Homework No. 3 Solution

1.

Graph to find $T = 14,\ \omega_o = \frac{\pi}{7}$

$$\begin{aligned} X[k] &= \frac{1}{14}\int_{-7}^{7} x(t)e^{-jk\frac{\pi}{7}t}\,dt \\ &\text{By the sifting property} \\ &= \frac{1}{14}\left[e^{j(k-1)\frac{6\pi}{7}} + e^{j(k-1)\frac{4\pi}{7}} + e^{j(k-1)\frac{2\pi}{7}} + 1 + e^{j(1-k)\frac{2\pi}{7}} + e^{j(1-k)\frac{4\pi}{7}} + e^{j(1-k)\frac{6\pi}{7}}\right] \\ &= \frac{1}{7}\left[\cos((k-1)\frac{6\pi}{7}) + \cos((k-1)\frac{4\pi}{7}) + \cos((k-1)\frac{2\pi}{7}) + \frac{1}{2}\right] \end{aligned}$$

2.

$X[k] = (-\frac{1}{3})^{|k|}, \quad \omega_o = 1$

$$\begin{aligned} x(t) &= \sum_{m=-\infty}^{\infty} (-\frac{1}{3})^{|k|} e^{jkt} \\ &= \sum_{m=0}^{\infty} (-\frac{1}{3}e^{jt})^k + \sum_{m=1}^{\infty} (-\frac{1}{3}e^{-jt})^k \\ &= \frac{1}{1+\frac{1}{3}e^{jt}} - \frac{\frac{1}{3}e^{-jt}}{1+\frac{1}{3}e^{-jt}} \\ &= \frac{8}{10+6\cos(t)} \end{aligned}$$

3.

$x(t) = te^{-t}u(t)$

$$\begin{aligned} X(j\omega) &= \int_0^{\infty} te^{-t}e^{-j\omega t}\,dt \\ &= \frac{1}{(1+j\omega)^2} \end{aligned}$$

4.

$x(t) = e^{-3t}u(t), \quad y(t) = e^{-3(t-2)}u(t-2)$

$$\begin{aligned} X(j\omega) &= \frac{1}{3+j\omega} \\ Y(j\omega) &= e^{-j2\omega}\frac{1}{3+j\omega} \end{aligned}$$

$$\begin{aligned} H(j\omega) &= e^{-j2\omega} \\ h(t) &= \delta(t-2) \end{aligned}$$

5.

$$\begin{aligned}\frac{d^3}{dt^3}y(t) - 3\frac{d}{dt}y(t) - 2y(t) &= 3\frac{d^2}{dt^2}x(t) + 8\frac{d}{dt}x(t) - 10x(t) \\ ((j\omega)^3 - 3j\omega - 2)Y(\omega) &= (3(j\omega)^2 + 8j\omega - 10)X(\omega)\end{aligned}$$

$$\begin{aligned}\Rightarrow H(\omega) &= \frac{Y(\omega)}{X(\omega)} = \frac{-3\omega^2 + 8j\omega - 10}{-j\omega^3 - 3j\omega - 2} \\ &= \frac{-3\omega^2 + 8j\omega - 10}{(j\omega+1)^2(j\omega-2)} \\ &= \frac{A}{(j\omega+1)^2} + \frac{B}{j\omega+1} + \frac{C}{j\omega-2}\end{aligned}$$

$$\begin{aligned}A &= \{\frac{A}{(j\omega+1)^2}(j\omega+1)^2 + \frac{B}{j\omega+1}(j\omega+1)^2 + \frac{C}{j\omega-2}(j\omega+1)^2\}|_{\omega=j} \\ &= \{(j\omega+1)^2 H(\omega)\}|_{\omega=j} \\ &= \{(j\omega+1)^2 \times \frac{-3\omega^2 + 8j\omega - 10}{(j\omega+1)^2(j\omega-2)}\}|_{\omega=j} \\ &= 5\end{aligned}$$

$$\begin{aligned}B &= \frac{1}{j}\frac{d}{d\omega}\{\frac{A}{(j\omega+1)^2}(j\omega+1)^2 + \frac{B}{j\omega+1}(j\omega+1)^2 + \frac{C}{j\omega-2}(j\omega+1)^2\}|_{\omega=j} \\ &= \{\frac{1}{j}\frac{d}{d\omega}(j\omega+1)^2 H(\omega)\}|_{\omega=j} \\ &= \{\frac{1}{j}\frac{d}{d\omega}(\frac{-3\omega^2 + 8j\omega - 10}{(j\omega+1)^2(j\omega-2)})\}|_{\omega=j} \\ &= 1\end{aligned}$$

$$\begin{aligned}C &= \{(j\omega-2)H(\omega)\}|_{\omega=-2j} \\ &= 2\end{aligned}$$

$$\begin{aligned}\Rightarrow H(\omega) &= \frac{5}{(j\omega+1)^2} + \frac{1}{j\omega+1} + \frac{2}{j\omega-2} \\ \Rightarrow h(t) &= 5te^{-t}u(t) + e^{-t}u(t) - 2e^{2t}u(-t)\end{aligned}$$

Because $h(t)$ has nonzero value when $t < 0$, this kind of system is noncausal.

6.

(a)

$x(t) = \int_{-\infty}^{t} \frac{\sin(2\pi\tau)}{\pi\tau} d\tau$

$$\frac{\sin(2\pi t)}{\pi t} \xleftrightarrow{FT} \begin{cases} 1 & \omega \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{t} s(\tau)d\tau \xleftrightarrow{FT} \frac{S(j\omega)}{j\omega} + \pi S(j0)\delta(\omega)$$

$$X(j\omega) = \begin{cases} \pi\delta(\omega) & \omega = 0 \\ \frac{1}{j\omega} & |\omega| \leq 2\pi,\, \omega \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$x(t) = \left(\frac{\sin(t)}{\pi t}\right) * \frac{d}{dt}\left[\left(\frac{\sin(2t)}{\pi t}\right)\right]$

$$x(t) = a(t) * b(t) \xleftrightarrow{FT} X(j\omega) = A(j\omega)B(j\omega)$$

$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{FT} \begin{cases} 1 & \omega \leq W \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{d}{dt}s(t) \xleftrightarrow{FT} j\omega S(j\omega)$$

$$X(j\omega) = \begin{cases} j\omega & |\omega| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$