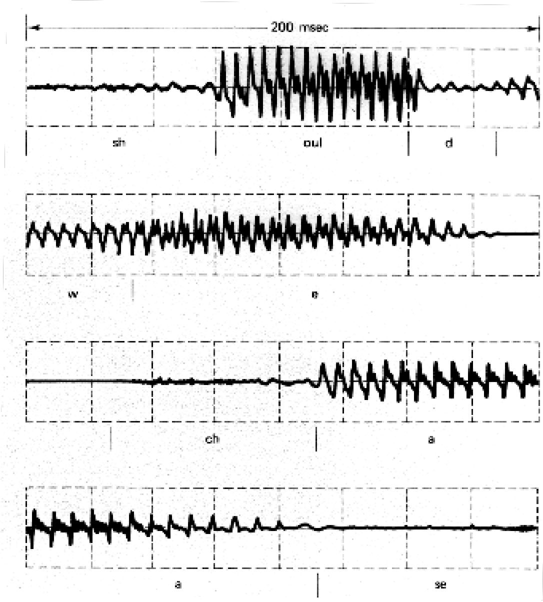
Chapter 1 Introduction

1-1 Signals

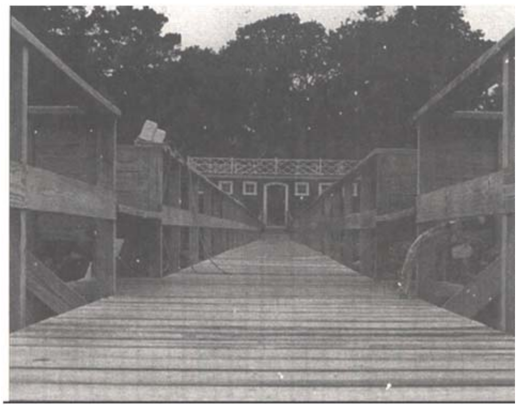
1. Information in a signal is contained in a pattern of variations of some form.

   ***Example* 1**: The human vocal mechanism produces speech by creating fluctuations in acoustic pressure.



■ **Figure 1.** Example of a recording of speech [1].

***Example* 2**: Monochromatic picture: variation in brightness.



■ **Figure 2.** A monochromatic picture [1].

2. Signals are represented mathematically as functions of one or more independent variables that convey information on the nature of a physical phenomenon.

   ***Example* 3**: Speech signal $\rightarrow$ acoustic pressure: a function of time (one-dimensional).

***Example* 4**: Picture → brightness: a function of two spatial variables (two-dimensional).

Note: For convenience, we will generally refer to the independent variable as time, although it may not in fact represent time in specific applications.

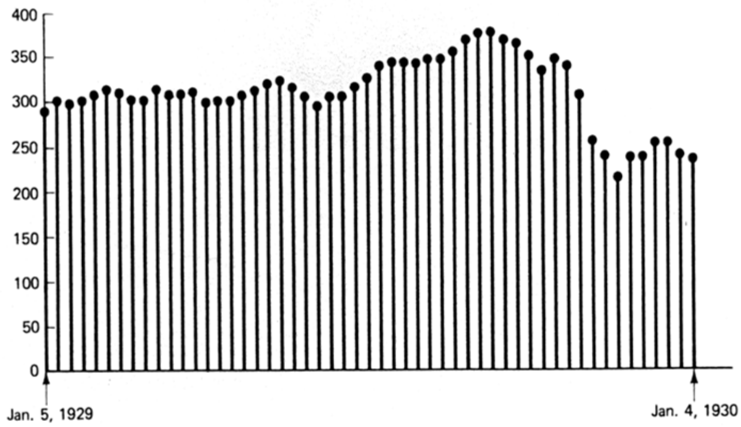
3. Classification of signals

Five methods of classifying signals, based on different features, are common:

(1) Continuous-time and discrete-time signals

(a) A signal $x(t)$ is said to be a continuous-time signal if it is defined for all time t. The amplitude or value varies continuously with time, e.g., speech signal.

(b) A discrete-time signal is defined only at discrete instants of time. Thus, the independent variable has discrete values only, which are usually uniformly spaced, e.g., stock market index $x[n]$.



■ **Figure 3.** An example of a discrete-time signal: the weekly Dow-Jones stock market index from January 5, 1929 to January 4, 1930 [1].

(c) Digital signal: discrete-time and discrete-state signal

(d) Analog signal: continuous-time and continuous-state signal

Note:

- If the signal amplitude is continuous, the signal is called "continuous-state" signal; otherwise, it is called "discrete-state" signal.
- A discrete-time signal is often referred to as a discrete-time sequence.
- continuous-time signal $\xrightarrow{\text{sampling}}$ discrete-time signal

(2) Even and odd signals

(a) Even: $x(t) = x(-t)$ for all t → symmetric about vertical axis

(b) Odd: $x(-t) = -x(t)$ for all $t \rightarrow$ anti-symmetric about vertical axis

***Example* 5**:

$$x(t) = \begin{cases} \sin\left(\frac{\pi t}{T}\right), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

$$x(-t) = \sin\left(-\frac{\pi t}{T}\right) = -\sin\left(\frac{\pi t}{T}\right) = -x(t) \Rightarrow \text{odd signal}$$

■

(c) Any signal can be broken into a sum of an odd signal and an even signal

$$x(t) = x_e(t) + x_o(t) \tag{1}$$

where $x_e(t)$ and $x_o(t)$ mean even and odd signals, respectively.

$\because x_e(t) = x_e(-t)$ and $x_o(t) = -x_o(-t)$

$\therefore x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t)$

$$\Rightarrow x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] \text{ and } x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right] \tag{2}$$

***Example* 6**:

$$x(t) = e^{-2t}\cos(t)$$
$$x(-t) = e^{2t}\cos(-t) = e^{2t}\cos(t)$$

$$\therefore \begin{cases} x_e(t) = \frac{1}{2}\left[e^{-2t}\cos(t) + e^{2t}\cos(t)\right] = \cosh(2t)\cos(t) \\ x_o(t) = -\sinh(2t)\cos(t) \end{cases}$$

■

(d) A complex-valued signal $x(t)$ is said to be conjugate symmetric if

$$x(-t) = x^*(t) \tag{3}$$

$$\begin{aligned} \Rightarrow x(-t) &= a(-t) + jb(-t) \\ &= a(t) - jb(t) = x^*(t) \\ \Rightarrow \underbrace{a(-t) = a(t)}_{even}&, \ \underbrace{b(-t) = -b(t)}_{odd} \end{aligned} \tag{4}$$

(3) Periodic signals and non-periodic signals

(a) Periodic signal: $x(t+T) = x(t)$ for all t

where T is a positive constant. $T = T_0,\ 2T_0,\ 3T_0, \ldots$

- $T=T_0$: fundamental period
- $1/T$: fundamental frequency, $f = 1/T$ Hz or cycles/sec
- $\omega = 2\pi f = 2\pi/T$: angular frequency (radians/sec)

Note: $x(t)$ is a constant

(1) The fundamental period is undefined.

(2) The fundamental frequency is defined to be zero.

(b) $x[n] = x[n+N]$ for integer n

where N is a positive integer.

- fundamental period: smallest N samples
- fundamental angular frequency: $\Omega = 2\pi/N$ (radians/sample)

(4) Deterministic signals and random signals

(a) A deterministic signal is a signal about which there is no uncertainty with respect to its value at any time.

