Homework No. 8 Solution

1. (5%)

$$\int_{-\infty}^{\infty} e^{-10|t|}e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{0} e^{10t}e^{-(\sigma+j\omega)t}dt + \int_{0}^{\infty} e^{-10t}e^{-(\sigma+j\omega)t}dt$$
$$= \int_{-\infty}^{0} e^{-(-10+\sigma)t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-(10+\sigma)t}e^{-j\omega t}dt$$

The first integral converges for $-10+\sigma<0 \Rightarrow \sigma<10$. The second integral converges if $10+\sigma>0 \Rightarrow \sigma>-10$. Therefore, the given integral converges when $|\sigma|<10$.

2.

(1) $x(t)=e^{-t}u(t+3)$ (7%)

$$X(s)=\int_{-\infty}^{\infty} e^{-t}u(t+3)e^{-st}dt = \int_{-3}^{\infty} e^{-t}e^{-st}dt$$
$$= \int_{-3}^{\infty} e^{-t(1+s)}dt = \left.\frac{-e^{-t(1+s)}}{1+s}\right|_{-3}^{\infty} = \frac{e^{3(1+s)}}{1+s},\ \mathrm{Re}\{s+1\}>0 \Rightarrow \mathrm{ROC}:\mathrm{Re}\{s\}>-1$$

(2) $x(t)=\sin(t)u(t)$ (8%)

$$X(s)=\int_{0}^{\infty} \frac{1}{2j}\left(e^{jt}-e^{-jt}\right)e^{-st}dt = \int_{0}^{\infty} \frac{1}{2j}e^{t(j-s)}dt - \int_{0}^{\infty} \frac{1}{2j}e^{-t(j+s)}dt$$
$$= \frac{1}{2j}\left(\frac{-1}{j-s}-\frac{1}{j+s}\right) = \frac{1}{1+s^2}$$

$\mathrm{Re}\{j-s\}<0$ and $\mathrm{Re}\{j+s\}>0 \Rightarrow \mathrm{ROC}:\mathrm{Re}\{s\}>0$

3.

(1) $X(s)=e^{5s}\dfrac{1}{s+3}$ with ROC $\mathrm{Re}\{s\}<-3$ (7%)

$$\underset{\text{left-sided}}{A(s)=\frac{1}{s+3}} \xleftrightarrow{\mathcal{L}} a(t)=-e^{-3t}u(-t)$$
$$X(s)=e^{5s}A(s) \xleftrightarrow{\mathcal{L}} x(t)=a(t+5)=-e^{-3(t+5)}u(-t-5)$$

(2) $X(s)=s^{-1}\dfrac{d}{ds}\left(\dfrac{e^{-3s}}{s}\right)$ with ROC $\mathrm{Re}\{s\}>0$ (8%)

$$\underset{\text{right-sided}}{A(s)=\frac{1}{s}} \xleftrightarrow{\mathcal{L}} a(t)=u(t)$$
$$B(s)=e^{-3s}A(s) \xleftrightarrow{\mathcal{L}} b(t)=a(t-3)=u(t-3)$$

$$C(s)=\frac{d}{ds}B(s)\overset{\mathcal{L}}{\longleftrightarrow}c(t)=-tb(t)=-tu(t-3)$$

$$X(s)=\frac{1}{s}C(s)\overset{\mathcal{L}}{\longleftrightarrow}\begin{aligned}x(t)&=u(t)*c(t)=\int_{-\infty}^{t}c(\tau)d\tau=-\int_{3}^{t}\tau d\tau\\&=-\frac{1}{2}(t^2-9)u(t-3)\quad(\because u(t)*c(t),\ 3\le t<\infty)\end{aligned}$$

4. $X(s)=\dfrac{2s^2+2s-2}{s^2-1}=\dfrac{2s^2-2}{s^2-1}+\dfrac{2s}{s^2-1}=2+\dfrac{1}{s-1}+\dfrac{1}{s+1}$

(1) With ROC $\mathrm{Re}\{s\}<-1$ (5%): Left-sided: $x(t)=2\delta(t)-(e^t+e^{-t})u(-t)$

(2) With ROC $\mathrm{Re}\{s\}>1$ (5%): Right-sided: $x(t)=2\delta(t)+(e^t+e^{-t})u(t)$

(3) With ROC $-1<\mathrm{Re}\{s\}<1$ (5%):

Two-sided: $x(t)=2\delta(t)-e^t u(-t)+e^{-t}u(t)$

5.

