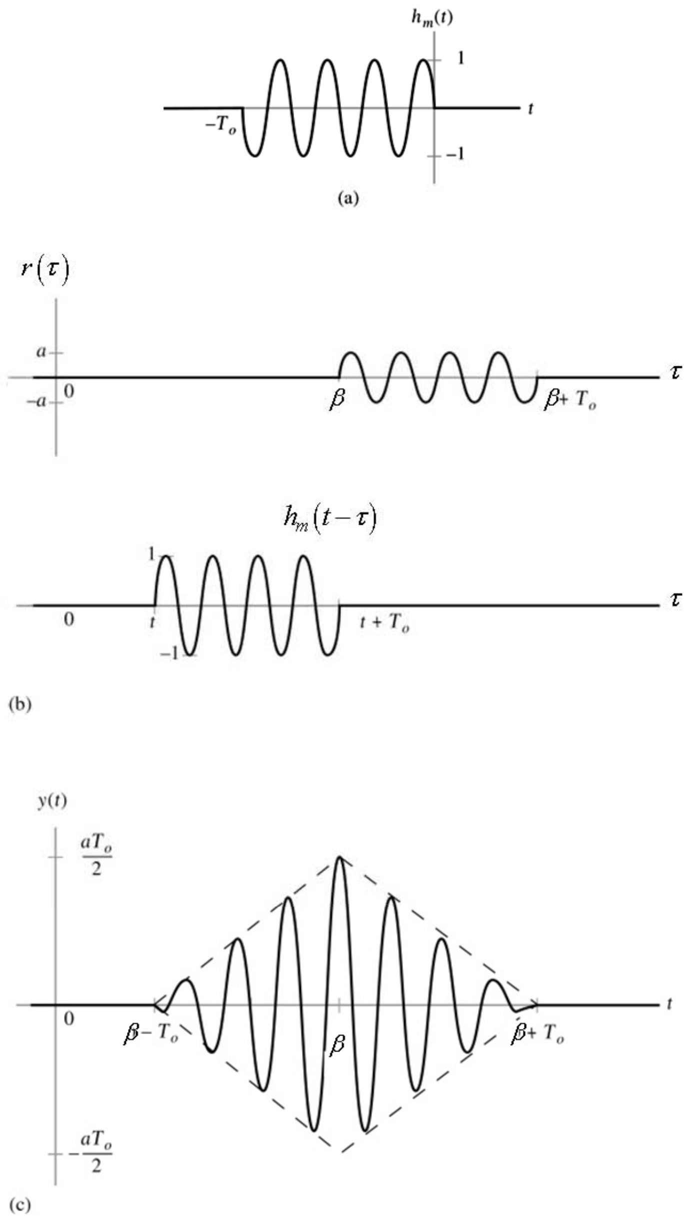
$$\approx\frac{a}{2}\cos\big(\omega_c(t-\beta)\big)[t+T_0-\beta]\ \big(\because \omega_c>10^6\text{rad/s}\big)$$

$\beta < t < \beta + T_0$,

$$w_t(\tau) = \begin{cases} a\sin\left(\omega_c(\tau-\beta)\right)\sin\left(\omega_c(\tau-t)\right), & t<\tau<\beta+T_0 \\ 0 & \text{, otherwise} \end{cases}$$

$$\Rightarrow \begin{aligned} y(t) &= \int_t^{\beta+T_0} a\sin\left(\omega_c(\tau-\beta)\right)\sin\left(\omega_c(\tau-t)\right)d\tau \\ &\approx \frac{a}{2}\cos\left(\omega_c(t-\beta)\right)\left[\beta+T_0-t\right] \end{aligned}$$

$\beta + T_0 < t,\ w_t(\tau) = 0 \Rightarrow y(t) = 0$



■ **Figure 15**. (a) Impulse response of the matched filter. (b) The received signal $r(\tau)$ superimposed on the reflected and time-shifted matched filter impulse response $h_m(t-\tau)$, depicted as function of τ. (c) Matched filter output $y(t)$.

2-4 Properties of Linear Time-Invariant Systems

1. LTI system with or without memory

   (1) Discrete-time memoryless system: $y[n]$ depends only on $x[n]$.

   $\Rightarrow h[n] = 0$ for $n \neq 0$ (27)

   $\Rightarrow h[n] = c\delta[n]$, $c = h[0]$ (28)

   $y[n] = cx[n]$ (29)

   If a discrete-time LTI system has an impulse response $h[n]$ which is not identically zero for $n \neq 0$, then the system has memory.

   ***Example* 17**: $y[n] = x[n] + x[n-1]$

   $$h[n] = \begin{cases} 1 & , n = 0,1 \\ 0, & \text{otherwise} \end{cases}$$ ■

   (2) Continuous-time memoryless system:

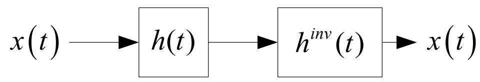
   $$\begin{cases} h(t) = 0 \text{ for } t \neq 0 \\ y(t) = cx(t) \Rightarrow h(t) = c\delta(t) \end{cases} \quad (30)$$

   $h(t) \neq 0$ for some value of $t \Rightarrow$ "memory" system

   If $c = 1$, then the convolution sum and convolution integral formulas of memoryless LTI system imply that

   $$\begin{aligned} x[n] &= x[n] * \delta[n] \\ x(t) &= x(t) * \delta(t) \end{aligned} \quad (31)$$

2. Invertibility of LTI systems



■ **Figure 16**. Concept of an inverse system for continuous-time LTI systems.

$$\begin{cases} h(t) * h^{inv}(t) = \delta(t) \\ h[n] * h^{inv}[n] = \delta[n] \end{cases} \quad (32)$$

The process of recovering $x(t)$ from $h(t) * x(t)$ is termed *deconvolution*, since it corresponds to recovering or undoing the convolution operation.

***Example* 18**:

$$y(t) = x(t - t_0)$$

$$\begin{cases} h(t) = \delta(t - t_0) \\ x(t - t_0) = x(t) * \delta(t - t_0) \ (\text{delay by } t_0) \end{cases}$$

$$\begin{cases} h^{inv}(t) = \delta(t+t_0) \\ x(t+t_0) = x(t) * \delta(t+t_0) \ \ (\text{advance by } t_0) \end{cases}$$

$$\begin{aligned} h(t) * h^{inv}(t) &= \delta(t-t_0) * \delta(t+t_0) \\ &= \int_{-\infty}^{\infty} \delta(\tau - t_0)\delta(t-\tau+t_0)\,d\tau \\ &= \delta(t)\int_{-\infty}^{\infty} \delta(\tau - t_0)\,d\tau = \delta(t) \end{aligned}$$

■

***Example* 19**: Multipath communication channels: compensation by means of an inverse system

$$\begin{aligned} &y[n] = x[n] + ax[n-1] \\ &\Rightarrow h[n] = \delta[n] + a\delta[n-1] \end{aligned}$$

Find a causal and stable inverse system that recovers $x[n]$ from $y[n]$.

$h[n] * h^{inv}[n] = \delta[n]$

$\Rightarrow \sum_{k=-\infty}^{\infty} h[k]h^{inv}[n-k] = h^{inv}[n] + ah^{inv}[n-1] = \delta[n]$

For $n < 0$, $h^{inv}[n] = 0 \Rightarrow$ causal

For $n = 0$, $h^{inv}[0] + ah^{inv}[-1] = 1 \Rightarrow h^{inv}[0] = 1$

For $n > 0$, $h^{inv}[n] + ah^{inv}[n-1] = 0 \Rightarrow h^{inv}[n] = -ah^{inv}[n-1]$

$\Rightarrow h^{inv}[1] = -a,\ h^{inv}[2] = a^2,\ h^{inv}[3] = -a^3, \ldots$

$\therefore h^{inv}[n] = (-a)^n u[n]$

$\sum_{k=-\infty}^{\infty} \left|h^{inv}[k]\right| = \sum_{k=-\infty}^{\infty} |a|^k < \infty$ when $|a| < 1$

■

3. Causality for LTI systems

The output of a causal system depends only on the present and past values of the input.

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \Rightarrow y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \Rightarrow y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] \end{aligned} \tag{33}$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \Rightarrow y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \Rightarrow y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)\,d\tau \end{aligned} \tag{34}$$

$$\Rightarrow \begin{cases} h[n] = 0 \text{ for } n < 0 \\ h(t) = 0 \text{ for } t < 0 \end{cases} \tag{35}$$

***Example* 20**:

$$\left.\begin{aligned} h[n] &= u[n] \\ h[n] &= \delta[n] - \delta[n-1] \end{aligned}\right\} \Rightarrow \text{causal}$$

$h(t) = \delta(t - t_0)$ is causal for $t_0 \geq 0$ and noncausal for $t_0 < 0$. ■

4. Stability for LTI systems

BIBO stability: bounded input → bounded output

(1) The impulse response $h[n]$ is absolutely summable, i.e., $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$. ⇔ The discrete-time system is BIBO stable.

