Homework No. 2 Solution

1. Find and sketch $y[n] = x[n] * h[n]$ of the following signals:

(a) (10%) $x[n] = (-1)^n \left(u[n] - u[n-5]\right)$ and $h[n] = u[n+2]$.

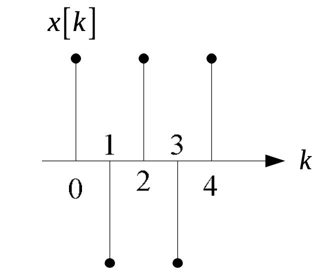
$n+2<0,\ n<-2,\ w_n[k]=0,\ y[n]=0$

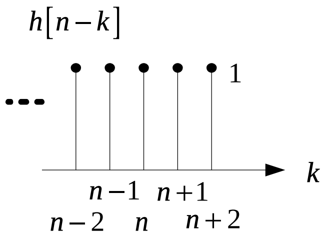
$0 \le n+2 \le 4,\ -2 \le n \le 2,\quad w_n[k] = (-1)^k,\ 0 \le k \le n+2$

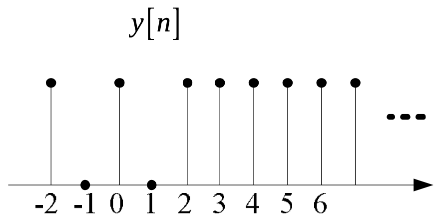
$$y[n] = \sum_{k=0}^{n+2} (-1)^k = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$4 < n+2,\ 2 < n,\quad w_n[k] = (-1)^k,\ 0 \le k \le 4$

$$y[n] = \sum_{k=0}^{4} (-1)^k = 1$$







(b) (10%) $x[n] = u[n] - u[-n]$ and $h[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \ge 0 \\ 4^n, & n < 0 \end{cases}$.

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \ge 0 \\ 4^n, & n < 0 \end{cases} = \left(\frac{1}{2}\right)^n u[n] + 4^n u[-n-1]$$

$$y[n] = x[n] * h[n] = u[n] * h[n] - u[-n] * h[n]$$

$$u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$

$n \ge 0$,

$$\sum_{k=-\infty}^{n} h[k] = \sum_{k=-\infty}^{-1} 4^k + \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

$$= \left(4^{-1} + 4^{-2} + \cdots\right) + \left[1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^n\right]$$

$$= \frac{1}{3} + 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] = \frac{7}{3} - \left(\frac{1}{2}\right)^n$$

$n < 0$,

$$\sum_{k=-\infty}^{n} h[k] = \sum_{k=-\infty}^{n} 4^k = 4^n + 4^{n-1} + \cdots$$

$$= 4^n \left(1 + 4^{-1} + \cdots\right) = \frac{4}{3} 4^n$$

$$u[-n] * h[n] = \sum_{k=n}^{\infty} h[k]$$

$n \ge 0$,

$$\sum_{k=n}^{\infty} h[k] = \sum_{k=n}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n+1} + \cdots = \left(\frac{1}{2}\right)^n \left(1 + \frac{1}{2} + \cdots\right) = 2\left(\frac{1}{2}\right)^n$$

