# Homework No. 4 Solution

**1.** Let $x[n]$ be a periodic signal with period N and Fourier coefficients $a_k$.

(1) Express the Fourier coefficients $b_k$ of $\left|x[n]\right|^2$ in terms of $a_k$. (10%)

Since $x[n] \overset{F.S.}{\longleftrightarrow} a_k$ and $x[n] \overset{F.S.}{\longleftrightarrow} a_{-k}^*$. By using the convolution property, we have: $x[n]x^*[n] = \left|x[n]\right|^2 \overset{F.S.}{\longleftrightarrow} b_k = \sum_{l=<N>} a_l a_{l+k}^*$.

(2) If the coefficients $a_k$ are real, is it guaranteed that the coefficients $b_k$ are also real? (10%)

From (1), it is clear that the answer is yes.

**2.** When the impulse train $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$ is the input to a particular LTI system with frequency response $H\left(e^{j\Omega}\right)$, the output of the system is found to be $y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right)$. Determine the values of $H\left(e^{jk\pi/2}\right)$ for $k$ = 0, 1, 2, and 3. (20%)

The F.S. of $x[n]$ are $a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j2\pi kn/4} = \frac{1}{4}$ for all $k$. The output signal $y[n]$ can be express as:

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{3} a_k H\left(e^{j2\pi k/4}\right)e^{j2\pi kn/4} \\
&= \frac{1}{4}\left(H\left(e^{j0}\right)e^{j0} + H\left(e^{j\pi/2}\right)e^{jn\pi/2} + H\left(e^{j\pi}\right)e^{jn\pi} + H\left(e^{j3\pi/2}\right)e^{j3n\pi/2}\right) \\
&= \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) = \frac{e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}}{2} \\
&= \frac{e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} + e^{j\left(\frac{3\pi}{2}n-\frac{\pi}{4}\right)}}{2} \quad \left(\because e^{-j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} = e^{j\left(\left(2\pi-\frac{\pi}{2}\right)n-\frac{\pi}{4}\right)}\right)
\end{aligned}
$$

$$\Rightarrow H\left(e^{j0}\right) = H\left(e^{j\pi}\right) = 0,\ H\left(e^{j\pi/2}\right) = 2e^{j\pi/4},\ \text{and } H\left(e^{j3\pi/2}\right) = 2e^{-j\pi/4}.$$

**3.** You are given $x[n] = n(1/2)^{|n|} \xleftrightarrow{DTFT} X(\Omega)$. Without evaluating $X(\Omega)$, find $y[n]$ if

(1) $Y(\Omega) = \text{Re}\{X(\Omega)\}$ (5%)

$\Rightarrow$ Since $x[n]$ is real and odd, $X(\Omega)$ is pure imaginary, thus $y[n] = 0$.

(2) $Y(\Omega) = dX(\Omega)/d\Omega$ (5%)

$\Rightarrow y[n] = -jnx[n] = -jn^2(1/2)^{|n|}$.

(3) $Y(\Omega) = X(\Omega) + X(-\Omega)$ (5%)

$\Rightarrow y[n] = x[n] + x[-n] = n(1/2)^{|n|} - n(1/2)^{|n|} = 0$

(4) $Y(\Omega) = e^{-4j\Omega} X(\Omega)$ (5%)

$\Rightarrow y[n] = x[n-4] = (n-4)(1/2)^{|n-4|}$

**4.** Let $x[n]$ and $h[n]$ be the signals with the following Fourier transforms:

$$X(e^{j\Omega}) = 3e^{-j\Omega} + 1 - e^{j\Omega} + 2e^{j3\Omega}$$
$$H(e^{j\Omega}) = 2e^{-j2\Omega} - e^{-j\Omega} + e^{j4\Omega}$$

Determine $y[n] = x[n] * h[n]$. (15%)

$$y[n] = x[n] * h[n]$$
$$= (3d[n-1] + d[n] - d[n+1] + 2d[n+3]) * (2d[n-2] - d[n-1] + d[n+4])$$
$$= 6d[n-3] - d[n-2] - 3d[n-1] + d[n] + 4d[n+1] - 2d[n+2] + 3d[n+3]$$
$$+ d[n+4] - d[n+5] + 2d[n+7]$$

**5.** Consider the finite-length sequence $x[n] = 2d[n] + d[n-1] + d[n-3]$.

(1) Compute the five-point DFT $X[k]$. (10%)

$\Rightarrow X[k] = 2 + e^{-j\frac{2p}{5}k} + e^{-j3\frac{2p}{5}k}$.

(2) If $Y[k] = X^2[k]$, determine the sequence $y[n]$ with five-point inverse DFT for $n$ = 0 ~ 4. (10%)

$$Y[k] = X^2[k] = 4 + 4e^{-j\frac{2p}{5}k} + e^{-j2\frac{2p}{5}k} + 4e^{-j3\frac{2p}{5}k} + 2e^{-j4\frac{2p}{5}k} + e^{-j6\frac{2p}{5}k}$$
$$= 4 + 5e^{-j\frac{2p}{5}k} + e^{-j2\frac{2p}{5}k} + 4e^{-j3\frac{2p}{5}k} + 2e^{-j4\frac{2p}{5}k}$$

$\therefore y[n] = 4d[n] + 5d[n-1] + d[n-2] + 4d[n-3] + 2d[n-4]$

(3) If $N$-point DFTs are used here, how should we choose $N$ such that $y[n] = x[n] * x[n]$, for $0 \le n \le N-1$. (5%)

$\Rightarrow$ N $\geqq$ 4+4-1=7.