HW3 Solution

(1.)(a.)

$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t)$

(i) $x(t) = e^{-3t}$

$$\begin{aligned} y^{(p)}(t) &= ke^{-3t} \\ 9ke^{-3t} - 6ke^{-3t} + ke^{-3t} &= -3e^{-3t} \\ k &= -\frac{3}{4} \\ y^{(p)}(t) &= -\frac{3}{4}e^{-3t} \end{aligned}$$

(ii)

$x(t) = 2\sin(t)$

$$\begin{aligned} y^{(p)}(t) &= A\cos(t) + B\sin(t) \\ \frac{d}{dt}y^{(p)}(t) &= -A\sin(t) + B\cos(t) \\ \frac{d^2}{dt^2}y^{(p)}(t) &= -A\cos(t) - B\sin(t) \end{aligned}$$

$$-A\cos(t) - B\sin(t) - 2A\sin(t) + 2B\cos(t) + A\cos(t) + B\sin(t) = 2\cos(t)$$

$-A + 2B + A = 2$

$-B - 2A + B = 0$

$A = 0$

$B = 1$

$y^{(p)}(t) = \sin(t)$

(b)(i)

$x[n] = -(\frac{1}{2})^{-n}u[n]$

$y^{(p)}[n] = A(\frac{1}{2})^{-n}u[n]; y^{(p)}[n-1] = A(\frac{1}{2})^{-(n-1)} = \frac{A}{2}(\frac{1}{2})^{-n}$

$A(\frac{1}{2})^{-n} - \frac{A}{2}(\frac{1}{2})^{-n} = -2(\frac{1}{2})^{-n}$

$A = -\frac{5}{2}$

$y^{(p)}[n] = -\frac{5}{2}(\frac{1}{2})^{-n}u[n]$

(ii)

$x[n] = \cos(\frac{\pi}{5}n)$

$$\begin{aligned}
y^{(p)}[n] &= A\cos(\frac{\pi}{5}n) + B\sin(\frac{\pi}{5}n) \\
2\cos(\frac{\pi}{5}n) &= A\cos(\frac{\pi}{5}n) + B\sin(\frac{\pi}{5}n) - \frac{2}{5}\left[A\cos(\frac{\pi}{5}(n-1)) + B\sin(\frac{\pi}{5}(n-1))\right) \\
&\quad \text{Using the trig identities} \\
\sin(\theta \pm \phi) &= \sin\theta\cos\phi \pm \cos\theta\sin\phi \\
\cos(\theta \pm \phi) &= \cos\theta\cos\phi \mp \sin\theta\sin\phi
\end{aligned}$$

$$A - \frac{2}{5}A\cos\frac{\pi}{5} + \frac{2}{5}B\sin\frac{\pi}{5} = 2$$

$$B - \frac{2}{5}A\sin\frac{\pi}{5} - \frac{2}{5}B\cos\frac{\pi}{5} = 0$$

$$y^{(p)}[n] = 2.6381\cos(\frac{\pi}{5}n) + 0.9170\sin(\frac{\pi}{5}n)$$

(2.)(a.)

(a) $\frac{d}{dt}y(t) + 10y(t) = 2x(t), \quad y(0^-) = 1, x(t) = u(t)$

$$\begin{aligned}
t \geq 0 &\quad \text{natural: characteristic equation} \\
r + 10 &= 0 \\
r &= -10 \\
y^{(n)}(t) &= ce^{-10t} \\
&\quad \text{particular} \\
y^{(p)}(t) &= ku(t) = \frac{1}{5}u(t) \\
y(t) &= \frac{1}{5} + ce^{-10t} \\
y(0^-) = 1 &= \frac{1}{5} + c \\
c &= \frac{4}{5} \\
y(t) &= \frac{1}{5}\left[1 + 4e^{-10t}\right]u(t)
\end{aligned}$$

(b.)

$\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 2x(t), \quad y(0^-) = -1, \ \frac{d}{dt}y(t)\Big|_{t=0^-} = 1, x(t) = e^{-t}u(t)$

$$\begin{aligned}
t \geq 0 & \qquad \text{natural: characteristic equation} \\
r^2 + 6r + 8 &= 0 \\
r &= -4, \ \ -2 \\
y^{(n)}(t) &= c_1 e^{-2t} + c_2 e^{-4t} \\
& \qquad \text{particular} \\
y^{(p)}(t) &= k e^{-t} u(t) \\
&= \frac{2}{3} e^{-t} u(t) \\
y(t) &= \frac{2}{3} e^{-t} u(t) + c_1 e^{-2t} + c_2 e^{-4t} \\
y(0^-) = -1 &= \frac{2}{3} + c_1 + c_2 \\
\frac{d}{dt} y(0)\Big|_{t=0^-} = 1 &= -\frac{2}{3} - 2c_1 - 4c_2 \\
c_1 &= -\frac{5}{2} \\
c_2 &= \frac{5}{6} \\
y(t) &= \frac{2}{3} e^{-t} u(t) - \frac{5}{2} e^{-2t} + \frac{5}{6} e^{-4t}
\end{aligned}$$

(c.)