$n < 0$,

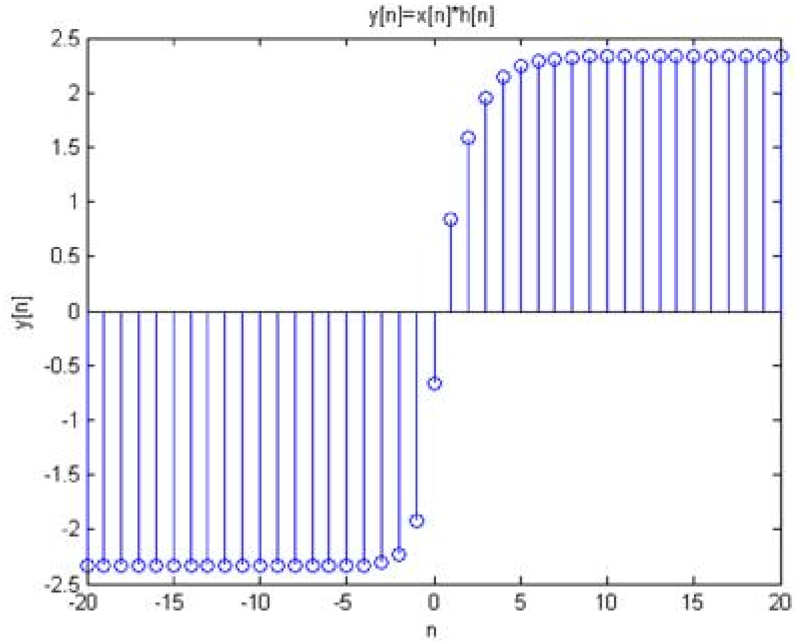
$$\sum_{k=n}^{\infty} h[k] = \sum_{k=n}^{-1} 4^k + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= 4^{-1} + 4^{-2} + \cdots + 4^n + \left(1 + \frac{1}{2} + \cdots\right)$$

$$= 4^{-1}\left(1 + 4^{-1} + \cdots + 4^{n+1}\right) + 2$$

$$= 4^{-1} \times \frac{4}{3} \times \left(1 - 4^n\right) + 2 = \frac{1}{3}\left(1 - 4^n\right) + 2 = \frac{7}{3} - \frac{4^n}{3}$$

$$
\begin{aligned}
y[n] &= \left[\frac{7}{3} - \left(\frac{1}{2}\right)^n\right]u[n] + \frac{4}{3}4^n u[-n-1] - \left\{2\left(\frac{1}{2}\right)^n u[n] + \left(\frac{7}{3} - \frac{4^n}{3}\right)u[-n-1]\right\} \\
&= \left[\frac{7}{3} - \left(\frac{1}{2}\right)^n - 2\left(\frac{1}{2}\right)^n\right]u[n] + \left\{\frac{4}{3}4^n - \left(\frac{7}{3} - \frac{4^n}{3}\right)\right\}u[-n-1] \\
&= \left[\frac{7}{3} - 3\left(\frac{1}{2}\right)^n\right]u[n] + \left(\frac{5}{3}4^n - \frac{7}{3}\right)u[-n-1]
\end{aligned}
$$



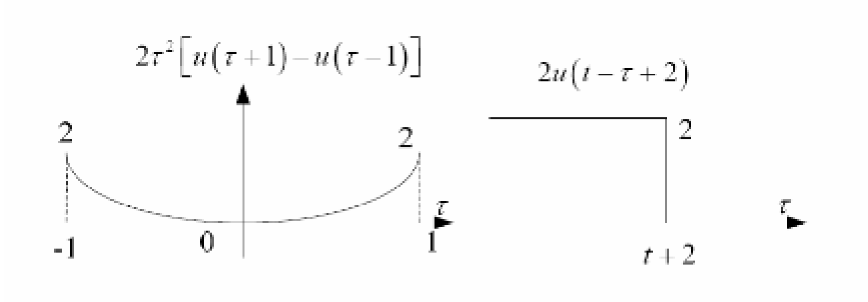
2. (20%)

$y(t) = 2t^2[u(t+1) - u(t-1)] * 2u(t+2).$

For $t+2 < -1,\ t < -3,\ y(t) = 0.$

For $t+2 < 1,\ -3 < t < -1,\ y(t) = 2\int_{-1}^{t+2} 2t^2 dt = \frac{4}{3}t^3\Big|_{-1}^{t+2} = \frac{4}{3}[(t+2)^3 + 1].$