(a) $\sum_{k=-\infty}^{\infty} |h[k]| < \infty \;\Rightarrow\;$ BIBO stable

Consider $|x[n]| \leq M_x$ for all n

$$\begin{aligned} |y[n]| &= \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \quad (\because |a+b| \leq |a| + |b|) \\ &\leq M_x \sum_{k=-\infty}^{\infty} |h[k]| \quad \text{for all } n \end{aligned} \tag{36}$$

Thus, if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, then the system is BIBO stable.

(b) BIBO stable $\Rightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty$

≡ $\sum_{k=-\infty}^{\infty} |h[k]| \to \infty \;\Rightarrow\;$ not BIBO stable (i.e., there exists a bounded input that can generate an unbounded output.)

The system output at index $n = n_0$ for the input $x[n]$ is

$$y[n_0] = \sum_{k=-\infty}^{\infty} h[k]x[n_0 - k] \tag{37}$$

Consider a bounded input of the form

$$x[n] = \pm B_1, \text{ for all } n \tag{38}$$

and let $x[n_0-k]=\text{sign}(h[k])B_1$, then

$$y[n_0]=B_1\sum_{k=-\infty}^{\infty}|h[k]| \quad \left(\because \text{sign}(h[k])h[k]=|h[k]|\right) \tag{39}$$

$\because \sum_{k=-\infty}^{\infty}|h[k]|\to\infty \quad \therefore y[n_0]\to\infty$

Therefore, "$h[n]$ is absolutely summable" is a sufficient and necessary condition to guarantee the BIBO stability of a discrete-time LTI system.

(2) The impulse response $h(t)$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty$. $\Leftrightarrow$ The continuous-time system is BIBO stable.

(a) $\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty \Rightarrow$ BIBO stable

Consider $|x(t)|\le M_x$ for all t

$$\begin{aligned}|y(t)|&=\left|\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau\right|\\&\le\int_{-\infty}^{\infty}|h(\tau)||x(t-\tau)|d\tau\\&\le M_x\int_{-\infty}^{\infty}|h(\tau)|d\tau\end{aligned} \tag{40}$$

Thus, if $\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty$, then the system is BIBO stable.

(b) BIBO stable $\Rightarrow \int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty$

$\equiv \int_{-\infty}^{\infty}|h(\tau)|d\tau\to\infty \Rightarrow$ not BIBO stable (i.e., there exists a bounded input that can generate an unbounded output.)

The system output at time $t=t_0$ for the input $x(t)$ is

$$y(t_0)=\int_{-\infty}^{\infty}h(\tau)x(t_0-\tau)d\tau \tag{41}$$

Consider a bounded input of the form

$$x(t)=\pm B_2,\ \text{for all } t \tag{42}$$

and let $x(t_0-\tau)=\text{sign}(h(\tau))B_2$, then

$$y(t_0) = B_2 \int_{-\infty}^{\infty} |h(\tau)| d\tau \quad \left(\because \operatorname{sign}(h(\tau)) h(\tau) = |h(\tau)|\right) \tag{43}$$

$$\because \int_{-\infty}^{\infty} |h(\tau)| d\tau \to \infty \quad \therefore y(t_0) \to \infty$$

The system is BIBO stable if and only if the impulse response is absolutely integrable,

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \tag{44}$$

***Example* 21**: $h[n] = \delta[n - n_0]$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n - n_0]| = 1 \Rightarrow \text{ stable}$$
$$h[n] = u[n] \Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{n} x[k] \Rightarrow \text{accumulator}$$
$$\sum_{k=-\infty}^{\infty} |u[n]| = \sum_{k=0}^{\infty} |u[n]| = \infty \Rightarrow \text{ unstable}$$

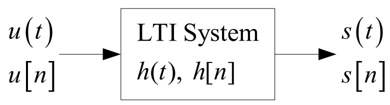
■

***Example* 22**: Properties of the first-order recursive system

$$y[n] = \rho y[n-1] + x[n], \ |\rho| < 1$$
$$\Rightarrow h[n] = \rho^n u[n]$$
$$\begin{cases} h[n] = 0 \text{ for } n < 0 \Rightarrow \text{causal} \\ h[n] \neq 0 \text{ for } n > 0 \Rightarrow \text{memory} \\ \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |\rho^k| = \sum_{k=-\infty}^{\infty} |\rho|^k < \infty \end{cases}$$

■

5. The unit step response of an LTI system



■ **Figure 17**. Step response of an LTI system *H*.

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k] \tag{45}$$

$$h[n] = s[n] - s[n-1] \tag{46}$$

$$s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau) d\tau \tag{47}$$

$$h(t) = \frac{d}{dt}s(t) = s'(t) \tag{48}$$

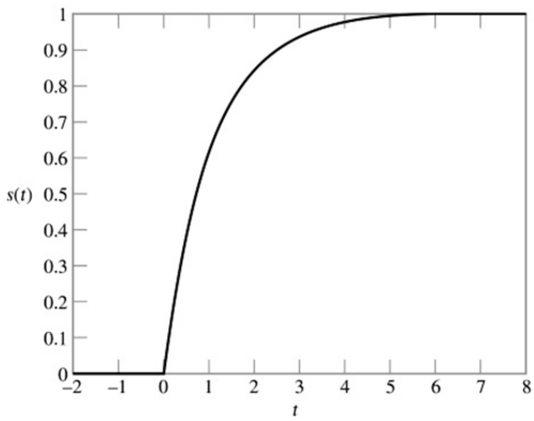
In both continuous and discrete time, the unit step response can also be used to characterize an LTI system to sudden changes in the input.

***Example* 23**: RC circuit

From chapter 1, the impulse response of a RC circuit is $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$. The step response of the circuit is

$$s(t) = \int_{-\infty}^{t} \frac{1}{RC}e^{-\tau/RC}u(\tau)d\tau = \begin{cases} \frac{1}{RC}\int_{0}^{t} e^{-\tau/RC}u(\tau)d\tau, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$= \begin{cases} 1 - e^{-t/RC}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$



■ **Figure 18**. RC circuit step response for $RC = 1$ s.

■

2-5 Systems Described by Differential or Difference Equations

1. The general form of a linear constant-coefficient *differential* equation (for continuous-time systems) is

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{49}$$

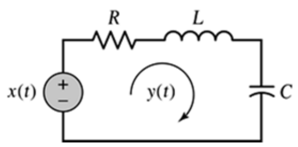
where the a_k and b_k are constant coefficients of the system, $x(t)$ is the input applied to the system and $y(t)$ is the resulting output.

2. A linear constant-coefficient *difference* equation (for discrete-time systems) has a similar form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{50}$$

The order of the differential or difference equation is (N, M), representing the number of energy storage devices in the system. Often, $N \geq M$, and the order is described using only N.

***Example* 24**: RLC circuit



$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

$$\overset{\frac{d}{dt}}{\Rightarrow} R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) + \frac{1}{C}y(t) = \frac{d}{dt}x(t)$$

Here the order is $N = 2$. This implies that the circuit contains two energy storage devices: capacitor and inductor. ■

***Example* 25**: $y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$

The order is $N = 2$. This implies a maximum memory of 2 in the system output. ■

3. Computing the current output of the system from the input signal and past outputs:

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\} \tag{51}$$

***Example* 26**: $y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$

$$y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4}y[n-2]$$

$$y[0] = x[0] + 2x[-1] - y[-1] - \frac{1}{4}y[-2]$$

$$y[1] = x[1] + 2x[0] - y[0] - \frac{1}{4}y[-1]$$

$$\vdots$$

In order to begin this process at time $n = 0$, we must know the two most recent past values of the output. These values are known as initial conditions.

Note:

- The number of initial conditions required to determine the output is equal to the maximum memory of the system. It's common to choose $n = 0$ or $t = 0$ as the starting time for solving a difference or differential equation, respectively. For example, the initial conditions for an Nth-order difference and differential equation are the N values

$$y[-N], y[-N+1], \ldots, y[-1] \tag{52}$$

and

$$y(t)\Big|_{t=0^-}, \left.\frac{dy(t)}{dt}\right|_{t=0^-}, \ldots, \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=0^-} \tag{53}$$

2-6 Solving Differential and Difference Equations

1. The output of a system described by a differential or difference equation may be expressed as the sum of two components.
   (1) Homogeneous solution, $y^{(h)}$: a solution of the homogeneous form (by setting all terms involving the input to zero) of the differential or difference equation.
   (2) Particular solution, $y^{(p)}$: any solution of the original equation for the given input.

