Homework 7-2 Solution

1.

(1) (8%)

$$x[n]=\cos^2\left(\frac{6\pi}{17}n+\frac{\pi}{3}\right)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)=x_1[n]+x_2[n]$$

$$\left.\begin{array}{l} x_1[n]=\frac{1}{2}\Rightarrow N_1=1 \\ x_2[n]=\frac{1}{2}\cos\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)\Rightarrow N_2=2\pi\Big/\frac{12\pi}{17}=\frac{17}{6} \end{array}\right\}\Rightarrow\therefore N=17$$

$$x[n]=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)=\frac{1}{2}+\frac{1}{4}\left[e^{j\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)}+e^{-j\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)}\right]$$

$$=\frac{1}{2}+\frac{1}{4}\left[e^{j\frac{2\pi}{3}}e^{j6\frac{2\pi}{17}n}+e^{-j\frac{2\pi}{3}}e^{-j6\frac{2\pi}{17}n}\right]$$

$$X[k]=a_k=\begin{cases}\frac{1}{2},\ k=0\\ \frac{1}{4}e^{j\frac{2\pi}{3}},\ k=6\\ \frac{1}{4}e^{-j\frac{2\pi}{3}},\ k=-6\\ 0,\text{ otherwise on } k=\{-8,-7,\ldots,8\}\end{cases}$$

(2) $x[n] = x[n + 8]$. (7%)

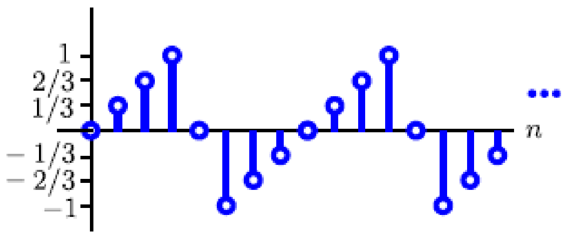


Figure 1

$$X[k]=a_k=\frac{1}{8}\sum_{n=5}^{11}\frac{1}{3}(n-8)e^{-j\frac{2\pi}{8}kn}$$

$$=\frac{1}{8}\sum_{n=-3}^{3}\frac{1}{3}ne^{-j\frac{\pi}{4}kn}$$

$$=\frac{-1}{8}e^{j\frac{3\pi}{4}k}+\frac{-1}{12}e^{j\frac{\pi}{2}k}+\frac{-1}{24}e^{j\frac{\pi}{4}k}+\frac{1}{24}e^{-j\frac{\pi}{4}k}+\frac{1}{12}e^{-j\frac{\pi}{2}k}+\frac{1}{8}e^{-j\frac{3\pi}{4}k}$$

$$=-\frac{1}{4}j\sin\left(\frac{3\pi}{4}k\right)-\frac{1}{6}j\sin\left(\frac{\pi}{2}k\right)-\frac{1}{12}j\sin\left(\frac{\pi}{4}k\right)$$

2.

(1) $X[k] = a_k = \cos\left(\frac{8\pi}{21}k\right)$ (7%)

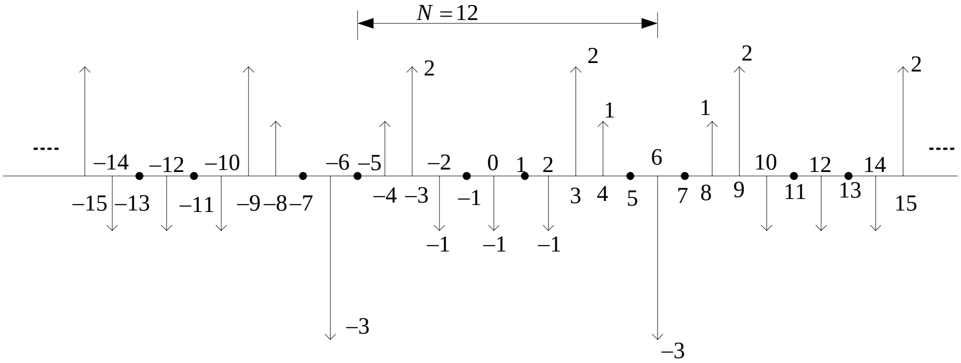
$$N = 21,\ \Omega_0 = \frac{2\pi}{21},$$

$$X[k] = a_k = \cos\left(\frac{8\pi}{21}k\right) = \frac{1}{2}\left[e^{j(4)\frac{2\pi}{21}} + e^{-j(4)\frac{2\pi}{21}}\right]$$

By inspection,

$$x[n] = \begin{cases} 21/2, & n = \pm 4 \\ 0, & \text{otherwise on } n \in \{-10, -9, \ldots, 10\} \end{cases}$$

(2) $X[k] = a_k = \sum_{m=-\infty}^{\infty} (-1)^m \left(\delta[k-2m] - 2\delta[k+3m]\right)$ (8%)



$$N = 12,\ \Omega_0 = \frac{\pi}{6},$$

$$x[n] = \sum_{k=-5}^{6} X[k] e^{jk\frac{\pi}{6}n}$$

$$= e^{j(-4)\frac{\pi}{6}n} + 2e^{j(-3)\frac{\pi}{6}n} - e^{j(-2)\frac{\pi}{6}n} - 1 - e^{j(2)\frac{\pi}{6}n} + 2e^{j(3)\frac{\pi}{6}n} + e^{j(4)\frac{\pi}{6}n} - 3e^{j(6)\frac{\pi}{6}n}$$

$$= 2\cos\left(\frac{2\pi}{3}n\right) + 4\cos\left(\frac{\pi}{2}n\right) - 2\cos\left(\frac{\pi}{3}\right) - 1 - 3(-1)^n$$

3.