For $t+2 \geq 1,\ -1 < t,\ y(t) = 2\int_{-1}^{1} 2t^2 dt = \frac{4}{3}t^3\Big|_{-1}^{1} = \frac{4}{3}[1+1] = \frac{8}{3}.$



3. Homogeneous solution

$$r^2+4=0 \Rightarrow r=\pm j2$$
$$y^h(t)=c1e^{j2t}+c2e^{-j2t}$$

(a) (5%) $x(t)=t$

$$y^p(t)=p_1t+p_2$$

$$4p_1t+4p_2=3 \Rightarrow p_1=0, p_2=\frac{3}{4}$$

$$\therefore y^p(t)=\frac{3}{4}$$

$$\therefore y(t)=y^h(t)+y^p(t)=c_1e^{j2t}+c_2e^{-j2t}+\frac{3}{4}$$

$$\therefore y(t)=b_1\sin(2t)+b_2\cos(2t)+\frac{3}{4}$$

From $y(0^-)=-1, \frac{d}{dt}y(t)|_{t=0^-}=1$

We get $\Rightarrow b_1=\frac{1}{2}, b_2=-\frac{7}{4}$

$$y(t)=-\frac{7}{4}\cos(2t)+\frac{1}{2}\sin(2t)+\frac{3}{4}$$

(b) (5%) $x(t)=e^{-t}$

$$y^p(t)=pe^{-t}$$

$$pe^{-t}+4pe^{-t}=-3e^{-t} \Rightarrow p=-\frac{3}{5}$$

$$\therefore y^p(t)=-\frac{3}{5}e^{-t}$$

$$y(t)=y^h(t)+y^p(t)=c_1e^{j2t}+c_2e^{-j2t}-\frac{3}{5}e^{-t}$$

$$\therefore y(t)=b_1\sin(2t)+b_2\cos(2t)-\frac{3}{5}e^{-t}$$

From $y(0^-)=-1, \frac{d}{dt}y(t)|_{t=0^-}=1 \Rightarrow b_1=\frac{1}{5}, b_2=-\frac{2}{5}$

$$\therefore y(t)=-\frac{2}{5}\cos(2t)+\frac{1}{5}\sin(2t)-\frac{3}{5}e^{-t}$$

(c) (10%) $x(t)=\sin(t)+\cos(t)$

$$y^p(t)=p_1\cos(t)+p_2\sin(t)$$

$$y'^{p}(t) = -p_1 \sin(t) + p_2 \cos(t)\ , y''^{p}(t) = -p_1 \cos(t) - p_2 \sin(t)$$

We get $p_1 = 1, p_2 = -1$

$$\therefore y^p(t) = \cos(t) - \sin(t)$$

$$\therefore y(t) = b1\cos(2t) + b2\sin(2t) + \cos(t) - \sin(t)$$

From $y(0^-) = -1, \frac{d}{dt} y(t)|_{t=0^-} = 1 \Rightarrow b_1 = -2, b_2 = 1$

$$\therefore y(t) = -2\cos(2t) + \sin(2t) + \cos(t) - \sin(t)$$

4. Homogeneous solution

$$r^2 - \frac{1}{4}r - \frac{1}{8} = 0 \Rightarrow r = \frac{1}{2}, -\frac{1}{4}$$

$$y^h[n] = c_1(\frac{1}{2})^n + c_2(-\frac{1}{4})^n$$

(a) (5%) $x[n] = nu[n]$

$$y^p[n] = (p_1 n + p_2)u[n]$$

$$p_1 n + p_2 - \frac{1}{4}(p_1(n-1) + p_2) - \frac{1}{8}(p_1(n-2) + p_2) = n + n - 1 \Rightarrow p_1 = \frac{16}{5}, p_2 = -\frac{104}{25}$$

$$y[n] = c_1(\frac{1}{2})^n + c_2(-\frac{1}{4})^n + (\frac{16}{5}n - \frac{104}{25})u[n]$$