(b) A random signal is a signal about which there is uncertainty before it occurs.

(5) Energy signals and power signals

(a) The instantaneous power dissipated in the resistor R is defined by

$$\begin{aligned} p(t) &= v^2(t)/R = i^2(t)\cdot R \\ &= v^2(t) = i^2(t),\ R = 1 \text{ ohm} \end{aligned} \tag{5}$$

We may express the instantaneous power of the signal as

$$p(t) = x^2(t) \tag{6}$$

the total energy of the non-periodic continuous-time signal $x(t)$ as

$$E = \lim_{T\to\infty}\int_{-T/2}^{T/2} x^2(t)\,dt = \int_{-\infty}^{\infty} x^2(t)\,dt \tag{7}$$

and its time-averaged, or average, power as

$$\begin{aligned} P &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt \\ &\left(= \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt, \text{ for periodic signal}\right) \end{aligned} \tag{8}$$

$\sqrt{P}$ means root mean-squared (rms) value of the periodic signal $x(t)$.

(b) For non-periodic discrete-time signal, the total energy of $x[n]$ is defined by

$$E = \sum_{n=-\infty}^{\infty} x^2[n] \tag{9}$$

and its average power is defined by

$$P = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N+1}^{N} x^2[n]$$
$$(= \frac{1}{N} \sum_{n=0}^{N-1} x^2[n], \text{ for periodic signals}) \tag{10}$$

(c) A signal is referred to as an energy signal if and only if the total energy of the signal satisfies the condition

$$0 < E < \infty \tag{11}$$

A signal is referred to as a power signal if and only if the average power of the signal satisfies the condition

$$0 < P < \infty \tag{12}$$

(d) Energy signal has zero time-average power and power signal has infinite energy. They are mutually exclusive.

Note:

- Periodic signals and random signals are usually viewed as power signals, whereas signals that are both deterministic and non-periodic are usually viewed as energy signals.
- Signals that satisfy neither property are referred to as neither energy signals nor power signals [3].

1-2 Basic Operations on Signals

1. Operations performed on dependent variables

(1) Amplitude scaling

$$y(t) = cx(t) \tag{13}$$

where c is a scaling factor.

$$y[n] = cx[n] \tag{14}$$

(2) Addition, e.g., mixer

$$y(t) = x_1(t) + x_2(t) \tag{15}$$

$$y[n] = x_1[n] + x_2[n] \tag{16}$$

(3) Multiplication, e.g., amplitude modulation (AM) radio signal

$$y(t) = x_1(t)x_2(t) \tag{17}$$

$$y[n]=x_1[n]x_2[n] \tag{18}$$

(4) Differentiation: $\frac{d}{dt}x(t)$, e.g., inductor, $v(t)=L\frac{d}{dt}i(t)$.

(5) Integration: $y(t)=\int_{-\infty}^{t}x(\tau)d\tau$, e.g., capacitor, $v(t)=\frac{1}{C}\int_{-\infty}^{t}i(\tau)d\tau$.

2. Operations performed on the independent variable

(1) Time scaling:

$$y(t)=x(at) \tag{19}$$

$$\begin{cases} a>1 \Rightarrow y(t) \text{ is a compressed version of } x(t). \\ 0<a<1 \Rightarrow y(t) \text{ is an expanded (stretched) version of } x(t). \end{cases}$$

$$y[n]=x[kn],\ k>0 \tag{20}$$

$k > 1 \Rightarrow$ some values of $x[n]$ are lost.

(2) Reflection:

$$y(t)=x(-t) \tag{21}$$

$y(t)$ represents a reflected version of $x(t)$ about $t = 0$. An even signal is the same as its reflected version. An odd signal is the negative of its reflected version.

(3) Time shifting:

$$y(t)=x(t-t_0) \tag{22}$$

$$\begin{cases} t_0>0 \Rightarrow y(t) \text{ is obtained by shifting } x(t) \text{ toward the right.} \\ t_0<0 \Rightarrow y(t) \text{ is obtained by shifting } x(t) \text{ toward the left.} \end{cases}$$

$$y[n]=x[n-m] \tag{23}$$

where the shift m must be a positive or negative integer.

3. Precedence rule for time shifting and time scaling

Let $y(t)$ is derived from another signal $x(t)$ through a combination of time shifting and time scaling; that is,

$$y(t)=x(at-b) \tag{24}$$

To obtain $y(t)$ from $x(t)$, the time-shifting and time-scaling operations must be performed in the correct order: time-shifting $\rightarrow$ time-scaling

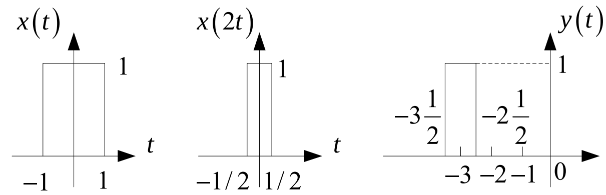
(1) The time-shifting operation always replaces *t* by *t–b*.

(2) The scaling operation always replaces *t* by *at*.

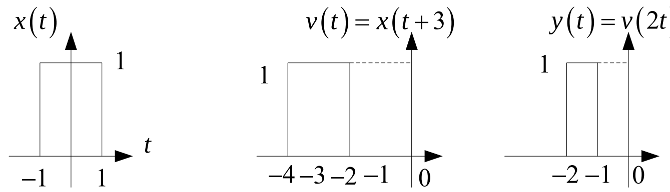
$$v(t) = x(t-b)$$
$$y(t) = v(at) = x(at-b) \quad (25)$$

Example 7: $y(t) = x(2t+3)$

Time scaling → time shifting: $y(t) = v(t+3) = x(2(t+3)) \neq x(2t+3)$



Time shifting → time scaling: $y(t) = v(2t) = x(2t+3)$



■

1-3 Basic Continuous-Time Signals

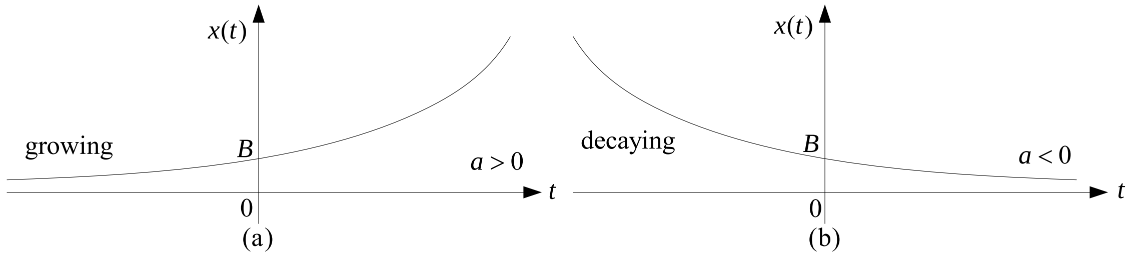
1. Complex exponential signal

$$x(t) = Be^{at} \quad (26)$$

Sinusoidal signal

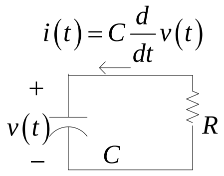
$$x(t) = A\cos(\omega t + \phi) \quad (27)$$

(1) *B* and *a* are real ⇒ real exponential signal



■ **Figure 4.** Continuous-time real exponential $x(t) = Be^{at}$: (a) $a > 0$; (b) $a < 0$ [1].