   Thus, the complete solution is

$$y(t) = y^{(h)}(t) + y^{(p)}(t) \quad \text{or} \quad y[n] = y^{(h)}[n] + y^{(p)}[n] \tag{54}$$

2. Linear constant-coefficient differential equations

   Consider a continuous-time system described by

$$\frac{d}{dt}y(t) + 2y(t) = x(t) \tag{55}$$

where $x(t)$ and $y(t)$ are the input and output of the system respectively. The complete solution to the above differential equation can be expressed as

$$y(t) = y^{(p)}(t) + y^{(h)}(t) \tag{56}$$

where

$$\begin{cases} y^{(p)}(t): \text{ particular solution} \\ y^{(h)}(t): \text{ homogeneous solution, i.e., solution of } \frac{d}{dt}y(t) + 2y(t) = 0 \end{cases}$$

(1) Determination of the particular solution

(A particular solution is usually obtained by assuming an output of the same general form as the input.)

Consider $x(t) = k\cos(\omega_0 t)u(t) = \mathrm{Re}\{ke^{j\omega_0 t}\}u(t)$. For $t \geq 0$ we can hypothesize a particular solution of the form

$$y^{(p)}(t) = \mathrm{Re}\left\{Ye^{j\omega_0 t}\right\} \tag{57}$$

$$\frac{d}{dt}y^{(p)}(t) + 2y^{(p)}(t) = \mathrm{Re}\left\{j\omega_0 Ye^{j\omega_0 t} + 2Ye^{j\omega_0 t}\right\} = \mathrm{Re}\left\{ke^{j\omega_0 t}\right\} \tag{58}$$

$$\Rightarrow j\omega_0 Y + 2Y = k \tag{59}$$

$$Y = \frac{k}{j\omega_0 + 2} = \frac{k}{\sqrt{4+\omega_0^2}}e^{-j\theta},\ \theta = \tan^{-1}\left(\frac{\omega_0}{2}\right) \tag{60}$$

$$y^{(p)}(t) = \mathrm{Re}\left\{Ye^{j\omega_0 t}\right\} = \frac{k}{\sqrt{4+\omega_0^2}}\cos\left(\omega_0 t - \theta\right),\ t > 0 \tag{61}$$

(2) Determination of the homogeneous solution

(a) In order to determine $y^{(h)}(t)$, we hypothesize a solution of the form

$$y^{(h)}(t) = Ae^{st} \tag{62}$$

$$\frac{d}{dt}y^{(h)}(t) + 2y^{(h)}(t) = sAe^{st} + 2Ae^{st} = 0 \tag{63}$$

$$s + 2 = 0 \Rightarrow s = -2 \tag{64}$$

$$y^{(h)}(t) = Ae^{-2t},\ t > 0 \tag{65}$$

(b) The homogeneous solution of a general linear constant coefficient differential equation can be found in a way given in the Appendix.

(3) Determination of the complete solution

From (1) and (2), we have

$$y(t) = y^{(p)}(t) + y^{(h)}(t) = Ae^{-2t} + \frac{k}{\sqrt{4+\omega_0^2}}\cos(\omega_0 t - \theta),\ t > 0 \quad (66)$$

(a) Determination of the constant A by specifying initial (or auxiliary) conditions on the differential equation

If we specify $y(0) = y_0$, then

$$A = y_0 - \frac{k}{\sqrt{4+\omega_0^2}}\cos\theta \quad (67)$$

$$y(t) = y_0 e^{-2t} + \frac{k}{\sqrt{4+\omega_0^2}}\left[\cos(\omega_0 t - \theta) - \cos\theta e^{-2t}\right],\ t > 0 \quad (68)$$

(b) Solution of the differential equation for $t < 0$

For $t < 0$, $x(t) = 0$ and $y(t) = y^{(h)}(t) = Be^{-2t}$,

$$y(t) = y_0 e^{-2t},\ t < 0\ \left(\because y(0) = y_0\right) \quad (69)$$

(c) Complete solution

$$y(t) = y_0 e^{-2t} + \frac{k}{\sqrt{4+\omega_0^2}}\left[\cos(\omega_0 t - \theta) - \cos\theta e^{-2t}\right]u(t) \quad (70)$$

Note:

- The above system is linear if the initial condition is zero.

  Let $x_1(t)$ and $x_2(t)$ be two input signals, and let $y_1(t)$ and $y_2(t)$ be the corresponding responses with $y_1(0) = y_2(0) = 0$, i.e.,

$$\frac{d}{dt}y_1(t) + 2y_1(t) = x_1(t),\ y_1(0) = 0 \quad (71)$$

$$\frac{d}{dt}y_2(t) + 2y_2(t) = x_2(t),\ y_2(0) = 0 \quad (72)$$

  Consider next the input $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, where α and β are any complex numbers.

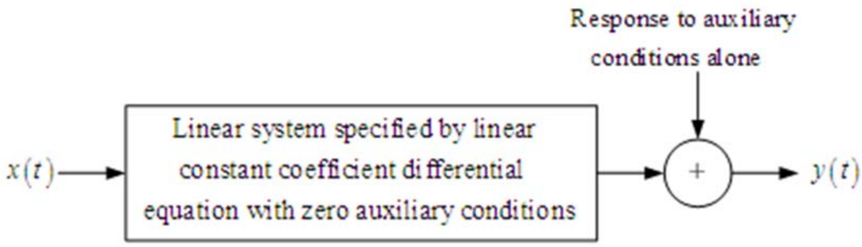
$$\frac{d}{dt}y_3(t) + 2y_3(t) = x_3(t),\ y_3(t) = \alpha y_1(t) + \beta y_2(t),\ y_3(0) = 0 \quad (73)$$

where $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ is the response corresponding to $x_3(t) = \alpha x_1(t) + \beta x_2(t)$.

$\Rightarrow$ The system is linear.

- The above system is incrementally linear if the initial condition is not zero.

$$y(t) = \underbrace{y_0 e^{-2t}}_{\text{due to the nonzero auxiliary condition alone}} + \underbrace{\frac{k}{\sqrt{4+\omega_0^2}}\left[\cos(\omega_0 t - \theta) - \cos\theta e^{-2t}\right]u(t)}_{\text{the linear response of the system assuming that the auxiliary condition is zero}} \quad (74)$$



■ **Figure 19**. Incrementally linear structure of a system specified by a linear constant-coefficient equation.

- A general *N*th-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad (75)$$

  - The solution $y(t) = y^{(p)}(t) + y^{(h)}(t)$

$$\begin{cases} y^{(p)}(t): \text{ particular solution} \\ y^{(h)}(t): \text{ homogeneous solution} \end{cases}$$

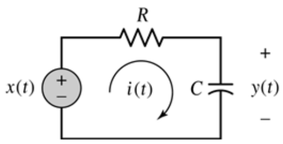
  - Initial conditions correspond to the values of

$$y(t)\Big|_{t=0^-},\ \frac{dy(t)}{dt}\Big|_{t=0^-}, \dots, \frac{d^{N-1}y(t)}{dt^{N-1}}\Big|_{t=0^-} \quad (76)$$

"The system will be linear only if all of these initial conditions are zero."

  - A necessary and sufficient condition for the initial conditions at $t = t_0^+$ (e.g., $t_0^+ = 0^+$) to equal the initial conditions at $t = t_0^-$ for a given input is that the right-hand side of the differential equation in (75), $\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$, contain no impulses or derivatives of impulses.

***Example* 27**: RC circuit



$$y(t)+RC\frac{d}{dt}y(t)=x(t)=\cos(t)u(t)$$

$R=1\Omega,\ C=1F,$ and $y(0^-)=2V$.

The homogeneous solution is

$$y(t)+RC\frac{d}{dt}y(t)=0$$

The order $N=1$. $y^{(h)}(t)=ce^{r_1 t}$

where r_1 is the root of the characteristic equation

$$1+RCr_1=0\Rightarrow r_1=-\frac{1}{RC}$$

$$\therefore y^{(h)}(t)=ce^{-t/RC}=ce^{-t}\ \ (RC=1)$$

Assume $y^{(p)}(t)=c_1\cos(t)+c_2\sin(t)$

$$c_1\cos(t)+c_2\sin(t)-RCc_1\sin(t)+RCc_2\cos(t)=\cos(t)$$

$$\begin{cases}c_1+RCc_2=1\\-RCc_1+c_2=0\end{cases}\Rightarrow\begin{cases}c_1=\dfrac{1}{1+(RC)^2}\\c_2=\dfrac{RC}{1+(RC)^2}\end{cases}$$

$$y^{(p)}(t)=\frac{1}{1+(RC)^2}\cos(t)+\frac{RC}{1+(RC)^2}\sin(t)=\frac{1}{2}\cos(t)+\frac{1}{2}\sin(t),\ t>0$$

$\because$ No impulses are introduced. $\therefore y(0^+)=y(0^-)=2$

We have

$$2=y^{(h)}(t)+y^{(p)}(t)=ce^{-0^+}+\frac{1}{2}\cos(0^+)+\frac{1}{2}\sin(0^+)=c+\frac{1}{2}\Rightarrow c=\frac{3}{2}$$

$$y(t)=\frac{3}{2}e^{-t}+\frac{1}{2}\cos(t)+\frac{1}{2}\sin(t),\ t>0$$ ■

3. Linear constant-coefficient difference equations

(1) The *N*th-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{M}b_k x[n-k] \tag{77}$$

The solution $y[n]$ can be written as

$$y[n] = y^{(p)}[n] + y^{(h)}[n] \tag{78}$$

$$\begin{cases} y^{(p)}[n]: \text{ particular solution} \\ y^{(h)}[n]: \text{ homogeneous solution} \rightarrow \sum_{k=0}^{N} a_k y[n-k] = 0 \end{cases}$$

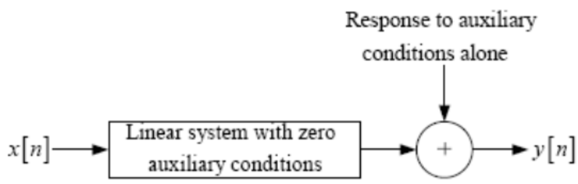
Note:

- $y^{(h)}[n]$ is the solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k y^{(h)}[n-k] = 0 \tag{79}$$

The homogeneous solution for a discrete-time system can be found in a way given in the Appendix.

