Homework No. 7 Solution

1.

(1) (8%)

$$x[n]=\cos^2\left(\frac{6\pi}{17}n+\frac{\pi}{3}\right)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)=x_1[n]+x_2[n]$$

$$\left.\begin{aligned}&x_1[n]=\frac{1}{2}\Rightarrow N_1=1\\&x_2[n]=\frac{1}{2}\cos\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)\Rightarrow N_2=2\pi\Big/\frac{12\pi}{17}=\frac{17}{6}\end{aligned}\right\}\Rightarrow\therefore N=17$$

$$x[n]=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)=\frac{1}{2}+\frac{1}{4}\left[e^{j\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)}+e^{-j\left(\frac{12\pi}{17}n+\frac{2\pi}{3}\right)}\right]$$

$$=\frac{1}{2}+\frac{1}{4}\left[e^{j\frac{2\pi}{3}}e^{j6\frac{2\pi}{17}n}+e^{-j\frac{2\pi}{3}}e^{-j6\frac{2\pi}{17}n}\right]$$

$$X[k]=a_k=\begin{cases}\frac{1}{2},\ k=0\\ \frac{1}{4}e^{j\frac{2\pi}{3}},\ k=6\\ \frac{1}{4}e^{-j\frac{2\pi}{3}},\ k=-6\\ 0,\ \text{otherwise on } k=\{-8,-7,\ldots,8\}\end{cases}$$

(2) $x[n] = x[n + 8]$. (7%)

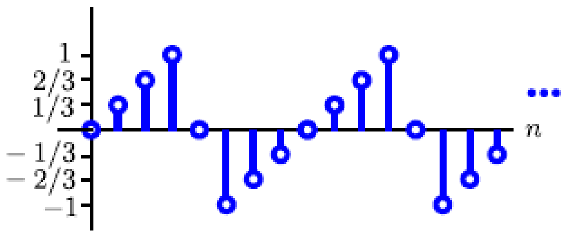

Figure 1

$$X[k]=a_k=\frac{1}{8}\sum_{n=5}^{11}\frac{1}{3}(n-8)e^{-j\frac{2\pi}{8}kn}$$

$$=\frac{1}{8}\sum_{n=-3}^{3}\frac{1}{3}ne^{-j\frac{\pi}{4}kn}$$

$$=\frac{-1}{8}e^{j\frac{3\pi}{4}k}+\frac{-1}{12}e^{j\frac{\pi}{2}k}+\frac{-1}{24}e^{j\frac{\pi}{4}k}+\frac{1}{24}e^{-j\frac{\pi}{4}k}+\frac{1}{12}e^{-j\frac{\pi}{2}k}+\frac{1}{8}e^{-j\frac{3\pi}{4}k}$$

$$=-\frac{1}{4}j\sin\left(\frac{3\pi}{4}k\right)-\frac{1}{6}j\sin\left(\frac{\pi}{2}k\right)-\frac{1}{12}j\sin\left(\frac{\pi}{4}k\right)$$

2. (8%)

$$X[k] = a_k = 2\sin\left(\frac{14\pi k}{19}\right) + \cos\left(\frac{10\pi}{19}k\right) + 1$$
$$= -j\left(e^{j\frac{14\pi k}{19}} - e^{-j\frac{14\pi k}{19}}\right) + \frac{1}{2}\left(e^{j\frac{10\pi}{19}k} + e^{-j\frac{10\pi}{19}k}\right) + 1$$
$$= -j\left(e^{j7\frac{2\pi k}{19}} - e^{-j7\frac{2\pi k}{19}}\right) + \frac{1}{2}\left(e^{j5\frac{2\pi}{19}k} + e^{-j5\frac{2\pi}{19}k}\right) + 1$$

$$x[n] = \begin{cases} -j,\ n = 7 \\ j,\ n = -7 \\ 1/2,\ n = \pm 5 \\ 1,\ n = 0 \\ 0,\ \text{otherwise on } \{-9,-8,\ldots,9\} \end{cases}$$

3.