***Example* 8**: The operation of the capacitor for $t \geq 0$



$$RC\frac{d}{dt}v(t) + v(t) = 0$$

$$\Rightarrow v(t) = V_0 e^{-t/RC}$$

where V_0 denotes the initial value of the voltage developed across the capacitor. ■

(2) B is real and a is pure imaginary ⇒ periodic complex exponential

$$x(t) = Be^{j\omega_0 t} \text{ : periodic} \tag{28}$$

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T} \tag{29}$$

$\Rightarrow e^{j\omega_0 T} = 1 \Rightarrow$ the fundamental period is $T_0 = \dfrac{2\pi}{|\omega_0|}$

Euler's relation:

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t \tag{30}$$

$$e^{j(\omega_0 t+\phi)} = \cos(\omega_0 t + \phi) + j\sin(\omega_0 t + \phi) \tag{31}$$

$$e^{-j(\omega_0 t+\phi)} = \cos(\omega_0 t + \phi) - j\sin(\omega_0 t + \phi) \tag{32}$$

$$\cos(\omega_0 t + \phi) = \frac{1}{2}\left[e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right] \text{ or } \cos(\omega_0 t + \phi) = \text{Re}\left\{e^{j(\omega_0 t+\phi)}\right\} \tag{33}$$

where $\text{Re}\{\cdot\}$ denotes the real part of the complex quantity enclosed inside the braces.

$$\therefore A\cos(\omega_0 t + \phi) = \text{Re}\left\{Be^{j\omega_0 t}\right\},\ B = Ae^{j\phi} \tag{34}$$

Note:

- Fundamental period = T_0
- Fundamental frequency = $\omega_0 = 2\pi/T_0$
- The fundamental frequency of a constant signal is zero.
- Harmonically related complex exponentials:

$$\phi_k(t) = e^{jk\omega_0 t},\quad k = 0, \pm 1, \pm 2, \ldots \tag{35}$$

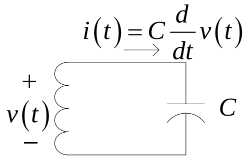
$k = 0 \Rightarrow \phi_k$ is a constant

$k \neq 0 \Rightarrow \phi_k$ is periodic with fundamental period $2\pi / (|k|\omega_0)$ or fundamental frequency $|k|\omega_0$.

$\phi_k(t)$ has a common period of $2\pi/\omega_0$.

***Example* 9**:

$$LC\frac{d^2}{dt^2}v(t) + v(t) = 0$$



$$\Rightarrow v(t) = V_0\cos(\omega_0 t),\ t \geq 0$$

where $\omega_0 = 1/\sqrt{LC}$ is the natural angular frequency of oscillation of the circuit. ■

(3) B is complex and a is complex: general complex exponential function

$$B = |B|e^{j\theta} \text{ and } a = r + j\omega_0 \tag{36}$$

$$\begin{aligned} Be^{at} &= |B|e^{j\theta}e^{(r+j\omega_0)t} = |B|e^{rt}\cdot e^{j(\omega_0 t+\theta)} \\ &= |B|e^{rt}\cos(\omega_0 t+\theta) + j|B|e^{rt}\sin(\omega_0 t+\theta) \\ &= |B|e^{rt}\cos(\omega_0 t+\theta) + j|B|e^{rt}\cos(\omega_0 t+\theta-\pi/2) \end{aligned} \tag{37}$$

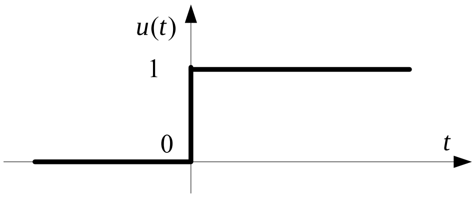
$r = 0 \Rightarrow$ the real and imaginary parts are sinusoidal.

$r > 0 \Rightarrow$ the real and imaginary parts are sinusoidal signals multiplied by a growing exponential.

$r < 0 \Rightarrow$ the real and imaginary parts are sinusoidal signals multiplied by a decaying exponential.

2. The continuous-time unit-step function

$$u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases} \tag{38}$$
$$(t = 0,\ \text{undefined})$$



■ **Figure 5.** Continuous-time version of the unit-step function of unit amplitude.

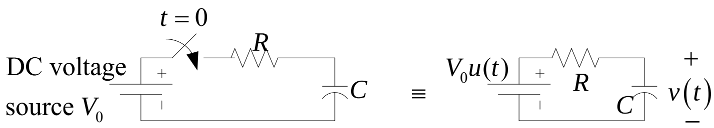
***Example* 10**: Rectangular pulse

$$x(t)=\begin{cases}A, 0\le |t|<0.5\\ 0 \quad ,|t|>0.5\end{cases}$$

$$x(t)=Au(t+0.5)-Au(t-0.5)$$

■

***Example* 11**: RC circuit



$$\left.\begin{array}{l}v(0)=0\\ v(\infty)=V_0\end{array}\right\}\Rightarrow v(t)=V_0\left(1-e^{-t/RC}\right)u(t)$$

■

3. The continuous-time unit impulse function

(1) The continuous-time version of the unit impulse is defined by the following pair of relations:

$$\delta(t)=0 \text{ for } t\neq 0 \text{ and } \int_{-\infty}^{\infty}\delta(t)\,dt=1 \qquad (39)$$

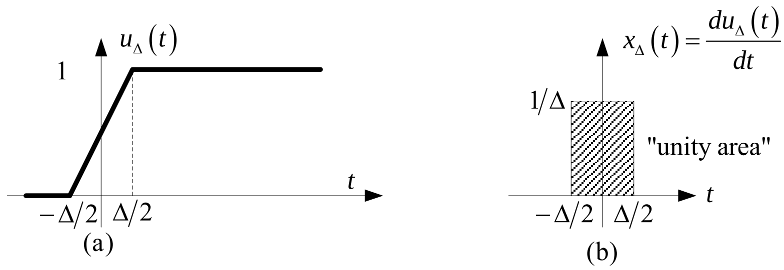
The impulse $\delta(t)$ is also referred to as the *Dirac delta function*.

(2) The impulse and the unit-step function are related to each other in that if we are given either one, we can uniquely determine the other.

$$\delta(t)\Rightarrow u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau \qquad (40)$$

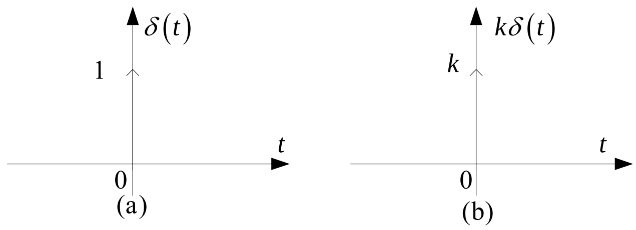
$$\delta(t)=\frac{du(t)}{dt} \text{ (in a restricted sense)} \qquad (41)$$

$u(t)$ is discontinuous at $t = 0$. $\Rightarrow$ not differentiable at $t = 0$



■ **Figure 6.** (a) Continuous approximation to the unit step; (b) derivative of $u_\Delta(t)$ [1].

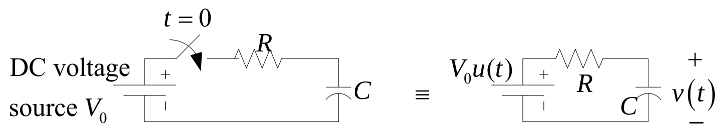
$$\Rightarrow \quad \delta(t)=\lim_{\Delta\to 0} x_\Delta(t) \tag{42}$$



■ **Figure 7.** (a) Unit impulse; (b) scaled impulse [1].