(2) A system described by the Nth-order linear constant-coefficient difference equation and some initial conditions is incrementally linear.



■ **Figure 20**. Incrementally linear structure of a system specified by a linear constant-coefficient difference equation.

(3) $\because y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$

$\therefore$ A set of initial conditions such as

$$\underbrace{y[-N], y[-N+1], \ldots, y[-1]}_{N} \tag{80}$$

are needed.

(4) The order $N > 0$,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\} \tag{81}$$

$\Rightarrow$ recursive equation

"We need initial conditions to determine $y[n]$."

$N = 0$, $y[n] = \sum_{k=0}^{M} (b_k / a_0) x[n-k]$: nonrecursive equation

"We do not need initial conditions to determine $y[n]$."

$$\Rightarrow h[n] = \begin{cases} b_n/a_0, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \quad (\text{let } x[n] = \delta[n], \text{ then } y[n] = h[n]) \tag{82}$$

***Example* 28**: Example of recursive difference equations

$$y[n] - \frac{1}{2} y[n-1] = x[n],\ y[-1] = a,\ x[n] = k\delta[n]$$

(i) Determine $y[n]$ for $n \ge 0$

$$y[n] = x[n] + \frac{1}{2} y[n-1]$$

$$y[0] = x[0] + \frac{1}{2} y[-1] = k + \frac{1}{2} a$$

$$y[1] = x[1] + \frac{1}{2} y[0] = \frac{1}{2}\left(k + \frac{1}{2} a\right)$$

$$y[2] = x[2] + \frac{1}{2} y[1] = \left(\frac{1}{2}\right)^2 \left(k + \frac{1}{2} a\right)$$

$$\vdots$$

$$y[n] = x[n] + \frac{1}{2} y[n-1]$$

$$= \left(\frac{1}{2}\right)^n \left(k + \frac{1}{2} a\right) = k\left(\frac{1}{2}\right)^n + a\left(\frac{1}{2}\right)^{n+1},\ n \ge 0$$

(ii) Determine $y[n]$ for $n < 0$

$$y[n-1] = 2\{y[n] - x[n]\}$$

$$y[-2] = 2\{y[-1] - x[-1]\} = 2a$$

$$y[-3] = 2\{y[-2] - x[-2]\} = 2^2 a$$

$$y[-4] = 2\{y[-3] - x[-3]\} = 2^3 a$$

$$\vdots$$

$$y[n] = 2\{y[n+1] - x[n+1]\} = 2^{-(n+1)} a = \left(\frac{1}{2}\right)^{n+1} a,\ n < 0$$

Thus, for all values of n,

$$y[n] = \underbrace{\left(\frac{1}{2}\right)^{n+1} a}_{y^{(h)}[n]} + \underbrace{k\left(\frac{1}{2}\right)^n u[n]}_{y^{(p)}[n]}$$

$a = 0 \Rightarrow$ The system is linear.

Note:

- Initial conditions are zero. ⇒ The system is linear.
- The recursive difference equation has an impulse response of infinite duration. ⇒ "infinite impulse response" (IIR) system
- The nonrecursive difference equation has an impulse response of finite duration. ⇒ "finite impulse response" (FIR) system ■

***Example* 29**: First-order recursive system

$$y[n]-\rho y[n-1]=x[n]=\left(\frac{1}{2}\right)^n u[n],\ \rho=\frac{1}{4},\ y[-1]=8$$

The homogeneous equation is

$$y[n]-\frac{1}{4}y[n-1]=0 \Rightarrow N=1,\ y^{(h)}[n]=cr_1^n \Rightarrow r_1-\frac{1}{4}=0 \Rightarrow r_1=\frac{1}{4}$$

Assume $y^{(p)}[n]=c_p\left(\frac{1}{2}\right)^n$,

$$c_p\left(\frac{1}{2}\right)^n-\frac{c_p}{4}\left(\frac{1}{2}\right)^{n-1}=\left(\frac{1}{2}\right)^n \Rightarrow c_p\left(1-\frac{2}{4}\right)=1 \Rightarrow c_p=2$$

$$\Rightarrow y^{(p)}[n]=2\left(\frac{1}{2}\right)^n$$

$$y[n]=2\left(\frac{1}{2}\right)^n+c\left(\frac{1}{4}\right)^n,\ n\geq 0$$

$$y[0]=x[0]+\frac{1}{4}y[-1]=3$$

$$3=2\left(\frac{1}{2}\right)^0+c\left(\frac{1}{4}\right)^0 \Rightarrow c=1 \Rightarrow y[n]=2\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n,\ n\geq 0$$

Note: If $\rho=\frac{1}{2}$, then no coefficient c_p satisfies $c_p(1-2\rho)=1$ and we assume a particular solution of the form $y^{(p)}[n]=c_p n(1/2)^n$.

$$c_p n(1-2\rho)+2\rho c_p=1 \Rightarrow c_p n\cdot 0+c_p=1 \Rightarrow c_p=1 \Rightarrow y^{(p)}[n]=n(1/2)^n$$ ■

4. General form of the particular solutions corresponding to some $x(t)$ and $x[n]$

$x(t)$	Particular solution
1	c
t^n	$c_1 t^n + c_2 t^{n-1} + \cdots + c_n t + c_{n+1}$
e^{at}	▪ ce^{at} if a is not a characteristic root. ▪ $c_1 te^{at} + c_2 e^{at}$ if a is a distinct characteristic root. ▪ $c_1 t^{k-1} e^{at} + c_2 t^{k-2} e^{at} + \cdots + c_k e^{at}$ if a is a (k–1)-multiple characteristic root.
$\cos(at)$	$c_1 \cos(at) + c_2 \sin(at)$
$\sin(at)$	$c_1 \cos(at) + c_2 \sin(at)$

$x[n]$	Particular solution
1	c
n^k	$c_1 n^k + c_2 n^{k-1} + \cdots + c_k n + c_{k+1}$
α^n	▪ $c\alpha^n$ if α is not a characteristic root. ▪ $c_1 n\alpha^n + c_2 \alpha^n$ if α is a distinct characteristic root. ▪ $c_1 n^{k-1}\alpha^n + c_2 n^{k-2}\alpha^n + \cdots + c_k \alpha^n$ if α is a (k–1)-multiple characteristic root.
$\cos(\Omega n + \phi)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

***Example* 30**: $\frac{d^2 y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 6y(t) = 6x(t)$

$$x(t) = \sin(2t),\ y(0) = 0,\ \left.\frac{d}{dt} y(t)\right|_{t=0} = y'(0) = 0$$

Characteristic equation:

$$r^2 + 7r + 6 = 0$$

$$(r+1)(r+6) = 0 \Rightarrow r_1 = -1,\ r_2 = -6$$

$$\Rightarrow y^{(h)}(t) = c_1 e^{-t} + c_2 e^{-6t}$$

$\because x(t) = \sin 2t$

$\therefore y^{(p)}(t) = p_1 \sin 2t + p_2 \cos 2t$

Substituting $y^{(p)}(t)$ into the differential equation, we obtain

$$-4p_1 \sin 2t - 4p_2 \cos 2t + 14p_1 \cos 2t - 14p_2 \sin 2t + 6p_1 \sin 2t + 6p_2 \cos 2t$$
$$= 6\sin 2t$$
$$(-4p_1 - 14p_2 + 6p_1 - 6)\sin 2t + (-4p_2 + 14p_1 + 6p_2)\cos 2t$$
$$= (2p_1 - 14p_2 - 6)\sin 2t + (14p_1 + 2p_2)\cos 2t = 0$$
$$\Rightarrow \begin{cases} 2p_1 - 14p_2 - 6 = 0 \\ 14p_1 + 2p_2 = 0 \end{cases} \Rightarrow \begin{cases} p_1 = 3/50 \\ p_2 = -21/50 \end{cases}$$
$$y(t) = y^{(h)}(t) + y^{(p)}(t) = c_1 e^{-t} + c_2 e^{-6t} + 3/50 \sin 2t - 21/50 \cos 2t$$
$\because y(0) = 0$ and $y'(0) = 0$
$$\therefore \begin{cases} c_1 + c_2 - 21/50 = 0 \\ -c_1 - 6c_2 + 6/50 = 0 \end{cases} \Rightarrow c_1 = 12/25,\ c_2 = -3/50$$
$$\Rightarrow y(t) = 12/25 e^{-t} - 3/50 e^{-6t} + 3/50 \sin 2t - 21/50 \cos 2t$$ ■