(1) $x[n] = \left(\frac{2}{5}\right)^n u[n+4]$. (5%)

$$X(\Omega) = \sum_{n=-\infty}^{\infty}\left(\frac{2}{5}\right)^n u[n+4]e^{-j\Omega n} = \sum_{n=-4}^{\infty}\left(\frac{2}{5}\right)^n e^{-j\Omega n}$$
$$= \sum_{n=-4}^{-1}\left(\frac{2}{5}\right)^n e^{-j\Omega n} + \sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n e^{-j\Omega n}$$
$$= \left(\frac{2}{5}\right)^{-4} e^{j4\Omega} + \left(\frac{2}{5}\right)^{-3} e^{j3\Omega} + \left(\frac{2}{5}\right)^{-2} e^{j2\Omega} + \left(\frac{2}{5}\right)^{-1} e^{j\Omega} + \frac{1}{1-\frac{2}{5}e^{-j\Omega}}$$
$$= \left(\frac{2}{5}\right)^{-1} e^{j\Omega}\left[1 + \left(\frac{2}{5}\right)^{-1} e^{j\Omega} + \left(\frac{2}{5}\right)^{-2} e^{j2\Omega} + \left(\frac{2}{5}\right)^{-3} e^{j3\Omega}\right] + \frac{1}{1-\frac{2}{5}e^{-j\Omega}}$$
$$= \left(\frac{2}{5}\right)^{-1} e^{j\Omega}\frac{1-\left(\frac{2}{5}\right)^{-4} e^{j4\Omega}}{1-\left(\frac{2}{5}\right)^{-1} e^{j\Omega}} + \frac{1}{1-\frac{2}{5}e^{-j\Omega}}$$

(2) $x[n] = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{N}n\right), & |n| \le N \\ 0, & \text{otherwise} \end{cases}$. (7%)

$$X(\Omega) = \sum_{n=-N}^{N} \left\{ \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{N}n\right) \right\} e^{-j\Omega n}$$
$$= \frac{1}{2}\sum_{n=-N}^{N} \left\{ 1 + \frac{1}{2}(e^{j\frac{\pi}{N}n} + e^{-j\frac{\pi}{N}n}) \right\} e^{-j\Omega n}$$
$$= \frac{1}{2}\cdot\frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)} + \frac{1}{4}\cdot\frac{\sin\left(\frac{2N+1}{2}\left(\Omega-\frac{\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega-\frac{\pi}{N}\right)\right)}$$
$$+\frac{1}{4}\cdot\frac{\sin\left(\frac{2N+1}{2}\left(\Omega+\frac{\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega+\frac{\pi}{N}\right)\right)}$$

$$\left(\begin{aligned}
&\sum_{n=-N}^{N} e^{-j\Omega n} \underset{m=n+N}{=} \sum_{m=0}^{2N} e^{-j\Omega(m-N)} = e^{j\Omega N}\sum_{m=0}^{2N} e^{-j\Omega m} \\
&= e^{j\Omega N}\frac{1-e^{-j\Omega(2N+1)}}{1-e^{-j\Omega}} = e^{j\Omega N}\frac{e^{-j\Omega\left(\frac{2N+1}{2}\right)}\left(e^{j\Omega\left(\frac{2N+1}{2}\right)} - e^{-j\Omega\left(\frac{2N+1}{2}\right)}\right)}{e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right)} \\
&= e^{j\Omega N - j\Omega\left(\frac{2N+1}{2}\right) + j\frac{\Omega}{2}}\frac{\sin\left(\Omega\left(\frac{2N+1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)} = \frac{\sin\left(\Omega\left(\frac{2N+1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)}
\end{aligned}\right)$$

4.