"The height of the arrow used to depict the scaled impulse will be chosen to be representative of its area."

***Example* 12**: RC circuit (continued)



$\because v(t)=V_0u(t)$

$\therefore i(t)=C\frac{d}{dt}v(t)=CV_0\delta(t)$

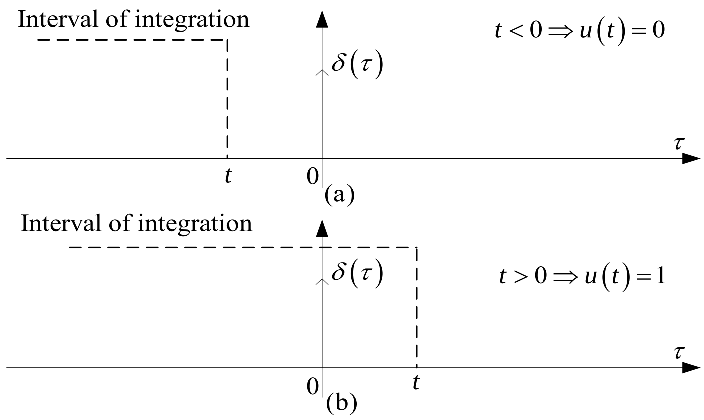
■

(3) Graphical interpretation of

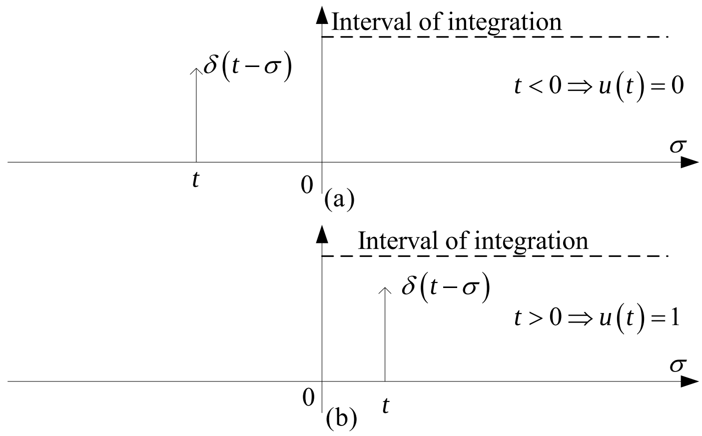
$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau \tag{43}$$

Alternative interpretation:

$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau \underset{(\tau=t-\sigma\Rightarrow d\tau=-d\sigma)}{=} \int_{\infty}^{0}\delta(t-\sigma)(-d\sigma)=\int_{0}^{\infty}\delta(t-\sigma)d\sigma \tag{44}$$



■ **Figure 8.** Running integral given in (43): (a) $t < 0$; (b) $t > 0$.

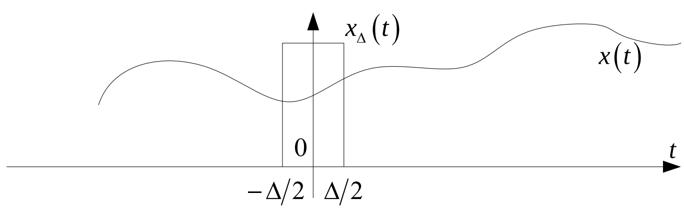


■ **Figure 9.** Relationship given in (44): (a) $t < 0$; (b) $t > 0$.

(4) Product of $x(t)$ and $\delta(t)$

$$x_1(t) = x(t)x_\Delta(t) \approx x(0)x_\Delta(t) \text{ and } \lim_{\Delta\to 0} x_\Delta(t) = \delta(t)$$
$$\Rightarrow x(t)\delta(t) = x(0)\delta(t) \tag{45}$$
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

“equivalence property”



■ **Figure 10.** The product $x(t)x_\Delta(t)$ [1].

(5) $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$

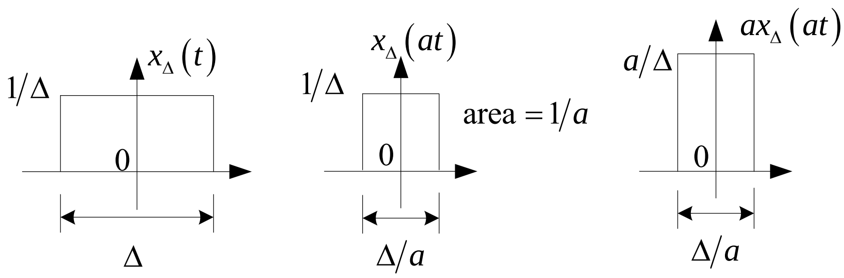
$$\left(= \left[\int_{-\infty}^{\infty} \delta(t - t_0) dt \right] x(t_0) \right), \text{ “sifting property”} \tag{46}$$

It is assumed that $x(t)$ is continuous at time $t = t_0$.

(6) Time-scaling property:

$$\delta(at) = \frac{1}{a}\delta(t) \tag{47}$$

$$\because \lim_{\Delta \to 0} x_\Delta(at) = \frac{1}{a}\delta(t) \tag{48}$$



■ **Figure 11.** Steps involved in proving the time-scaling property of the unit impulse.

4. Ramp function

The integral of the step function $u(t)$ is a ramp function of unit slope. The ramp function is defined as

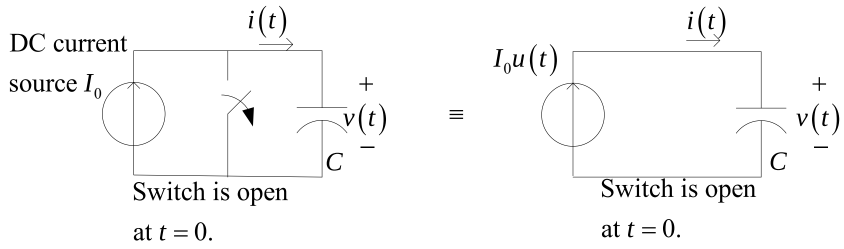
$$r(t) = \begin{cases} t\,, & t \ge 0 \\ 0\,, & t < 0 \end{cases} = tu(t) \tag{49}$$

***Example* 13**: Parallel circuit

$i(t) = I_0 u(t)$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau = \frac{1}{C}\int_{-\infty}^{t} I_0 u(\tau)\, d\tau = \begin{cases} 0 & , t < 0 \\ I_0 t/C\,, & t \ge 0 \end{cases}$$

$$= \frac{I_0}{C} tu(t) = \frac{I_0}{C} r(t)$$

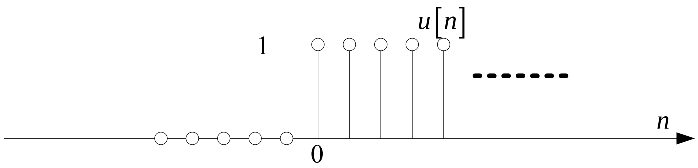


■

1-4 Basic Discrete-Time Signals

1. Discrete-time unit step sequence

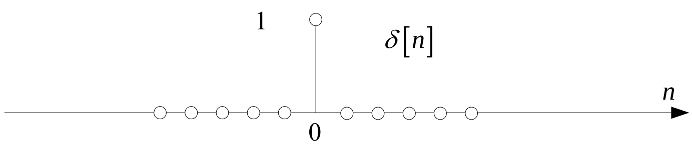
$$u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases} \tag{50}$$



■ **Figure 12.** Discrete-time version of step function of unit amplitude.