***Example* 31**: $y[n] + 2y[n-1] = x[n] - x[n-1],\ x[n] = n^2,\ y[0] = 1$

$r + 2 = 0 \Rightarrow r = -2$

$\therefore y^{(h)}[n] = c(-2)^n$

$\because x[n] = n^2 \quad \therefore x[n] - x[n-1] = 2n - 1$

$\Rightarrow y^{(p)}[n] = p_1 n + p_2$

Substituting $y^{(p)}[n]$ into the difference equation, we obtain

$$p_1 n + p_2 + 2[p_1(n-1) + p_2] = 2n - 1$$
$$\Rightarrow 3p_1 n + 3p_2 - 2p_1 = 2n - 1$$
$$\Rightarrow (3p_1 - 2)n + (3p_2 - 2p_1 + 1) = 0$$
$$\Rightarrow \begin{cases} 3p_1 - 2 = 0 \\ 3p_2 - 2p_1 + 1 = 0 \end{cases} \Rightarrow \begin{cases} p_1 = 2/3 \\ p_2 = 1/9 \end{cases}$$
$$\Rightarrow y[n] = c(-2)^n + 2n/3 + 1/9$$
$\because y[0] = 1 \therefore c = 8/9$
$$\Rightarrow y[n] = 8/9(-2)^n + 2n/3 + 1/9$$ ■

2-7 Characteristics of Systems Described by Differential or Difference Equations

1. It's informative to express the output of a system described by a differential or difference equation as the sum of two components:
   (1) One associated only with the initial conditions.
   $\Rightarrow$ natural response (zero-input response), $y^{(n)}$
   (2) One associated only with the input signal. $\Rightarrow$ forced response (zero-state response), $y^{(f)}$
2. The *natural response* is the system output for zero input and thus describes the manner in which the system dissipates any stored energy or memory of the past represented by non-zero initial conditions.
   zero input
   $\Rightarrow$ $y^{(h)}(t)$ or $y^{(h)}[n]$
   $\Rightarrow$ Choose the coefficient c_i such that the initial conditions are satisfied.

***Example* 32**: RC circuit (same as Example 27)

$$y(t)+RC\frac{d}{dt}y(t)=x(t),\ \ R=1\Omega,\ C=1F,\ \text{and}\ y(0^-)=2V$$

$$y^{(h)}(t)=ce^{-t}\Rightarrow y^{(n)}(0)=2\Rightarrow c=2\Rightarrow y^{(n)}(t)=2e^{-t},\ t\ge 0$$ ■

***Example* 33**: First-order recursive system (same as Example 29)

$$y[n]-\rho y[n-1]=x[n]=\left(\frac{1}{2}\right)^n u[n],\ \rho=\frac{1}{4},\ y[-1]=8$$

$$y^{(h)}[n]=c\left(\frac{1}{4}\right)^n\Rightarrow 8=c\left(\frac{1}{4}\right)^{-1}\Rightarrow c=2\Rightarrow y^{(n)}[n]=2\left(\frac{1}{4}\right)^n,\ n\ge -1$$ ■

3. The *forced response* is the system output due to the input signal assuming zero initial conditions. Thus, the forced response is of the same form as the complete solution of the differential or difference equation.
   zero initial conditions
   $\Rightarrow$ "at rest", no stored energy or memory in the system
   $\Rightarrow$ System behavior is "forced" by the input.

The forced response depends on the $y^{(p)}$, which is valid only for times $t > 0$ or $n \ge 0$.

Note: As before, we shall consider finding the forced response only for continuous-time systems and inputs that do not result in impulses on the right-hand side of the differential equation, i.e., $y(0^-)=y(0^+)$.

***Example* 34**: RC circuit (same as Example 27)

$$y(t)+RC\frac{d}{dt}y(t)=x(t)=\cos(t)u(t),\ \ R=1\Omega,\ C=1F$$

$$y(t)=ce^{-t}+\frac{1}{2}\cos(t)+\frac{1}{2}\sin(t),\ t>0$$

Assume that the system is initially at rest, $y(0)=0\Rightarrow c=-1/2$,

$$y^{(f)}(t)=-\frac{1}{2}e^{-t}+\frac{1}{2}\cos(t)+\frac{1}{2}\sin(t),\ t>0$$ ■

***Example* 35**: First-order recursive system (same as Example 29)

$$y[n]-\rho y[n-1]=x[n]=\left(\frac{1}{2}\right)^n u[n],\ \rho=\frac{1}{4}$$

$$y[0]=x[0]+\frac{1}{4}y[-1]=1+0=1$$

$$y[n]=2\left(\frac{1}{2}\right)^n+c\left(\frac{1}{4}\right)^n\Rightarrow c=-1$$

$$\therefore y^{(f)}[n]=2\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n$$ ■

4. The basic difference between impulse-response descriptions and differential- or difference-equation system descriptions:
   (1) Impulse response $\Rightarrow$ no provision for initial conditions
   It applies only to systems that are initially at rest or when the input is know for all time.
   (2) Differential- and difference-equation system descriptions are more flexible in this respect, since they apply to systems either at rest or with nonzero initial conditions.
5. Linearity and time-invariant
   (1) The forced response of an LTI system described by a differential or difference equation is linear with respect to the input.

$$\left.\begin{array}{l}x_1(t)\to y_1^{(f)}(t)\\ x_2(t)\to y_2^{(f)}(t)\end{array}\right\}\Rightarrow \alpha x_1(t)+\beta x_2(t)\to \alpha y_1^{(f)}(t)+\beta y_2^{(f)}(t) \qquad (83)$$

   (2) The natural response is linear with respect to the initial conditions:

$$\left.\begin{array}{l}\text{initial conditions } I_1 \to y_1^{(n)}(t) \\ \text{initial conditions } I_2 \to y_2^{(n)}(t)\end{array}\right\} \Rightarrow \alpha I_1 + \beta I_2(t) \to \alpha y_1^{(n)}(t) + \beta y_2^{(n)}(t) \tag{84}$$

(3) Time invariant

(a) The forced response is also time-invariant since the system is initially at rest.

(b) The complete response of an LTI system described by a differential or difference equation is not time-invariant, since the initial conditions will result in an output term that does not shift with a time shift of the input.

(4) The forced response is also *causal* since the system is initially at rest, i.e., the output does not begin prior to the time at which the input is applied to the system.

6. Roots of the characteristic equation

The roots of the characteristic equation afford considerable information about the LTI system behavior.

(1) The forced response depends on both the input and the roots of the characteristic equation, since it involves both the homogeneous and particular solution.

(2) The basic form of the natural response is dependent entirely on the roots of the characteristic equation.

(3) The impulse response of an LTI system also depends on the roots of the characteristic equation, since it contains the same term as the natural response.

(4) Stability

For a BIBO stable LTI system, the output must be bounded for any set of initial condition.

$\Rightarrow$ The natural response of the system must be bounded.

$\Rightarrow$ Each term in the natural response must be bounded.

(a) In discrete-time LTI systems,

$$\left|r_i^n\right| \text{ is bounded or } \left|r_i\right| < 1 \text{ for all } i \tag{85}$$

(b) In continuous-time LTI systems,

$$\left|e^{r_i t}\right| \text{ is bounded or } \operatorname{Re}\{r_i\} < 0 \tag{86}$$

$\operatorname{Re}\{r_i\} = 0$ means that the system is on the verge of instability.

In a stable LTI system with zero input, the stored energy eventually dissipates and the output approaches zero.

7. Response time

(1) The natural response has decayed to zero.

≡ The system behavior is governed only by the particular solution.

≡ The transition of the system from its initial condition to an equilibrium condition determined by the input.