From $y[-1] = 1, y[-2] = 0 \Rightarrow c_1 = \frac{1}{3}, c_2 = -\frac{1}{12}$

$$\therefore y[n] = \frac{1}{3}(\frac{1}{2})^n - \frac{1}{12}(-\frac{1}{4})^n + (\frac{16}{5}n - \frac{104}{25})u[n]$$

(b) (5%) $x[n] = (\frac{1}{8})^n u[n]$

$$y_p[n] = p(\frac{1}{8})^n u[n]$$

$$p(\frac{1}{8})^n - \frac{1}{4}p(\frac{1}{8})^{n-1} - \frac{1}{8}p(\frac{1}{8})^{n-2} = (\frac{1}{8})^n + (\frac{1}{8})^{n-1} \Rightarrow p = -1$$

$$\therefore y_p[n] = -(\frac{1}{8})^n u[n]$$

$$y[n] = c_1(\frac{1}{2})^n + c_2(-\frac{1}{4})^n - (\frac{1}{8})^n u[n]$$

$$\text{From } y[-1]=1, y[-2]=0 \ \Rightarrow c_1=\frac{1}{3}, c_2=-\frac{1}{12}$$

$$y[n]=\frac{1}{3}(\frac{1}{2})^n-\frac{1}{12}(-\frac{1}{4})^n-(\frac{1}{8})^n u[n]$$

(c) (10%) $x[n]=e^{j\frac{p}{4}n}u[n]$

$$y^p[n]=pe^{j\frac{p}{4}n}u[n]$$

$$pe^{j\frac{p}{4}n}-\frac{1}{4}pe^{j\frac{p}{4}(n-1)}-\frac{1}{8}pe^{j\frac{p}{4}(n-2)}=e^{j\frac{p}{4}n}+e^{j\frac{p}{4}(n-1)}$$

$$p=\frac{1+e^{-j\frac{p}{4}}}{1-\frac{1}{4}e^{-j\frac{p}{4}}-\frac{1}{8}e^{-j\frac{p}{2}}}$$

$$y^p[n]=-\frac{1+e^{-j\frac{p}{4}}}{1-\frac{1}{4}e^{-j\frac{p}{4}}-\frac{1}{8}e^{-j\frac{p}{2}}}e^{j\frac{p}{4}n}u[n]$$

$$y[n]=c_1(\frac{1}{2})^n+c_2(-\frac{1}{4})^n-\frac{1+e^{-j\frac{p}{4}}}{1-\frac{1}{4}e^{-j\frac{p}{4}}-\frac{1}{8}e^{-j\frac{p}{2}}}e^{j\frac{p}{4}n}u[n]$$

$$\text{From } y[-1]=1, y[-2]=0 \quad \begin{array}{l}\Rightarrow c_1=\frac{1}{3}\\ \Rightarrow c_2=-\frac{1}{12}\end{array}$$

5. (20%)

$$\frac{d}{dt}y(t)+2y(t)=e^{3t}u(t)$$

Homogeneous solution：

$$r+2=0\Rightarrow r=-2$$
$$y^h(t)=c_1e^{-2t}$$

Particular solution：

$$y^p(t)=pe^{3t}$$

$$3pe^{3t} + 2pe^{3t} = e^{3t}, t > 0$$

$$\Rightarrow p = \frac{1}{5}, t > 0$$

$$\therefore y^p(t) = \frac{1}{5}e^{3t}, t > 0$$

Complete solution： $y(t) = c_1 e^{-2t} + \frac{1}{5}e^{3t}, t > 0$

Because at rest $\Rightarrow \therefore y(0) = 0$

From $y(0) = 0 \Rightarrow c_1 = -\frac{1}{5}$

$$y(t) = -\frac{1}{5}e^{-2t} + \frac{1}{5}e^{3t}, t > 0$$

or $\Rightarrow y(t) = \frac{1}{5}(-e^{-2t} + e^{3t})u(t)$