2. Discrete-time unit impulse (or unit sample)

$$\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases} \tag{51}$$



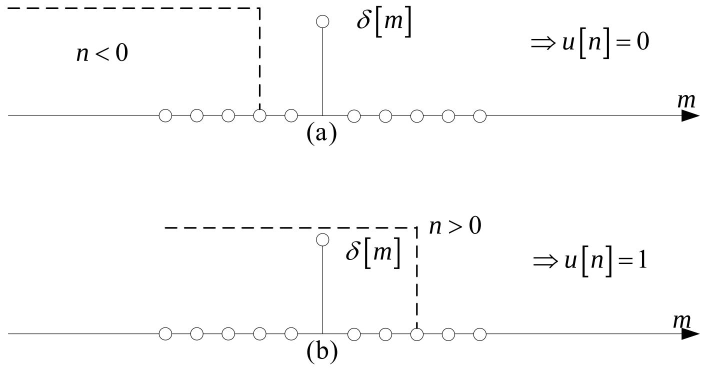
■ **Figure 13.** Discrete-time form of the unit impulse.

$$x[n]\delta[n] = x[0]\delta[n] \tag{52}$$

$$\delta[n] = u[n] - u[n-1] \tag{53}$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \tag{54}$$

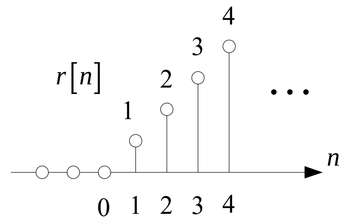
or $$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \tag{55}$$



■ **Figure 14.** Running sum of (54): (a) $n < 0$; (b) $n > 0$ [1].

3. Discrete-time ramp function

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases} = nu[n] \tag{56}$$



■ **Figure 15.** Discrete-time version of the ramp function.

4. Discrete-time complex exponential signals

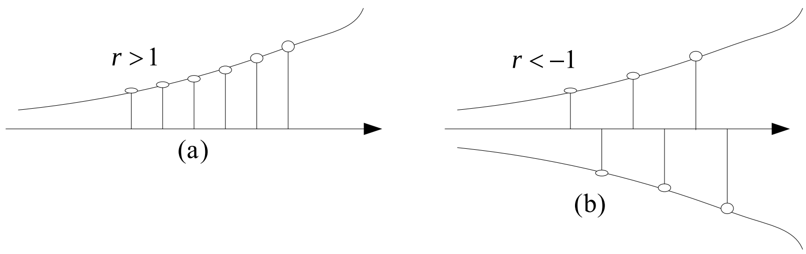
$$x[n] = Br^n = Be^{\alpha n} \quad \left(r = e^{\alpha},\ \alpha \text{ may be any complex number.}\right) \tag{57}$$

sinusoidal signals

$$x[n] = A\cos(\Omega n + \phi) \tag{58}$$

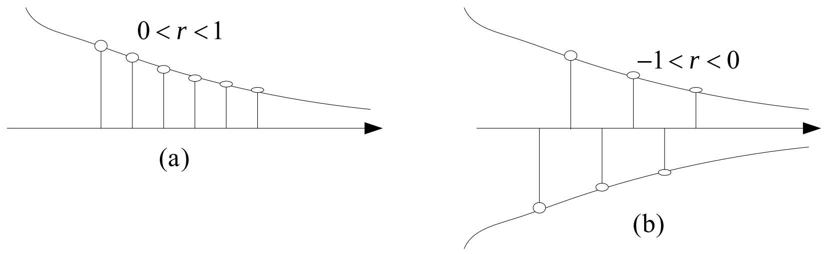
(1) B and r are real

$|r| > 1 \ \Rightarrow$ the signal grows exponentially with n.



■ **Figure 16.** $x[n] = Br^n$: (a) $r > 1$; (b) $r < -1$ [1].

$|r| < 1 \;\; \Rightarrow$ the signal decays exponentially with *n*.



■ **Figure 17.** $x[n] = Br^n$: (a) $0 < r < 1$; (b) $-1 < r < 0$ [1].

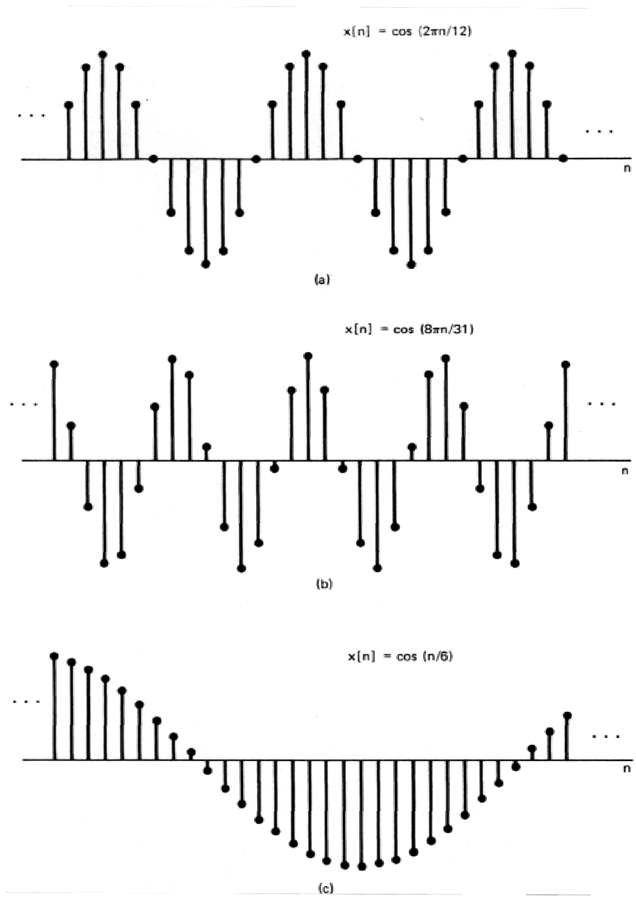
(2) α is pure imaginary

$$x[n] = e^{j\Omega_0 n} = \cos\Omega_0 n + j\sin\Omega_0 n \tag{59}$$

$$A\cos(\Omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\Omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\Omega_0 n} \tag{60}$$

Both Ω_0 and ϕ have units of radians.

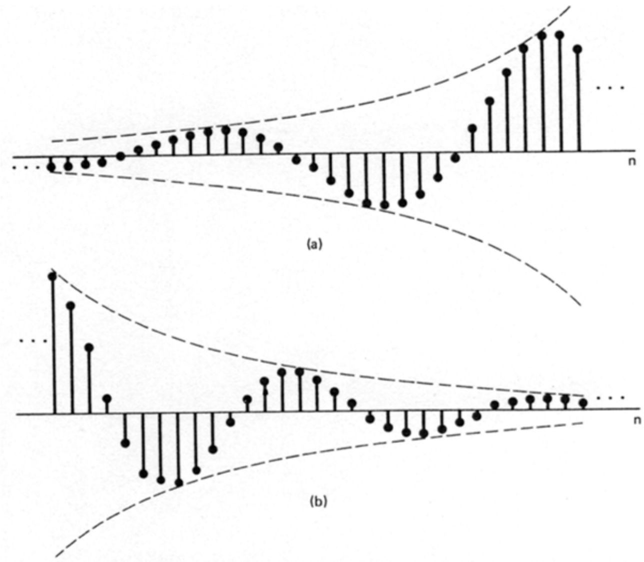
Examples of sinusoidal sequences:



■ **Figure 18.** Discrete-time sinusoidal signals [1].