(1) (8%) $H(s)=\dfrac{2s-1}{s^2+2s+1}=\dfrac{2}{s+1}+\dfrac{-3}{(s+1)^2}$, $\left(\dfrac{d}{ds}\left(\dfrac{1}{s+1}\right)=\dfrac{-1}{(s+1)^2}\right)$

(a) Causal system (right-sided system): $h(t)=(2e^{-t}-3te^{-t})u(t)$.

(b) Stable system (The ROC must include the $j\omega$–axis.):

$$h(t)=(2e^{-t}-3te^{-t})u(t)$$

(2) (7%) $x(t)=e^{-2t}u(t),\ y(t)=-2e^{-t}u(t)+2e^{-3t}u(t)$

$$X(s)=\frac{1}{s+2};\ Y(s)=\frac{-2}{s+1}+\frac{2}{s+3}$$

$$H(s)=Y(s)/X(s)=\frac{-4}{(s+1)(s+3)}\Big/\frac{1}{s+2}=\frac{-4(s+2)}{(s+1)(s+3)}=\frac{-2}{s+1}+\frac{-2}{s+3}$$

$$h(t)=(-2e^{-t}-2e^{-3t})u(t)$$

($\because$ For a stable system, the ROC must include the $j\omega$–axis.)

6.

(1) (7%) $x(t)=u(t)-u(t-10)\Rightarrow X(s)=\int_{0^-}^{10}e^{-st}dt=\dfrac{1-e^{-10s}}{s}$

(2) $x(t)=\begin{cases}\sin(\pi t), & 0<t<1\\ 0, & \text{otherwise}\end{cases}$ (8%)

$$X(s) = \int_{0^-}^{1} \frac{1}{2j}\left(e^{j\pi t} - e^{-j\pi t}\right)e^{-st}dt$$
$$= \frac{1}{2j}\left[\frac{1}{j\pi - s}\left(e^{j\pi - s} - 1\right) - \frac{1}{-j\pi - s}\left(e^{-j\pi - s} - 1\right)\right]$$
$$= \frac{1}{2j}\left[\frac{1}{j\pi - s}\left(-e^{-s} - 1\right) + \frac{1}{j\pi + s}\left(-e^{-s} - 1\right)\right]$$
$$= \frac{1}{2j}\frac{\left(-e^{-s} - 1\right)\left(j\pi + s + j\pi - s\right)}{-\pi^2 - s^2} = \frac{\pi\left(1 + e^{-s}\right)}{s^2 + \pi^2}$$

7. (20%)

(1) $x(t-3) \overset{\mathcal{L}}{\longleftrightarrow} e^{-3s}X(s) = e^{-3s}\frac{2s}{s^2+2}$

(2) $x(3t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{3}X\left(\frac{s}{3}\right) = \frac{1}{3} \times \frac{2\frac{s}{3}}{\frac{s^2}{9}+2} = \frac{2s}{s^2+18}$

(3) $x(t) * \frac{d}{dt}x(t)$

$b(t) = \frac{d}{dt}x(t) \overset{\mathcal{L}}{\longleftrightarrow} B(s) = sX(s)$

$y(t) = x(t) * b(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = X(s)B(s) = sX^2(s) = s\left(\frac{2s}{s^2+2}\right)^2$

(4) $e^{-2t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s+2) = \frac{2(s+2)}{(s+2)^2+2}$

(5) $\int_0^t x(3\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = \frac{X(s/3)}{3s} = \frac{2}{s^2+18}$

(From (2), $x(3t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{3}X\left(\frac{s}{3}\right)$.)