(1) $\left|X(\Omega)\right| = \begin{cases} 1, & \pi/4 < |\Omega| < 3\pi/4 \\ 0, & \text{otherwise} \end{cases}$, $\arg\{X(\Omega)\} = -4\Omega$. (5%)

$$x[n] = \frac{1}{2\pi}\int_{0.25\pi}^{0.75\pi} e^{j(n-4)\Omega}d\Omega + \frac{1}{2\pi}\int_{-0.75\pi}^{-0.25\pi} e^{j(n-4)\Omega}d\Omega]$$
$$= \frac{1}{2\pi}\cdot\frac{e^{j0.75\pi(n-4)} - e^{j0.25\pi(n-4)} + e^{-j0.25\pi(n-4)} - e^{-j0.75\pi(n-4)}}{j(n-4)}$$
$$= \frac{\sin(0.75\pi(n-4)) - \sin(0.25\pi(n-4))}{\pi(n-4)}$$

(2) $X(\Omega) = \sin\left(\frac{\Omega}{2}\right) + \cos(\Omega)$. (5%)

$$X(\Omega)=\sin\left(\frac{\Omega}{2}\right)+\cos(\Omega)$$

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{j0.5\Omega}-e^{-j0.5\Omega}}{2j}+\frac{e^{j\Omega}+e^{-j\Omega}}{2}\right)e^{j\Omega n}d\Omega$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{j0.5\Omega}-e^{-j0.5\Omega}}{2j}e^{j\Omega n}d\Omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{j(n+0.5)\Omega}-e^{-j(0.5-n)\Omega}}{2j}\right)d\Omega$$

$$=\frac{1}{2\pi}\cdot\left(\frac{1}{j(n+0.5)}\cdot\frac{e^{j(n+0.5)\Omega}}{2j}\Bigg|_{-\pi}^{\pi}+\frac{1}{j(0.5-n)}\cdot\frac{e^{-j(0.5-n)\Omega}}{2j}\Bigg|_{-\pi}^{\pi}\right)$$

$$=\frac{1}{2\pi}\cdot\left(\frac{1}{j(n+0.5)}\cdot\frac{e^{j(n+0.5)\pi}-e^{-j(n+0.5)\pi}}{2j}+\frac{1}{j(0.5-n)}\cdot\frac{e^{-j(0.5-n)\pi}-e^{j(0.5-n)\pi}}{2j}\right)$$

$$=\frac{1}{2\pi}\cdot\left(\frac{1}{j(n+0.5)}\cdot\frac{e^{jn\pi}e^{j0.5\pi}-e^{-jn\pi}e^{-j0.5\pi}}{2j}+\frac{1}{j(0.5-n)}\cdot\frac{e^{jn\pi}e^{-j0.5\pi}-e^{-jn\pi}e^{j0.5\pi}}{2j}\right)$$

$$=\frac{1}{2\pi}\cdot\left(\frac{\cos(n\pi)}{j(n+0.5)}-\frac{\cos(n\pi)}{j(0.5-n)}\right)$$

$$\therefore x[n]=\frac{1}{2\pi}\cdot\left(\frac{\cos(n\pi)}{j(n+0.5)}-\frac{\cos(n\pi)}{j(0.5-n)}\right)+\frac{1}{2}\delta[n+1]+\frac{1}{2}\delta[n-1]$$

5.

(1) $x[n]=(n-2)(u[n+4]-u[n-5])$. (5%)

$$s[n]=u[n+4]-u[n-5]\overset{F}{\longleftrightarrow}S(\Omega)=\frac{\sin(9\Omega/2)}{\sin(\Omega/2)}$$

$$ns[n]\overset{F}{\longleftrightarrow}j\frac{d}{d\Omega}S(\Omega)$$

$$x[n]=(n-2)s[n]\overset{F}{\longleftrightarrow}X(\Omega)=j\frac{d}{d\Omega}\frac{\sin(9\Omega/2)}{\sin(\Omega/2)}-2\frac{\sin(9\Omega/2)}{\sin(\Omega/2)}$$