(3) General complex exponential

$$B = |B|e^{j\theta}, \quad r = |r|e^{j\Omega_0}$$
$$Br^n = |B||r|^n \cos(\Omega_0 n + \theta) + j|B||r|^n \sin(\Omega_0 n + \theta) \tag{61}$$



■ **Figure 19.** (a) Growing discrete-time sinusoidal signal, $|r| > 1$; (b) decaying discrete-time sinusoid, $|r| < 1$ [1].

(4) Periodicity properties of discrete-time complex exponentials

Continuous-time $e^{j\omega_0 t}$

(a) The larger the magnitude of ω_0, the higher the rate of oscillation in the signal.

(b) $e^{j\omega_0 t}$ is periodic for any value of ω_0.

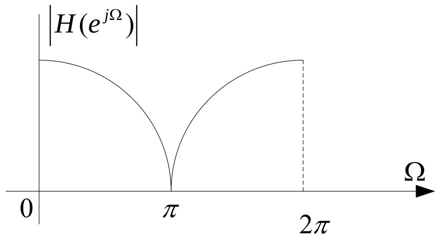
Discrete-time $e^{j\Omega_0 n}$

(a) $e^{j(\Omega_0+2\pi)n} = e^{j\Omega_0 n}e^{j2\pi n} = e^{j\Omega_0 n}$ (62)

- The signal with frequency Ω_0 is identical to the signals with frequencies $(\Omega_0 \pm 2\pi)$, $(\Omega_0 \pm 4\pi)$, and so on.
- We only need to consider an interval of 2π in which to choose Ω_0. ($0 \le \Omega_0 < 2\pi$ or $-\pi \le \Omega_0 < \pi$)
- The signal $e^{j\Omega_0 n}$ does not have a continually increasing rate of oscillation as Ω_0 is increasing in magnitude.

  $0 \to \pi$: signal with increasing rates of oscillation

  $\pi \to 2\pi$: signal with decreasing rates of oscillation



■ **Figure 20.** The magnitude of $H(e^{j\Omega})$ from $\Omega = 0$ to $\Omega = 2\pi$ radians.

(b) $$e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} \Rightarrow e^{j\Omega_0 N} = 1 \Rightarrow \Omega_0 N = 2\pi m \Rightarrow \frac{\Omega_0}{2\pi} = \frac{m}{N} \qquad (63)$$

The signal $e^{j\Omega_0 n}$ is periodic with period N only if $\Omega_0/2\pi$ is a rational number. (For example, Fig. 18(a) and (b) are periodic, T = 12 and 31. Fig. 18(c) is not periodic.)

- If $x[n]$ is periodic with fundamental period N, its fundamental frequency is $2\pi/N$ (radians/sample).
- N: the number of samples contained in a single cycle of $x[n]$.
- If N and m have no factors in common then the fundamental period of $x[n]$ is N.
- $\frac{2\pi}{N} = \frac{\Omega_0}{m}$ and $N = m \cdot \frac{2\pi}{\Omega_0}$
- Constant discrete-time signal: fundamental frequency = 0; fundamental period = undefined.

***Example* 14**: Discrete-time sinusoidal signals

A pair of sinusoidal signals with a common angular frequency is defined by $x_1[n] = \sin[5\pi n]$ and $x_2[n] = \sqrt{3}\cos[5\pi n]$. Find their fundamental period and express the composite sinusoidal signal

$$y[n] = x_1[n] + x_2[n].$$

$\Omega = 5\pi$ rad/sample $\Rightarrow N = 2\pi m/\Omega = 2m/5$

For $x_1[n]$ and $x_2[n]$ to be period, N must be an integer. This can be so only for m = 5, 10, 15,…, which results in N = 2, 4, 6,….

$$\begin{aligned} y[n] &= \sin(5\pi n) + \sqrt{3}\cos(5\pi n) \\ &= \sqrt{1+3}\left[\frac{1}{2}\sin(5\pi n) + \frac{\sqrt{3}}{2}\cos(5\pi n)\right] \\ &= 2\left[\sin(5\pi n)\cos\left(\frac{\pi}{3}\right) + \cos(5\pi n)\sin\left(\frac{\pi}{3}\right)\right] \\ &= 2\sin\left(5\pi n + \frac{\pi}{3}\right) \end{aligned}$$

■

(c) Differences between the signals $e^{j\omega_0 t}$ and $e^{j\Omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\Omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for exponentials at frequencies separated by 2π
Periodic for any choice of ω_0	Periodic only if $\Omega_0 = m \cdot \frac{2\pi}{N}$, $N > 0$. m and N are integers.
Fundamental frequency ω_0	Fundamental frequency $\frac{\Omega_0}{m} = \frac{2\pi}{N}$ (m and N have no factors in common.)
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period $\Omega_0 = 0$: undefined $\Omega_0 \neq 0$: $N = m \cdot \frac{2\pi}{\Omega_0}$

(d) Harmonically related periodic exponentials

$$\phi_k[n] = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \ldots \tag{64}$$

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} = e^{j2\pi n} \cdot e^{jk(2\pi/N)n} = \phi_k[n] \tag{65}$$

There are only N distinct periodic exponentials in the set given in the above equation.

(e) Discrete-time signal obtained by taking samples of continuous-time signal.

$$x[n] = e^{j\omega_0 nT} = e^{j(\omega_0 T)n} \Rightarrow \Omega_0 = \omega_0 T \tag{66}$$

$x[n]$ is periodic only if $\omega_0 T/2\pi$ is a rational number.

Similarly,

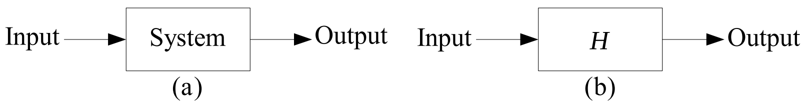
$$x(t) = \cos(2\pi t) \tag{67}$$

$$\begin{aligned} x[n] &= x[nT] = \cos(2\pi nT) = \cos(\Omega_0 n) \\ \Omega_0 &= 2\pi T \end{aligned} \tag{68}$$

***Example* 15**:
Fig. 18(a): $T = 1/12 \Rightarrow \Omega_0 = \pi/6 \Rightarrow \Omega_0/2\pi = 1/12 \Rightarrow N = 12$
Fig. 18(b): $T = 4/31 \Rightarrow \Omega_0 = 8\pi/31 \Rightarrow \Omega_0/2\pi = 4/31 \Rightarrow N = 31$
Fig. 18(c): $T = 1/12\pi \Rightarrow \Omega_0 = 1/6 \Rightarrow \Omega_0/2\pi = 1/12\pi$ ■

1-5 Systems

1. A system can be viewed as any process that results in the transformation of signals



■ **Figure 21.** (a) Block diagram of a system; (b) representation of operator *H*.

