Homework No. 7 Solution

1.

(1) $x(t)=e^{-t}u(t+4)$

$$X(s)=\int_{-\infty}^{\infty}e^{-t}u(t+4)e^{-st}dt=\int_{-4}^{\infty}e^{-t}e^{-st}dt$$
$$=\int_{-4}^{\infty}e^{-t(1+s)}dt=\left.\frac{-e^{-t(1+s)}}{1+s}\right|_{-4}^{\infty}=\frac{e^{4(1+s)}}{1+s},\quad \mathrm{Re}\{s+1\}>0\Rightarrow \mathrm{ROC}:\mathrm{Re}\{s\}>-1$$

(2) $x(t)=\sin(t)u(t)$

$$X(s)=\int_{0}^{\infty}\frac{1}{2j}\left(e^{jt}-e^{-jt}\right)e^{-st}dt=\int_{0}^{\infty}\frac{1}{2j}e^{t(j-s)}dt-\int_{0}^{\infty}\frac{1}{2j}e^{-t(j+s)}dt$$
$$=\frac{1}{2j}\left(\frac{-1}{j-s}-\frac{1}{j+s}\right)=\frac{1}{1+s^2}$$
$$\mathrm{Re}\{j-s\}<0 \text{ and } \mathrm{Re}\{j+s\}>0\Rightarrow \mathrm{ROC}:\mathrm{Re}\{s\}>0$$

2.

(1) $X(s)=e^{5s}\dfrac{1}{s+2}$ with ROC $\mathrm{Re}\{s\}<-2$

$$\underbrace{A(s)=\frac{1}{s+2}}_{\text{left-sided}}\overset{\mathcal{L}}{\longleftrightarrow}a(t)=-e^{-2t}u(-t)$$
$$X(s)=e^{5s}A(s)\overset{\mathcal{L}}{\longleftrightarrow}x(t)=a(t+5)=-e^{-2(t+5)}u(-t-5)$$

(2) $X(s)=s^{-1}\dfrac{d}{ds}\left(\dfrac{e^{-3s}}{s}\right)$ with ROC $\mathrm{Re}\{s\}>0$

$$\underbrace{A(s)=\frac{1}{s}}_{\text{right-sided}}\overset{\mathcal{L}}{\longleftrightarrow}a(t)=u(t)$$
$$B(s)=e^{-3s}A(s)\overset{\mathcal{L}}{\longleftrightarrow}b(t)=a(t-3)=u(t-3)$$
$$C(s)=\frac{d}{ds}B(s)\overset{\mathcal{L}}{\longleftrightarrow}c(t)=-tb(t)=-tu(t-3)$$
$$X(s)=\frac{1}{s}C(s)\overset{\mathcal{L}}{\longleftrightarrow}\begin{aligned}x(t)&=u(t)*c(t)=\int_{-\infty}^{t}c(\tau)d\tau=-\int_{3}^{t}\tau d\tau\\&=-\frac{1}{2}\left(t^2-9\right)u(t-3)\quad\left(\because u(t)*c(t),\ 3\le t<\infty\right)\end{aligned}$$

3. $X(s)=\dfrac{-s-4}{s^2+3s+2}=\dfrac{-3}{s+1}+\dfrac{2}{s+2}$

(1) With ROC $\operatorname{Re}\{s\}<-2$

Left-sided: $x(t)=\left(3e^{-t}-2e^{-2t}\right)u(-t)$

(2) With ROC $\operatorname{Re}\{s\}>-1$

Right-sided: $x(t)=\left(-3e^{-t}+2e^{-2t}\right)u(t)$

(3) With ROC $-2<\operatorname{Re}\{s\}<-1$

Two-sided: $x(t)=3e^{-t}u(-t)+2e^{-2t}u(t)$

4.

(1) $H(s)=\dfrac{2s^2+2s-2}{s^2-1}=2+\dfrac{1}{s+1}+\dfrac{1}{s-1}$

(a) Causal system : $h(t)=2\delta(t)+\left(e^{-t}+e^{t}\right)u(t)$.

(b) Stable system: $h(t)=2\delta(t)+e^{-t}u(t)-e^{t}u(-t)$

(2) $x(t)=e^{-2t}u(t),\ y(t)=-2e^{-t}u(t)+2e^{-3t}u(t)$

$X(s)=\dfrac{1}{s+2};\ Y(s)=\dfrac{-2}{s+1}+\dfrac{2}{s+3}$

$H(s)=Y(s)/X(s)=\dfrac{-4}{(s+1)(s+3)}\Big/\dfrac{1}{s+2}=\dfrac{-4(s+2)}{(s+1)(s+3)}=\dfrac{-2}{s+1}+\dfrac{-2}{s+3}$

$h(t)=\left(-2e^{-t}-2e^{-3t}\right)u(t)$

($\because$ For a stable system, the ROC must include the $j\omega$–axis.)

5.

(1) $x(t)=u(t)-u(t-6)\Rightarrow X(s)=\int_{0^-}^{6}e^{-st}dt=\dfrac{1-e^{-6s}}{s}$

(2)

$$X(s)=\int_{0^-}^{1}\frac{1}{2j}\left(e^{j\pi t}-e^{-j\pi t}\right)e^{-st}dt$$

$$=\frac{1}{2j}\left[\frac{1}{j\pi-s}\left(e^{j\pi-s}-1\right)-\frac{1}{-j\pi-s}\left(e^{-j\pi-s}-1\right)\right]$$

$$=\frac{1}{2j}\left[\frac{1}{j\pi-s}\left(-e^{-s}-1\right)+\frac{1}{j\pi+s}\left(-e^{-s}-1\right)\right]$$

$$=\frac{1}{2j}\frac{\left(-e^{-s}-1\right)\left(j\pi+s+j\pi-s\right)}{-\pi^2-s^2}=\frac{\pi\left(1+e^{-s}\right)}{s^2+\pi^2}$$

6.

(1) $x(t-1)\overset{\mathcal{L}}{\longleftrightarrow}e^{-1s}X(s)=e^{-1s}\dfrac{2s}{s^2+2}$

(2) $x(t)*\dfrac{d}{dt}x(t)$

$$b(t)=\frac{d}{dt}x(t)\overset{\mathcal{L}}{\longleftrightarrow}B(s)=sX(s)$$

$$y(t)=x(t)*b(t)\overset{\mathcal{L}}{\longleftrightarrow}Y(s)=X(s)B(s)=sX^2(s)=s\left(\frac{2s}{s^2+2}\right)^2$$

(3) $e^{-3t}x(t)\overset{\mathcal{L}}{\longleftrightarrow}X(s+3)=\dfrac{2(s+3)}{(s+3)^2+2}$

(4) $\int_0^t x(3\tau)d\tau\overset{\mathcal{L}}{\longleftrightarrow}Y(s)=\dfrac{X(s/3)}{3s}=\dfrac{2}{s^2+18}$