(1) (5%)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{3}{4}\right)^n e^{-j\Omega n} = \frac{\left(\frac{3}{4}e^{-j\Omega}\right)^4}{1 - \frac{3}{4}e^{-j\Omega}}$$

$$\left|X(\Omega)\right| = \frac{\left(\frac{3}{4}\right)^n}{\sqrt{\frac{25}{16} - \frac{3}{2}\cos(\Omega)}};\ \angle X(\Omega) = -4\Omega + \tan^{-1}\left(\frac{3\sin(\Omega)}{4 - 3\cos(\Omega)}\right)$$

(2) (5%)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \frac{1}{2} e^{-j\Omega}$$

$$\left| X(\Omega) \right| = \frac{1}{2}; \angle X(\Omega) = -\Omega$$

4.

(1) (5%)

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega} e^{j\Omega n} d\Omega$$

by orthogonality

$$= \delta[n+1]$$

(2) (5%)

Since $X(\Omega)$ doesn't periodic in Ω with period 2π, the DTFT is not exist. However, if you still evaluate it, you should get the following formula.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}}{2j} + \frac{e^{j\Omega} + e^{-j\Omega}}{2} \right) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1] + \frac{\cos(n\pi)}{2\pi j(n+0.5)} - \frac{\cos(n\pi)}{2\pi j(0.5-n)}$$

$$\left(\frac{1}{2\pi} \cdot \frac{1}{2j} \int_{-\pi}^{\pi} e^{j\Omega/2} e^{j\Omega n} d\Omega = \frac{1}{2\pi} \cdot \frac{1}{2j} \frac{e^{j(n+1/2)\Omega}}{j(n+0.5)} \Bigg|_{-\pi}^{\pi} = \frac{1}{2\pi} \cdot \frac{1}{2j} \frac{je^{jn\pi} + je^{-jn\pi}}{j(n+0.5)} = \frac{1}{2\pi j} \cdot \frac{\cos(n\pi)}{n+0.5} \right)$$

5.

(1) (5%) Since $x[n]$ is real and odd, $X(\Omega)$ is purely imaginary, thus $y[n]$=0.

(2) (5%) $y[n] = -jnx[n] = -jn^2 \left(\frac{3}{4} \right)^{|n|}$.

(3) (5%) $y[n] = 2\pi x_1[n]x[n] = 2\pi n^2 \left(\frac{3}{4} \right)^{2|n|} e^{j\frac{\pi}{2}n}$.

(4) (5%) $y[n] = -j\left[e^{-j\frac{\pi}{4}(n-4)} x[n-4] + e^{j\frac{\pi}{4}(n-4)} x[n-4] \right]$.

6.

(1) (10%)

$$y[n] = x[n] \otimes h[n] = x[n] \otimes \left(\delta[n] + 2\delta[n-1] + \delta[n-2] \right)$$

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$H(\Omega) = 1 + 2e^{-j\Omega} + e^{-j2\Omega} = 2e^{-j\Omega}\left(0.5e^{j\Omega} + 1 + 0.5e^{-j\Omega}\right) = 2e^{-j\Omega}\left(1 + \cos(\Omega)\right)$$

(2) (5%)

$$\begin{aligned} h_1[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(\Omega) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega+\pi) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega) e^{j(\Omega-\pi)n} d\Omega \\ &= \frac{e^{j\pi n}}{2\pi}\int_{-\pi}^{\pi} H(\Omega) e^{j\Omega n} d\Omega \\ &= -1^n h[n] \\ &= \delta[n] - 2\delta[n-1] + \delta[n-2] \end{aligned}$$

7.

(1) (7%) $h[n] = \delta[n] + 2\left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{2}\right)^n u[n]$

$$\begin{aligned} H(\Omega) &= 1 + \frac{2}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 + \frac{1}{2}e^{-j\Omega}} = \frac{1 - \frac{1}{4}e^{-j2\Omega} + 2 + e^{-j\Omega} + 1 - \frac{1}{2}e^{-j\Omega}}{1 - \frac{1}{4}e^{-j2\Omega}} \\ &= \frac{4 + \frac{1}{2}e^{-j\Omega} - \frac{1}{4}e^{-j2\Omega}}{1 - \frac{1}{4}e^{-j2\Omega}} \end{aligned}$$

$$Y(\Omega)\left(1 - \frac{1}{4}e^{-j2\Omega}\right) = X(\Omega)\left(4 + \frac{1}{2}e^{-j\Omega} - \frac{1}{4}e^{-j2\Omega}\right)$$

$$\therefore y[n] - \frac{1}{4}y[n-2] = 4x[n] + \frac{1}{2}x[n-1] - \frac{1}{4}x[n-2]$$

(2) (8%)

$$H(\Omega) = 1 + \frac{e^{-j\Omega}}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 + \frac{1}{4}e^{-j\Omega}\right)}$$

$$= \frac{1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega} + e^{-j\Omega}}{1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}} = \frac{1 + \frac{3}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}}$$

$$Y(\Omega)\left(1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}\right) = X(\Omega)\left(1 + \frac{3}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}\right)$$

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + \frac{3}{4}x[n-1] - \frac{1}{8}x[n-2]$$