(1) Continuous-time system: continuous-time input, continuous-time output

$$x(t) \to y(t),\quad y(t) = H\{x(t)\} \tag{69}$$

(2) Discrete-time system: discrete-time input, discrete-time output

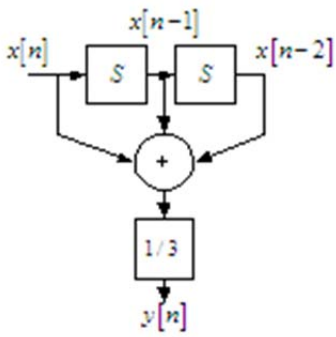
$$x[n] \to y[n],\quad y[n] = H\{x[n]\} \tag{70}$$

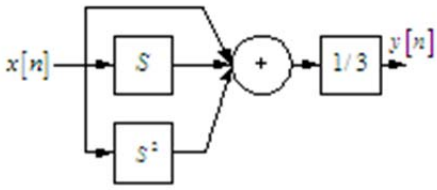
***Example* 16**: Moving-average (MA) system

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

$y[n]$ is the average of the sample values $x[n]$, $x[n-1]$, and $x[n-2]$. The value of $y[n]$ changes as n moves along the discrete-time axis. Let the operator S^k denote a system that shifts the input $x[n]$ by k time units to produce an output equal to $x[n-k]$. Accordingly, we may define the overall operator for the moving average system is

$$H = \frac{1}{3}\left(1 + S + S^2\right)$$

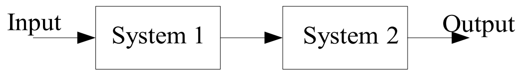




■

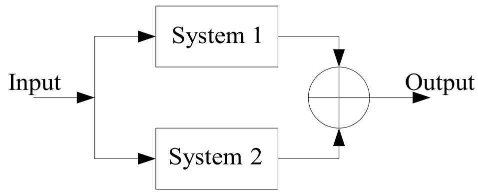
2. Interconnection of systems:

(1) Series interconnection (or cascade interconnection)



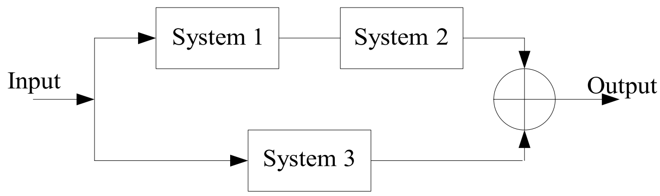
■ **Figure 22.** Series (cascade) interconnection [1].

(2) Parallel interconnection



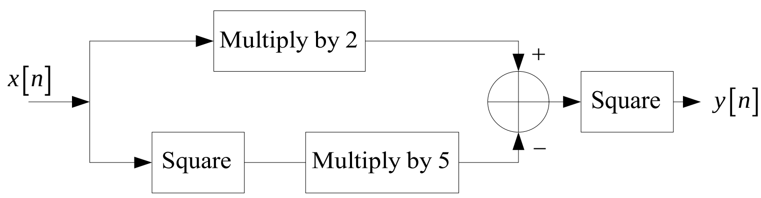
■ **Figure 23.** Parallel interconnection [1].

(3) Series/parallel interconnection



■ **Figure 24.** Series/parallel interconnection [1].

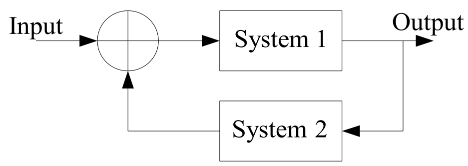
***Example* 17**: $y[n]=\left(2x[n]-5x^2[n]\right)^2$



■

Viewing a complex system in the above manner is often useful in facilitating the analysis of the properties of the system.

(4) Feedback interconnection



■ **Figure 25.** Feedback interconnection [1].

3. Properties of systems

(1) Systems with or without memory

(a) Memoryless system: the output at a given time is dependent only on the input at the same time.

(b) System with memory: the output at a given time is dependent on the inputs at some previous and/or future time instants other than (or in addition to) the input at the same time.

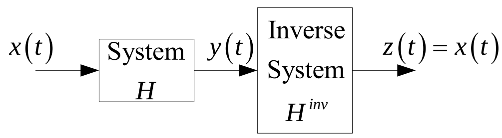
***Example* 18**:

- $y[n]=\sum_{k=-\infty}^{\infty} x[k]$ and $y(t)=x(t-1)$ have memory.
- A resistor is memoryless because of $i(t)=v(t)/R$.
- An inductor has memory because of $i(t)=\frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$.
- The MA system $y[n]=\frac{1}{3}(x[n]+x[n-1]+x[n-2])$ has memory.
- $y[n]=x^2[n]$ is memoryless. ■

(2) Invertibility

A system is said to be invertible if distinct inputs lead to distinct outputs. That is, there must be a one-to-one mapping between input and output signals for a system to be invertible.

⇒ Observing the system output, we can determine the system input.



■ **Figure 26.** The notion of system invertibility.

$$H^{inv}\{y(t)\}=H^{inv}\{H\{x(t)\}\}=H^{inv}H\{x(t)\} \tag{71}$$

For this output signal to equal the original input $x(t)$, we require that

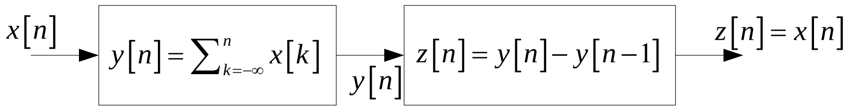
$$H^{inv}H = I \tag{72}$$

where *I* denotes the identity operator.

***Example* 19**: $y[n] = \sum_{k=-\infty}^{n} x[k]$ (invertible system)

The difference between two successive values of the output is precisely the last input value.

$z[n] = y[n] - y[n-1] = x[n]$



■

***Example* 20**: $y(t) = x^2(t) \;\Rightarrow\; x(t) = \sqrt{y(t)}$ or $-\sqrt{y(t)}$

⇒ a non-invertible system ■

***Example* 21**: $y(t) = x(t - t_0) = S^{t_0}\{x(t)\}$

$$S^{-t_0}\{y(t)\} = S^{-t_0}\{S^{t_0}\{x(t)\}\} = S^{-t_0}S^{t_0}\{x(t)\} = I\{x(t)\}$$

The inverse of the system is a time shift $-t_0$. ■

(3) Causality

A system is causal if the output at any time depends only on values of the input at the present time and in the past.

⇒ often referred to as non-anticipative

- If two inputs to a causal system are identical up to some time t_0 or n_0, the corresponding outputs must also be equal up to this same time.

  ***Example* 22**:

  $$\left.\begin{array}{l} y[n] = x[n] - x[n+1] \\ y(t) = x(t+1) \end{array}\right\} \text{noncausal;} \quad \left.\begin{array}{l} y[n] = \sum_{k=-\infty}^{n} x[k] \\ y(t) = x(t-1) \end{array}\right\} \text{causal}$$

  ■

- All memoryless systems are causal. (True)

  All systems with memory are causal. (False)

- Causality is not of fundamental importance in some applications, such as image processing, in which the independent variable is not time.
- The important point to note here is that causality is required for a system to be capable of operating in *real time*.

***Example* 23**: The MA system, $y[n]=\frac{1}{3}\left(x[n]+x[n-1]+x[n-2]\right)$, is causal. By contrast, the MA system described by $\frac{1}{3}\left(x[n+1]+x[n]+x[n-1]\right)$ is noncausal. ■

(4) Stability

Bounded input → bounded output (BIBO): "stable system"

Bounded input → unbounded output (the magnitude grows without bound): "unstable system"

The operator *H* is BIBO stable if the output signal *y*(*t*) satisfies the condition

$$|y(t)| \le M_y < \infty \text{ for all } t \tag{73}$$

whenever the input signals *x*(*t*) satisfy the condition

$$|x(t)| \le M_x < \infty \text{ for all } t \tag{74}$$

Both M_x and M_y represent some finite positive numbers.