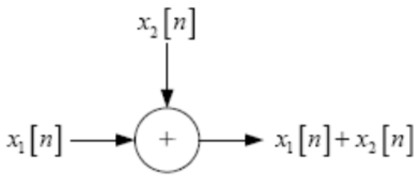
(2) The response time of an LTI system to a transient is therefore proportional to

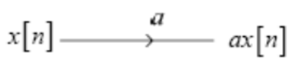
$$\begin{cases} \max|r_i| & \text{for the discrete-time case} \\ \max\left(\mathrm{Re}\{r_i\}\right) & \text{for the continuous-time case} \end{cases} \tag{87}$$

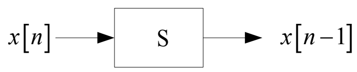
2-8 Block-Diagram Representations of LTI Systems Described by Differential or Difference Equations

1. Difference equation:

Basic elements:

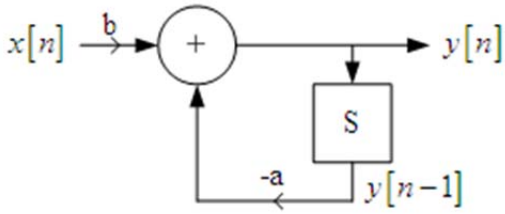
(1) Adder 

(2) Multiplication by a coefficient 

(3) Unit delay 

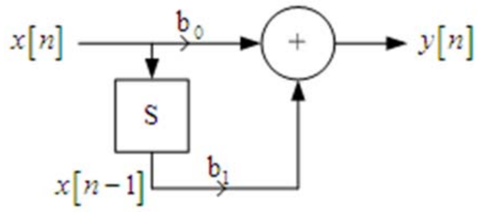
***Example* 36**: $y[n]+ay[n-1]=bx[n]$ (initial rest)

$$\Rightarrow y[n]=-ay[n-1]+bx[n]$$



■

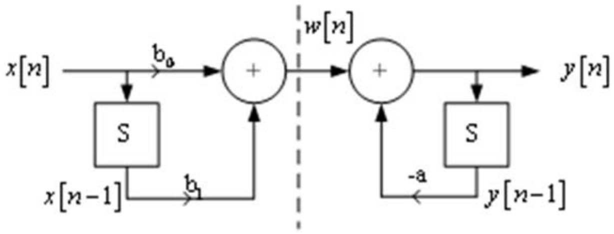
***Example* 37**: $y[n] = b_0 x[n] + b_1 x[n-1]$



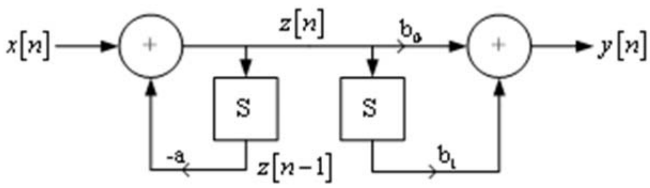
■

***Example* 38**: $y[n] + ay[n-1] = b_0 x[n] + b_1 x[n-1]$ (initial rest)

$$\Rightarrow y[n] = -ay[n-1] + \underbrace{b_0 x[n] + b_1 x[n-1]}_{w[n]=b_0 x[n]+b_1 x[n-1]} = -ay[n-1] + w[n]$$

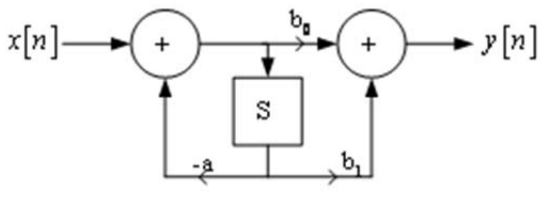


⇓ Interchange the order of cascade interconnection.



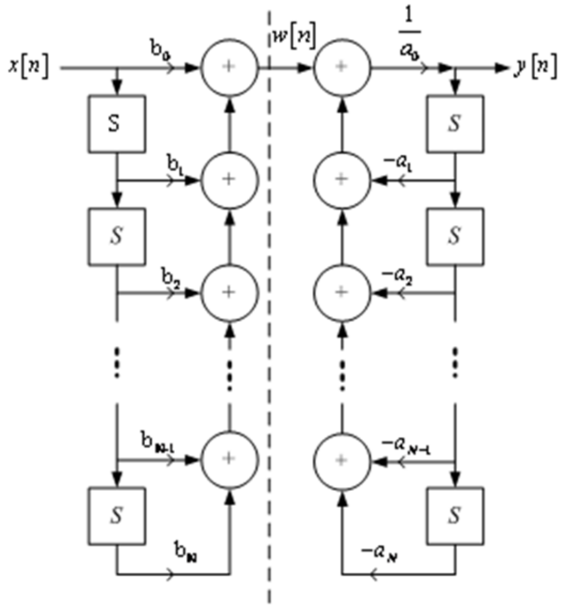
$$\begin{cases} z[n] = -az[n-1] + x[n] \\ y[n] = b_0 z[n] + b_1 z[n-1] \end{cases}$$

⇓

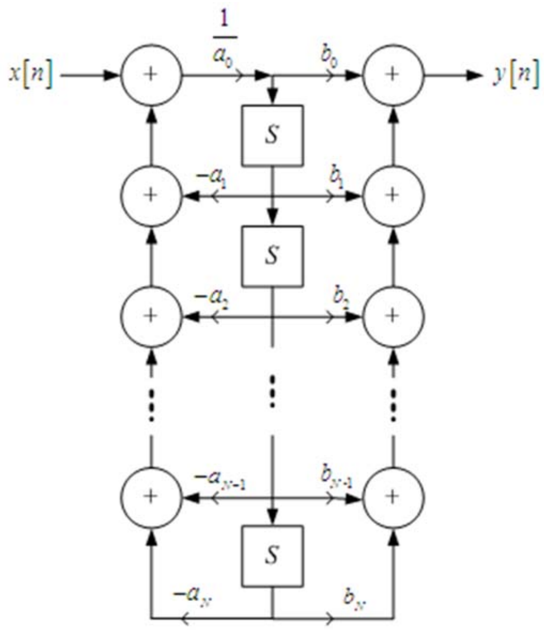


Requiring a single delay ■

***Example* 39**: $y[n]=\frac{1}{a_0}\left\{\underbrace{\sum_{k=0}^{N} b_k x[n-k]}_{w[n]}-\sum_{k=1}^{N} a_k y[n-k]\right\}$ (initial rest)



"Direct form I realization" (with $2N$ delay elements)



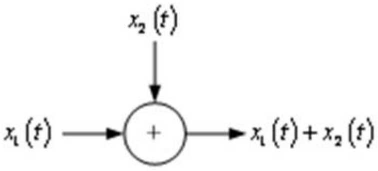
"Direct form II realization" or "canonic realization" (N delay elements)

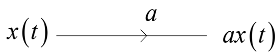
Note:

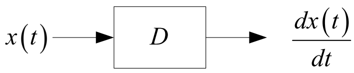
- The direct-form II realization requires much fewer delay elements than the direct-form I realization.
- In fact, the direct-form II realization requires the minimum number of delay elements.

2. Differential equation:

Basic elements:

(1) Adder 

(2) Multiplication by a coefficient 

(3) Differentiator 

$$y(t)=\frac{1}{a_0}\left\{\sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}-\sum_{k=1}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} \tag{88}$$

The direct-form I and direct-form II realizations of the differential equation are the same as those of the difference equation except that the delay elements used in the realizations are replaced by differentiators.

A differentiation element is often difficult to realize. Hence, we need some other realization method. Realization of the differential equation using integrators:

Let

$$\begin{aligned}
&y^{(0)}(t)=y(t)\\
&y^{(1)}(t)=y(t)*u(t)=\int_{-\infty}^{t} y(\tau)\,d\tau\\
&y^{(2)}(t)=y(t)*u(t)*u(t)=y^{(1)}(t)*u(t)=\int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} y(\sigma)\,d\sigma\right]d\tau\\
&\qquad\vdots\\
&y^{(k)}(t)=y^{(k-1)}(t)*u(t)=\int_{-\infty}^{t} y^{(k-1)}(\tau)\,d\tau
\end{aligned} \tag{89}$$

$$\begin{aligned}
&x^{(0)}(t)=x(t)\\
&x^{(1)}(t)=x(t)*u(t)=\int_{-\infty}^{t} x(\tau)\,d\tau\\
&x^{(2)}(t)=x(t)*u(t)*u(t)=x^{(1)}(t)*u(t)=\int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} x(\sigma)\,d\sigma\right]d\tau\\
&\qquad\vdots\\
&x^{(k)}(t)=x^{(k-1)}(t)*u(t)=\int_{-\infty}^{t} x^{(k-1)}(\tau)\,d\tau
\end{aligned} \tag{90}$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}=\sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k} \quad (M=N \text{ here}) \tag{91}$$

Assume initial rest, then the *N*th integral of $\frac{d^k y(t)}{dt^k}$ is precisely $y^{(N-k)}(t)$.

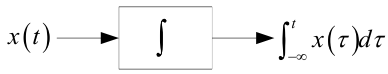
($\because$ The initial conditions for the integration are zero.)