$y[n] - \frac{1}{9}y[n-2] = x[n-1], y[-1] = 1, y[-2] = 0, x[n] = u[n]$

$$\begin{aligned}
n \geq 0 & \qquad \text{natural: characteristic equation} \\
r^2 - \frac{1}{9} &= 0 \\
r &= \pm\frac{1}{3} \\
y^{(n)}[n] &= c_1 \left(\frac{1}{3}\right)^n + c_2 \left(-\frac{1}{3}\right)^n \\
& \qquad \text{particular} \\
y^{(p)}[n] &= k u[n] \\
k - \frac{1}{9}k &= 1 \\
k &= \frac{9}{8} \\
y^{(p)}[n] &= \frac{9}{8} u[n]
\end{aligned}$$

$$
\begin{aligned}
y[n] &= \frac{9}{8}u[n] + c_1\left(\frac{1}{3}\right)^n + c_2\left(-\frac{1}{3}\right)^n \\
& \quad \text{Translate initial conditions} \\
y[n] &= \frac{1}{9}y[n-2] + x[n-1] \\
y[0] &= \frac{1}{9}0 + 0 = 0 \\
y[1] &= \frac{1}{9}1 + 1 = \frac{10}{9} \\
0 &= \frac{9}{8} + c_1 + c_2 \\
\frac{10}{9} &= \frac{9}{8} + \frac{1}{3}c_1 - \frac{1}{3}c_2 \\
y[n] &= \frac{9}{8}u[n] - \frac{7}{12}\left(\frac{1}{3}\right)^n - \frac{13}{24}\left(-\frac{1}{3}\right)^n
\end{aligned}
$$

(d.)

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1]; x[n] = (-1)^n u[n]$$

$$r^2 + \frac{1}{4}r - \frac{1}{8} = 0; r = \frac{1}{4}, -\frac{1}{2}$$

$$y^{(n)}[n] = A(\frac{1}{4})^n + B(-\frac{1}{2})^n$$

$$y^{(p)}[n] = k(-1)^n u[n]$$

$$k(-1)^n + \frac{1}{4}k(-1)^{n-1} - \frac{1}{8}k(-1)^{n-2} = (-1)^n + (-1)^{n-1} = 0; k = 0$$

$$y^{(p)}[n] = 0$$

$$y[n] = A(\frac{1}{4})^n + B(-\frac{1}{2})^n$$

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n] + x[n-1]$$

$$y[0] = -\frac{1}{4}y[-1] + \frac{1}{8}y[-2] + x[0] + x[-1] = -\frac{1}{4} = A + B$$

$$y[1] = -\frac{1}{4}y[0] + \frac{1}{8}y[-1] + x[1] + x[0] = -\frac{9}{16} = \frac{A}{4} - \frac{B}{2}$$

$$A = \frac{7}{12}; B = -\frac{5}{6}$$

$$y[n] = \frac{7}{12}(\frac{1}{4})^n - \frac{5}{6}(-\frac{1}{2})^n$$

(3.)(a)

(i) Natural Response

$$\begin{aligned}
y^{(n)}(t) &= c_1 e^{-4t} + c_2 e^{-t} \\
y(0^-) = 0 &= c_1 + c_2 \\
\left.\frac{d}{dt}y(t)\right|_{t=0^-} = 1 &= -4c_1 - c_2 \\
y^{(n)}(t) &= -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t}
\end{aligned}$$

(ii) Forced Response

$$\begin{aligned}
y^{(f)}(t) &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) + c_1 e^{-4t} + c_2 e^{-t} \\
y(0) = 0 &= \frac{3}{34} + c_1 + c_2 \\
\left.\frac{d}{dt}y(t)\right|_{t=0^-} = 0 &= \frac{5}{34} - 4c_1 - c_2 \\
y^{(f)}(t) &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) + \frac{4}{51}e^{-4t} - \frac{1}{6}e^{-t}
\end{aligned}$$

(b.)

(i) Natural Response

$$\begin{aligned}
y^{(n)}[n] &= c_1\left(\frac{1}{2}\right)^n u[n] + c_2\left(\frac{1}{4}\right)^n u[n] \\
y[-2] = -1 &= c_1\left(\frac{1}{2}\right)^{-2} + c_2\left(\frac{1}{4}\right)^{-2} \\
y[-1] = 1 &= c_1\left(\frac{1}{2}\right)^{-1} + c_2\left(\frac{1}{4}\right)^{-1} \\
y^{(n)}[n] &= \frac{5}{4}\left(\frac{1}{2}\right)^n u[n] - \frac{3}{8}\left(\frac{1}{4}\right)^n u[n]
\end{aligned}$$

(ii) Forced Response

$$
\begin{aligned}
y^{(f)}[n] &= \frac{32}{3}u[n] + c_1\left(\frac{1}{2}\right)^n u[n] + c_2\left(\frac{1}{4}\right)^n u[n] \\
& \quad \text{Translate initial conditions} \\
y[n] &= \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + 2x[n] \\
y[0] &= \frac{3}{4}0 - \frac{1}{8}0 + 2(2) = 4 \\
y[1] &= \frac{3}{4}4 - \frac{1}{8}0 + 2(2) = 7 \\
y[0] = 4 &= \frac{32}{3} + c_1 + c_2 \\
y[1] = 7 &= \frac{32}{3} + \frac{1}{2}c_1 + \frac{1}{4}c_2 \\
y^{(f)}[n] &= \frac{32}{3}u[n] - 8\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}\left(\frac{1}{4}\right)^n u[n]
\end{aligned}
$$