***Example* 24**: $y[n]=\frac{1}{3}\left(x[n]+x[n-1]+x[n-2]\right)$

$$\begin{aligned}|y[n]| &= \frac{1}{3}\left|\left(x[n]+x[n-1]+x[n-2]\right)\right| \\ &\le \frac{1}{3}\left(|x[n]|+|x[n-1]|+|x[n-2]|\right) \\ &\le \frac{1}{3}\left(M_x+M_x+M_x\right)=M_x\end{aligned}$$

MA system is stable. ■

***Example* 25**: $y[n]=\frac{1}{2M+1}\sum_{k=-M}^{M} x[n-k]$

x[*n*] is bounded. → *y*[*n*] is bounded. ⇒ "stable system" ■

***Example* 26**: $y[n]=r^n x[n],\ r>1$

$|y[n]|=|r^n||x[n]|\to\infty\ (\because r>1)$ ■

***Example* 27**: $y[n]=\sum_{k=-\infty}^{n}u[k]=(n+1)u[n]$

$y[0]=1,\ y[1]=2,\ y[2]=3,\ldots,y[n]$ grows without bound.

$\Rightarrow$ "unstable system" ■

(5) Time invariance

A system is time invariant if a time shift in the input signal causes an identical time shift in the output signal. Put another way, the characteristics of a time-invariant system do not change with time. Otherwise, the system is said to be time variant.

$$\begin{aligned} x(t)&\to y(t)\\ x(t-t_0)&\to y(t-t_0)\end{aligned} \tag{75}$$

***Example* 28**: $y(t)=\sin(x(t))$

$y_1(t)=\sin(x_1(t))\Rightarrow y_1(t-t_0)=\sin(x_1(t-t_0))$

Let $x_2(t)=x_1(t-t_0)$,

$y_2(t)=\sin(x_2(t))=\sin(x_1(t-t_0))=y_1(t-t_0)$, "time invariant" ■

***Example* 29**: $y(n)=nx(n)$

$y_1[n]=nx_1[n]\Rightarrow y_1[n-n_0]=(n-n_0)x_1[n-n_0]$

Let $x_2[n]=x_1[n-n_0]$, $y_2[n]=nx_2[n]=nx_1[n-n_0]\neq y_1[n-n_0]$

"time-varying" ■

***Example* 30**: Inductor

$$y_1(t) = i(t) = \frac{1}{L}\int_{-\infty}^{t} x_1(\tau)\,d\tau$$

$$y_1(t-t_0) = \frac{1}{L}\int_{-\infty}^{t-t_0} x_1(\tau)\,d\tau$$

$$y_2(t) = \frac{1}{L}\int_{-\infty}^{t} x_1(\tau - t_0)\,d\tau \overset{\tau'=\tau-t_0}{=} \frac{1}{L}\int_{-\infty}^{t-t_0} x_1(\tau')\,d\tau' = y_1(t-t_0)$$

"time invariant" ■

***Example* 31**: Thermistor

A thermistor, $y_1(t) = x_1(t)/R(t)$, has a resistance that varies with time due to temperature changes.

$$y_2(t) = x_1(t-t_0)/R(t)$$

$$y_1(t-t_0) = x_1(t-t_0)/R(t-t_0)$$

In general, $R(t) \neq R(t-t_0)$ for $t_0 \neq 0$. $y_1(t-t_0) \neq y_2(t)$ for $t_0 \neq 0$. ■

(6) Linearity

A linear system is one that possesses the two important properties of superposition and homogeneity. That is, if input consists of the weighted sum of several signals, then the output is simply the superposition, that is, the weighted sum, of the responses of the system to each of those signals.

Additivity property:

$$\begin{aligned} &y_1(t) = H\{x_1(t)\} \\ &y_2(t) = H\{x_2(t)\} \\ &y_1(t) + y_2(t) = H\{x_1(t) + x_2(t)\} \quad \text{"additivity property"} \end{aligned} \tag{76}$$

Then for a system to be linear, it's necessary that the composite input $x_1(t) + x_2(t)$ produce the corresponding output $y_1(t) + y_2(t)$.

Homogeneity or scaling property:

$$\begin{aligned} &ay(t) = H\{ax(t)\} \\ &\text{Superposition Principle: } ay_1(t) + by_2(t) = H\{ax_1(t) + bx_2(t)\} \end{aligned} \tag{77}$$

In general, $\sum_{i=1}^{N} a_i y_i(t) = \sum_{i=1}^{N} a_i H\{x_i(t)\}$.

When a system violates either the principle of superposition or the property of homogeneity, the system is said to be nonlinear.

- Linear systems have zero output when the input is zero. From the scaling property,

$$a x_1(t) = x_2(t) \Rightarrow a y_1(t) = y_2(t) \tag{78}$$

Let $a = 0$, then $x_2(t) = 0$ and $y_2(t) = 0$.

- An incrementally linear system is one that responds linearly to changes in the input

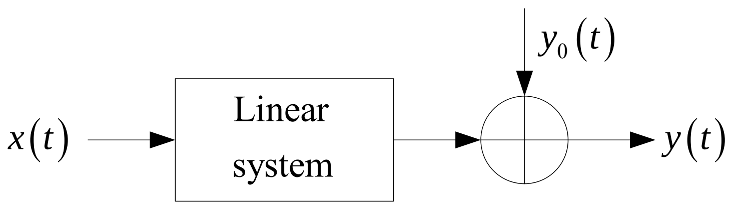
***Example* 32**: $y[n] = 2x[n] + 3$

$x[n] = 0 \Rightarrow y[n] = 3 \neq 0$, "not linear"

$$y_1[n] - y_2[n] = 2x_1[n] + 3 - \{2x_2[n] + 3\} = 2\{x_1[n] - x_2[n]\}$$

"incrementally linear" ■

- An incrementally linear system can be visualized as follows:



■ **Figure 27.** Structure of an incrementally linear system [1].

$y_0(t)$ is the zero-input response of the overall system.

***Example* 33**: $y[n] = n x[n]$ is linear.

$$x[n] = \sum_{i=1}^{N} a_i x_i[n]$$
$$y[n] = n \sum_{i=1}^{N} a_i x_i[n] = \sum_{i=1}^{N} a_i n x_i[n] = \sum_{i=1}^{N} a_i y_i[n]$$

■

***Example* 34**: $y(t) = x(t) x(t-1)$ is nonlinear.

$$\begin{aligned} x(t) &= \sum_{i=1}^{N} a_i x_i(t) \\ y(t) &= \sum_{i=1}^{N} a_i x_i(t) \sum_{j=1}^{N} a_j x_j(t-1) \\ &= \sum_{i=1}^{N} a_i \left[\sum_{j=1}^{N} a_j x_i(t) x_j(t-1) \right] \neq \sum_{i=1}^{N} a_i y_i(t) \end{aligned}$$

■

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[1] Alan V. Oppenheim and Alan S. Willsky, with S. Hamid Nawab, *Signals and Systems*, 2nd Ed., Prentice-Hall, 1997.

[2] S. Haykin and B. Van Veen, *Signals and Systems*, 2nd Ed., Hoboken, NJ: John Wiley & Sons, 2003.

[3] H. P. Hsu, *Schaum's Outline of Theory and Problems of Signals and Systems*, McGraw-Hill, 1995.