The *N*th integral of $\frac{d^k x(t)}{dt^k}$ is precisely $x^{(N-k)}(t)$.

$$\Rightarrow \sum_{k=0}^{N} a_k y^{(N-k)}(t) = \sum_{k=0}^{N} b_k x^{(N-k)}(t) \tag{92}$$

$$\because y^{(0)}(t) = y(t)$$
$$\therefore y(t) = \frac{1}{a_N}\left\{\sum_{k=0}^{N} b_k x^{(N-k)}(t) - \sum_{k=0}^{N-1} a_k y^{(N-k)}(t)\right\} \tag{93}$$

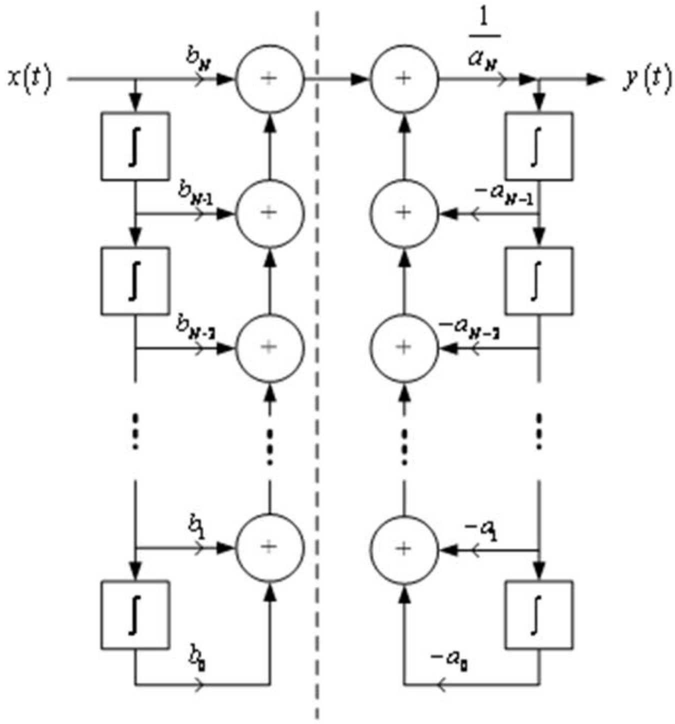
Let an integrator be expressed as



■ **Figure 21**. Pictorial representation of an integrator.

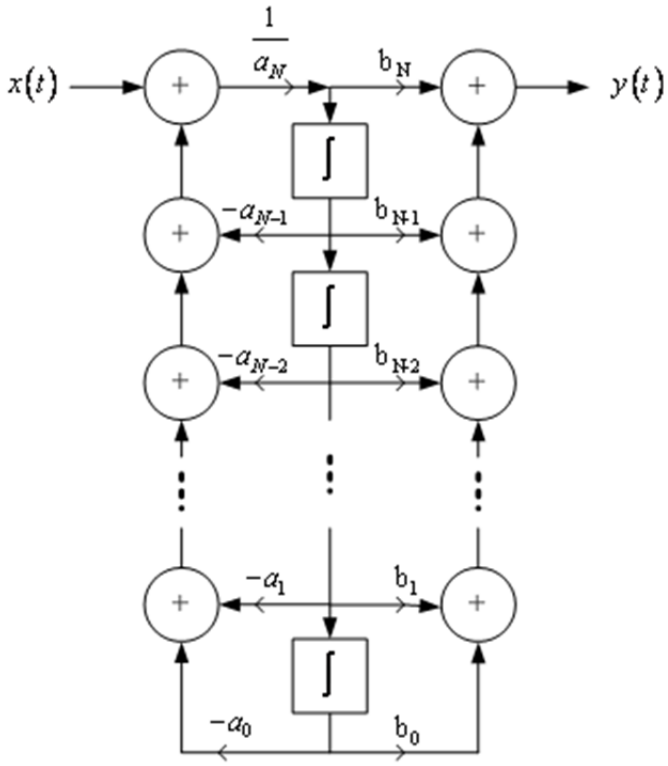
Then the corresponding direct-form I and direct-form II realizations are illustrated as follows:

Direct-form I realization:



■ **Figure 22**. Direct form I realization for the LTI system described by Eq. (93).

Direct-form II realization:



■ **Figure 23**. Direct form II realization for the LTI system described by Eq. (93).

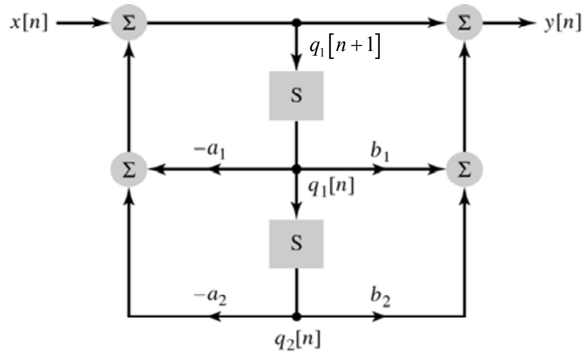
2-9 State-Variable Descriptions of LTI Systems

1. The *state* of a system may be defined as a minimal set of signals that represent the system's entire memory of the past. That is, given only the value of the state at an initial point in time, n_i(or t_i), and the input for times $n \geq n_i$ (or $t \geq t_i$), we can determine the output for all times $n \geq n_i$ (or $t \geq t_i$).
2. A general state-variable description with the direct form II implementation of a second-order LTI system is first considered. Using

   (1) the input for $n \geq n_i$ and

   (2) outputs of the time-shift operation labeled $q_1[n]$ and $q_2[n]$ at $n = n_i$

   to determine the output of the system for $n \geq n_i$.



■ **Figure 24**. Direct form II representation of a second-order discrete-time LTI system depicting state variables $q_1[n]$ and $q_2[n]$.

From Fig. 24, the next value of the state, $q_1[n+1]$ and $q_2[n+1]$, are obtained from the current state and the input via the two equations

$$\begin{cases} q_1[n+1] = -a_1 q_1[n] - a_2 q_2[n] + x[n] \\ q_2[n+1] = q_1[n] \end{cases} \tag{94}$$

$$\therefore \begin{aligned} y[n] &= x[n] - a_1 q_1[n] - a_2 q_2[n] + b_1 q_1[n] + b_2 q_2[n] \\ &= (b_1 - a_1) q_1[n] + (b_2 - a_2) q_2[n] + x[n] \end{aligned} \tag{95}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n] \tag{96}$$

$$\Rightarrow \mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \tag{97}$$

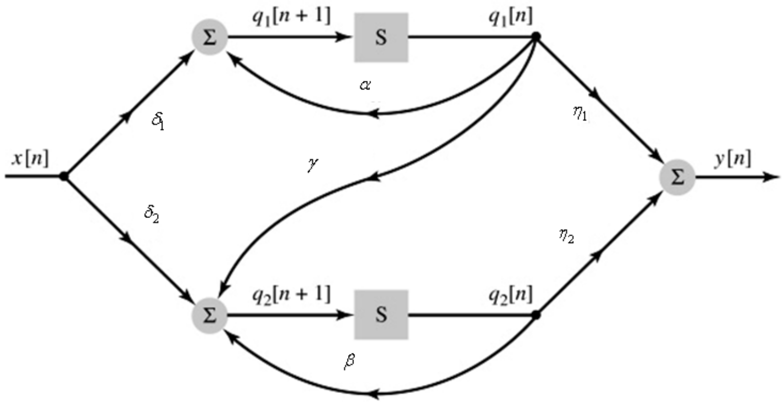
$$y[n] = \begin{bmatrix} b_1 - a_1 & b_2 - a_2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + x[n] = \mathbf{c}\mathbf{q}[n] + Dx[n] \tag{98}$$

where $\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} b_1 - a_1 & b_2 - a_2 \end{bmatrix}$, and $D = 1$.

Equations (97) and (98) are the general form of a state-variable description corresponding to a discrete-time system.

3. If the input-output characteristics of the system are described by an Nth-order difference equation, then $\mathbf{q}[n]$ is $N \times 1$, $\mathbf{b}$ is $N \times 1$, $\mathbf{A}$ is $N \times N$, and $\mathbf{c}$ is $1 \times N$.