(2) $x[n]=\cos\left(\frac{\pi}{4}n\right)\left(\frac{1}{2}\right)^n u[n-2]=\cos\left(\frac{\pi}{4}n\right)\frac{1}{4}\left(\frac{1}{2}\right)^{n-2}u[n-2]$. (8%)

$$a[n]=\left(\frac{1}{2}\right)^n u[n]\overset{F}{\longleftrightarrow}A(\Omega)=\frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$b[n]=a[n-2]\overset{F}{\longleftrightarrow}B(\Omega)=e^{-j2\Omega}A(\Omega)$$

$$x[n]=\frac{1}{4}\cos\left(\frac{\pi}{4}n\right)b[n]\xleftrightarrow{F}X(\Omega)=\frac{1}{8}\left\{B\left(\Omega-\frac{\pi}{4}\right)+B\left(\Omega+\frac{\pi}{4}\right)\right\}$$

$$X(\Omega)=\frac{1}{8}\left\{B\left(\Omega-\frac{\pi}{4}\right)+B\left(\Omega+\frac{\pi}{4}\right)\right\}$$

$$=\frac{1}{8}\left\{e^{-j2\left(\Omega-\frac{\pi}{4}\right)}A\left(\Omega-\frac{\pi}{4}\right)+e^{-j2\left(\Omega+\frac{\pi}{4}\right)}A\left(\Omega+\frac{\pi}{4}\right)\right\}$$

$$=\frac{1}{8}\left\{e^{-j2\left(\Omega-\frac{\pi}{4}\right)}\frac{1}{1-\frac{1}{2}e^{-j\left(\Omega-\frac{\pi}{4}\right)}}+e^{-j2\left(\Omega+\frac{\pi}{4}\right)}\frac{1}{1-\frac{1}{2}e^{-j\left(\Omega+\frac{\pi}{4}\right)}}\right\}$$

(3) $X(\Omega)=\left[e^{-j2\Omega}\frac{\sin(15\Omega/2)}{\sin(\Omega/2)}\right]\circledast\left[\frac{\sin(7\Omega/2)}{\sin(\Omega/2)}\right]=A(\Omega)\circledast B(\Omega)$. (7%)

$$X(\Omega)\xleftrightarrow{F}2\pi a[n]b[n]$$

$$a[n]=\begin{cases}1, & |n-2|\le 7\\ 0, & \text{otherwise}\end{cases};\ b[n]=\begin{cases}1, & |n|\le 3\\ 0, & \text{otherwise}\end{cases}$$

$$\therefore x[n]=\begin{cases}2\pi, & |n|\le 3\\ 0, & \text{otherwise}\end{cases}$$

6. Use the duality property to evaluate the DTFS of $\frac{\sin(11\pi n/20)}{\sin(\pi n/20)}$. (8%)

$$\frac{\sin(11\pi n/20)}{\sin(\pi n/20)}=\frac{\sin((2\times 5+1)\pi n/(2\times 10))}{\sin(\pi n/(2\times 10))}$$

$$\Omega_0=\frac{\pi}{10}\Rightarrow N=2\pi\Big/\frac{\pi}{10}=20$$

$$\because\begin{cases}1, & |n|\le 5\\ 0, & 5<|n|\le 10\end{cases}\xleftrightarrow{F}\frac{1}{20}\cdot\frac{\sin(11\pi k/20)}{\sin(\pi k/20)}$$

$$\therefore\frac{1}{20}\cdot\frac{\sin(11\pi n/20)}{\sin(\pi n/20)}\xleftrightarrow{F}X[k]=\frac{1}{20}\begin{cases}1, & |k|\le 5\\ 0, & 5<|k|\le 10\end{cases}=X[k+20i]$$

$$\frac{\sin(11\pi n/20)}{\sin(\pi n/20)}\xleftrightarrow{F}X[k]=\begin{cases}1, & |k|\le 5\\ 0, & 5<|k|\le 10\end{cases}=X[k+20i]$$

where k and i are integers.

7. You are given $x[n]=n(3/4)^{|n|}\xleftrightarrow{DTFT}X(\Omega)$. Without evaluating $X(\Omega)$, find $y[n]$ if

(1) $Y(\Omega)=\mathrm{Im}\left\{X(\Omega)\right\}$. (5%)

Since $x[n]$ is real and odd, $X(\Omega)$ is purely imaginary.

$\therefore y[n]=x[n]$

(2) $Y(\Omega)=\dfrac{d}{d\Omega}\left\{e^{-j4\Omega}\left[X\left(\Omega+\dfrac{\pi}{4}\right)+X\left(\Omega-\dfrac{\pi}{4}\right)\right]\right\}$. (7%)

$$y[n]=-jn\left\{e^{-j\frac{\pi}{4}(n-4)}x[n-4]+e^{j\frac{\pi}{4}(n-4)}x[n-4]\right\}$$
$$=-jn\left\{2\cos\left(\frac{\pi}{4}(n-4)\right)x[n-4]\right\}$$
$$=-jn\left\{2\cos\left(\frac{\pi}{4}(n-4)\right)(n-4)\left(\frac{3}{4}\right)^{|n-4|}\right\}$$
$$\left(\frac{d}{d\Omega}\Rightarrow -jn;\ e^{-j4\Omega}\Rightarrow n-4;\ \Omega+\frac{\pi}{4}\Rightarrow e^{-j\frac{\pi}{4}n};\ \Omega-\frac{\pi}{4}\Rightarrow e^{j\frac{\pi}{4}n}\right)$$

8.

(1) (5%) $y[n]+\frac{1}{2}y[n-1]=x[n]-2x[n-1]$

$$\left(1+\frac{1}{2}e^{-j\Omega}\right)Y(\Omega)=\left(1-2e^{-j\Omega n}\right)X(\Omega)$$
$$H(\Omega)=\frac{Y(\Omega)}{X(\Omega)}=\frac{1-2e^{-j\Omega}}{1+\frac{1}{2}e^{-j\Omega}}$$
$$h[n]=\left(-\frac{1}{2}\right)^{n}u[n]-2\left(-\frac{1}{2}\right)^{n-1}u[n-1]$$

(2) (10%)

(a) $h[n]=\delta[n]+2\left(\dfrac{1}{2}\right)^{n}u[n]+\left(\dfrac{-1}{2}\right)^{n}u[n]$.

$$H(\Omega)=1+\frac{2}{1-\frac{1}{2}e^{-j\Omega}}+\frac{1}{1+\frac{1}{2}e^{-j\Omega}}=\frac{1-\frac{1}{4}e^{-j2\Omega}+2+e^{-j\Omega}+1-\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{4}e^{-j2\Omega}}$$

$$=\frac{4+\frac{1}{2}e^{-j\Omega}-\frac{1}{4}e^{-j2\Omega}}{1-\frac{1}{4}e^{-j2\Omega}}$$

$$Y(\Omega)\left(1-\frac{1}{4}e^{-j2\Omega}\right)=X(\Omega)\left(4+\frac{1}{2}e^{-j\Omega}-\frac{1}{4}e^{-j2\Omega}\right)$$

$$\therefore y[n]-\frac{1}{4}y[n-2]=4x[n]+\frac{1}{2}x[n-1]-\frac{1}{4}x[n-2]$$

(b) $H(\Omega)=1+\dfrac{e^{-j\Omega}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1+\frac{1}{4}e^{-j\Omega}\right)}$.

$$H(\Omega)=1+\frac{e^{-j\Omega}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1+\frac{1}{4}e^{-j\Omega}\right)}$$

$$=\frac{1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}+e^{-j\Omega}}{1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}}=\frac{1+\frac{3}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}}{1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}}$$

$$Y(\Omega)\left(1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}\right)=X(\Omega)\left(1+\frac{3}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}\right)$$

$$y[n]-\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n]+\frac{3}{4}x[n-1]-\frac{1}{8}x[n-2]$$