***Example* 40**: State-variable description of a second-order system



$$\begin{cases} q_1[n+1] = \alpha q_1[n] + \delta_1 x[n] \\ q_2[n+1] = \gamma q_1[n] + \beta q_2[n] + \delta_2 x[n] \end{cases}$$

$$y[n] = \eta_1 q_1[n] + \eta_2 q_2[n]$$

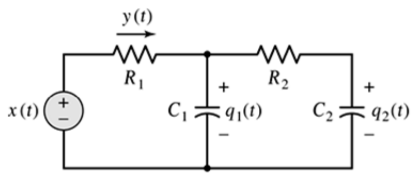
$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix},\ \mathbf{c} = [\eta_1 \quad \eta_2],\ \text{and } D = 0$$ ■

4. The state-variable description of continuous-time system is analogous to that of discrete-time systems, with the exception that the state equation given by (97) and (98) is expressed in terms of a derivative

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t) \tag{99}$$

$$\text{and } \ y(t) = \mathbf{c}\mathbf{q}(t) + Dx(t) \tag{100}$$

***Example* 41**: State-variable description of an electrical circuit



$$x(t) = y(t)R_1 + q_1(t) \Rightarrow y(t) = -\frac{1}{R_1}q_1(t) + \frac{1}{R_1}x(t)$$

Let $i_2(t)$ be the current through R_2,

$$q_1(t) = R_2 i_2(t) + q_2(t) \Rightarrow i_2(t) = \frac{1}{R_2}q_1(t) - \frac{1}{R_2}q_2(t)$$

$$\because i_2(t) = C_2 \frac{d}{dt} q_2(t)$$

$$\therefore \frac{d}{dt} q_2(t) = \frac{1}{R_2 C_2} q_1(t) - \frac{1}{R_2 C_2} q_2(t)$$

We need a state equation for $q_1(t)$. Let $i_1(t)$ be the current through C_1, we have $y(t) = i_1(t) + i_2(t)$ and $i_1(t) = C_1 \frac{d}{dt} q_1(t)$.

$$-\frac{1}{R_1} q_1(t) + \frac{1}{R_1} x(t) = C_1 \frac{d}{dt} q_1(t) + C_2 \frac{d}{dt} q_2(t)$$

$$\Rightarrow \frac{d}{dt} q_1(t) = -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} \right) q_1(t) + \frac{1}{R_2 C_1} q_2(t) + \frac{1}{R_1 C_1} x(t)$$

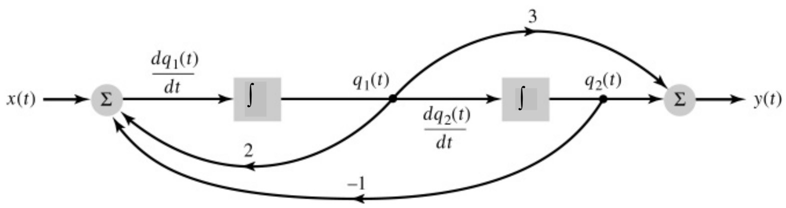
$$\therefore \mathbf{A} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} \right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \frac{1}{R_1 C_1} \\ 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -\frac{1}{R_1} & 0 \end{bmatrix},$$

and $D = \frac{1}{R_1}$

■

In a block diagram representation of a continuous-time system, the state-variable correspond to the outputs of the integrators.

***Example* 42**: State-variable description from a block diagram



$$\frac{d}{dt} q_1(t) = 2q_1(t) - q_2(t) + x(t)$$

$$\frac{d}{dt} q_2(t) = q_1(t)$$

$$y(t) = 3q_1(t) + q_2(t)$$

$$\therefore \mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 3 & 1 \end{bmatrix}, \text{ and } D = 0$$

■

5. Transformations of the state
   (1) There is no unique state-variable description of a system with a given input-output characteristic. Different state-variable descriptions may be obtained by transforming the state-variables.

***Example* 43**: Consider Example 42 again

Let $q_1'(t) = q_2(t)$ and $q_2'(t) = q_1(t)$,

$$\mathbf{A}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},\ \mathbf{b}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathbf{c}' = \begin{bmatrix} 1 & 3 \end{bmatrix},\ \text{and } D' = 0. \qquad \blacksquare$$

We may define $\mathbf{T}$ as the state-transformation matrix, then the new state vector

$$\mathbf{q}' = \mathbf{T}\mathbf{q} \tag{101}$$

where $\mathbf{T}$ must be a nonsingular matrix, i.e., $\mathbf{T}^{-1}$ exists and $\mathbf{q} = \mathbf{T}^{-1}\mathbf{q}'$.

The original state-variable description is given by

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{A}\mathbf{q} + \mathbf{b}x \\ y &= \mathbf{c}\mathbf{q} + Dx \end{aligned} \tag{102}$$

where $\dot{\mathbf{q}}$ denotes differentiation in continuous time or time advance ([n+1]) in discrete time.

$$\begin{aligned} \dot{\mathbf{q}}' &= \mathbf{T}\dot{\mathbf{q}} = \mathbf{T}\mathbf{A}\mathbf{q} + \mathbf{T}\mathbf{b}x = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{q}' + \mathbf{T}\mathbf{b}x = \mathbf{A}'\mathbf{q}' + \mathbf{b}'x \\ y &= \mathbf{c}\mathbf{q} + Dx = \mathbf{c}\mathbf{T}^{-1}\mathbf{q}' + Dx = \mathbf{c}'\mathbf{q}' + D'x \end{aligned} \tag{103}$$

where $\mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$, $\mathbf{b}' = \mathbf{T}\mathbf{b}$, $\mathbf{c}' = \mathbf{c}\mathbf{T}^{-1}$, and $D' = D$.

***Example* 44**: Transforming the state

$\mathbf{A} = \frac{1}{10}\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{c} = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}$, and $D = 2$. Find $\mathbf{A}'$, $\mathbf{b}'$, $\mathbf{c}'$, and D' corresponding to the new states

$$\begin{cases} q_1'[n] = -\frac{1}{2}q_1[n] + \frac{1}{2}q_2[n] \\ q_2'[n] = \frac{1}{2}q_1[n] + \frac{1}{2}q_2[n] \end{cases}$$

$\mathbf{T} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $\det(\mathbf{T}) \neq 0 \Rightarrow$ nonsingular

$\therefore \mathbf{T}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore \mathbf{A}' = \mathbf{TAT}^{-1} = \begin{bmatrix} -1/2 & 0 \\ 0 & 3/10 \end{bmatrix}$, $\mathbf{b}' = \mathbf{Tb} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{c}' = \mathbf{cT}^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}$,

and $D = 2$.

Note: $\mathbf{A}'$ is a diagonal matrix and thus separates the state update into the two decoupled first-order difference equations

$$\begin{cases} q_1[n+1] = -\frac{1}{2}q_1[n] + x[n] \\ q_2[n+1] = \frac{3}{10}q_2[n] + 3x[n] \end{cases}$$

■

(2) Both the block diagram and state-variable descriptions represent the internal structure of an LTI system. Advantages of the state-variable descriptions:

(a) Powerful tool from linear algebra may be used to systematically study and design the internal structure of the system.

(b) Transform the internal structure without changing the input-output characteristics of the system is used to optimize some performance criteria by transformation not directly related to input-output behavior.

Appendix
Characteristic Equations of Differential and Difference Equations

1. Differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \text{(A1)}$$

Characteristic equation:

$$a_N r^N + a_{N-1} r^{N-1} + \cdots + a_1 r + a_0 = 0 \quad \text{(A2)}$$

The roots of the characteristic equation, $r_1, r_2, \ldots, r_N$, are called the characteristic roots of the differential equation.

Note:

- When the characteristic roots are all distinct, the homogeneous solution $y^{(h)}(t)$ will be

$$y^{(h)}(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_N e^{r_N t} \quad \text{(A3)}$$

- Suppose that r_1 is a *k*-multiple root of the characteristic equation. Then, corresponding to r_1, there will be *k* terms in the homogeneous solution:

$$c_1 t^{k-1} e^{r_1 t} + c_2 t^{k-2} e^{r_1 t} + \cdots + c_{k-1} t e^{r_1 t} + c_k e^{r_1 t} \quad \text{(A4)}$$

***Example* A1**: $\frac{d^3 y(t)}{dt^3} + 7\frac{d^2 y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 12 y(t) = x(t)$

Characteristic equation:

$r^3 + 7r^2 + 16r + 12 = 0$

$(r+2)^2 (r+3) = 0 \Rightarrow \therefore y_h(t) = c_1 t e^{-2t} + c_2 e^{-2t} + c_3 e^{-3t}$ ■

2. Difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{(A5)}$$

Characteristic equation:

$$a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-1} r + a_N = 0 \quad \text{(A6)}$$

The roots of the characteristic equation, $r_1, r_2, \ldots, r_N$, are called the characteristic roots of the difference equation.

Note:

- When $r_1, r_2, \ldots, r_N$ are all distinct , the homogeneous solution $y^{(h)}[n]$ will be

$$y^{(h)}[n] = c_1 r_1^n + c_2 r_2^n + \cdots + c_N r_N^n \tag{A7}$$

- When the characteristic equation contains multiple roots, the homogeneous solution of a difference equation will be of slightly different form. Specifically, let r_1 be a k-multiple characteristic root; then its corresponding terms in the homogeneous solution are

$$c_1 n^{k-1} r_1^n + c_2 n^{k-2} r_1^n + \cdots + c_{k-1} n r_1^n + c_k r_1^n \tag{A8}$$

***Example* A2**: $y[n] + 6y[n-1] + 12y[n-1] + 8y[n-3] = x[n]$

Characteristic equation:

$r^3 + 6r^2 + 12r + 8 = 0$

$(r+2)^3 = 0 \Rightarrow r = -2, -2, -2$

$\therefore y^{(h)}[n] = (c_1 n^2 + c_2 n + c_3)(-2)^n$

■

References:

[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, *Signals and Systems*, 2nd Ed., Prentice-Hall, 1997.

[2] S. Haykin and B. Van Veen, *Signals and Systems*, 2nd Ed., Hoboken, NJ: John Wiley & Sons, 2003.