Homework No. 6 Solution
Due 10:10am, May 9, 2006

3.58 (e) (10%)

$x(t) = \int_{-\infty}^{t} \frac{\sin(2\pi\tau)}{\pi\tau} d\tau$

$$\frac{\sin(2\pi t)}{\pi t} \xleftrightarrow{FT} \begin{cases} 1 & \omega \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{t} s(\tau) d\tau \xleftrightarrow{FT} \frac{S(j\omega)}{j\omega} + \pi S(j0)\delta(\omega)$$

$$X(j\omega) = \begin{cases} \pi\delta(\omega) & \omega = 0 \\ \frac{1}{j\omega} & |\omega| \leq 2\pi,\ \omega \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(f) (10%)

$x(t) = e^{-t+2}u(t-2)$

$$e^{-t}u(t) \xleftrightarrow{FT} \frac{1}{1+j\omega}$$

$$s(t-2) \xleftrightarrow{FT} e^{-j2\omega}S(j\omega)$$

$$X(j\omega) = e^{-j2\omega}\frac{1}{1+j\omega}$$

(g) (10%)

$x(t) = \left(\frac{\sin(t)}{\pi t}\right) * \frac{d}{dt}\left[\left(\frac{\sin(2t)}{\pi t}\right)\right]$

$$\begin{array}{ccc} x(t) = a(t) * b(t) & \stackrel{FT}{\longleftrightarrow} & X(j\omega) = A(j\omega)B(j\omega) \\ \frac{\sin(Wt)}{\pi t} & \stackrel{FT}{\longleftrightarrow} & \begin{cases} 1 & \omega \leq W \\ 0, & \text{otherwise} \end{cases} \\ \frac{d}{dt}s(t) & \stackrel{FT}{\longleftrightarrow} & j\omega S(j\omega) \\ X(j\omega) & = & \begin{cases} j\omega & |\omega| \leq 1 \\ 0, & \text{otherwise} \end{cases} \end{array}$$

3.59 (e) (10%)

$X(j\omega) = \frac{2\sin(\omega)}{\omega(j\omega+2)}$

$$S_1(j\omega) = \frac{2\sin(\omega)}{\omega} \stackrel{FT}{\longleftrightarrow} s_1(t) = \begin{cases} 1 & |t| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{array}{ccc} S_2(j\omega) & = & \frac{1}{(j\omega+2)} \stackrel{FT}{\longleftrightarrow} s_2(t) = e^{-2t}u(t) \\ x(t) & = & s_1(t) * s_2(t) \end{array}$$

$$x(t) = \begin{cases} 0 & t < -1 \\ \frac{1}{2}[1 - e^{-2(t+1)}] & -1 \leq t < \quad 1 \\ \frac{e^{-2t}}{2}[e^2 - e^{-2}] & t \geq 1 \end{cases}$$

(f) (10%)

$X(j\omega) = \frac{4\sin^2(\omega)}{\omega^2}$

$$S(j\omega) = \frac{2\sin(\omega)}{\omega} \quad \overset{FT}{\longleftrightarrow} \quad s(t) = \begin{cases} 1 & |t| \le 1 \\ 0, & \text{otherwise} \end{cases}$$
$$x(t) \quad = \quad s(t) * s(t)$$

$$x(t) = \begin{cases} 2 - |t| & t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

3.68m(a.m) (10%)

$$2\frac{d}{dt}y(t) - 5y(t) = 8x(t)$$
$$\Rightarrow (2j\omega - 5)Y(\omega) = 8X(\omega)$$
$$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{8}{2j\omega - 5}$$

$$\begin{aligned} \Rightarrow h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{8}{2j\omega - 5}e^{j\omega t}d\omega \end{aligned}$$

Using Cauchy Integral Theorem:

$$\oint \frac{f(z)}{z-a}dz = j2\pi f(a)$$

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\oint \frac{8}{2j\omega - 5}e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\oint \frac{4}{j\omega - \frac{5}{2}}e^{j\omega t}d\omega \\ &= \frac{-j}{2\pi}\oint \frac{4}{\omega + j\frac{5}{2}}e^{j\omega t}d\omega \end{aligned}$$

There is a single pole at $\omega = -j\frac{5}{2}$ in the lower-half complex plane.

(i) When $t > 0$, we must choose the integral path in the upper-half complex plane.(See Figure 1)
Because there is no pole $\Rightarrow h(t) = 0$

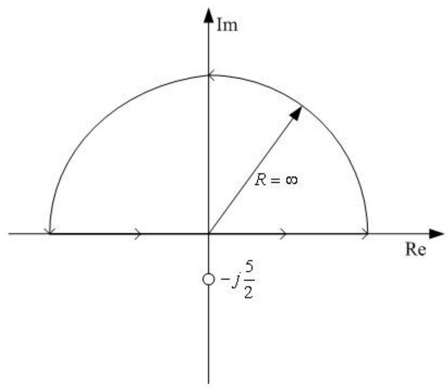

Figure 1: The integral path when $t > 0$

(ii) When $t \leq 0$, we must choose the integral path in the lower-half complex plan, and there is a single pole at $\omega = -j\frac{5}{2}$. (See Figure 2)

$$\begin{aligned} h(t) &= \frac{-j}{2\pi} \oint \frac{4}{\omega + j\frac{5}{2}} e^{j\omega t} d\omega \\ &= \frac{-j}{2\pi} \times (-j2\pi) \times 4e^{j(-j\frac{5}{2})t} \\ &= -4e^{\frac{5}{2}t} \end{aligned}$$

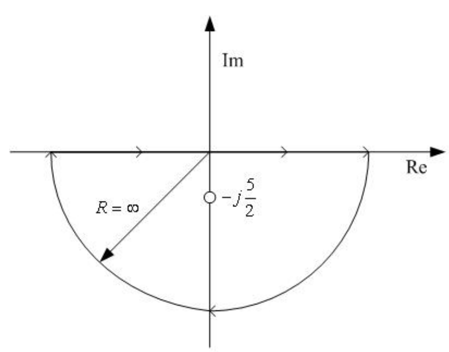

Figure 2: The integral path when $t \leq 0$

With (i) and (ii), we obtain $h(t) = -4e^{\frac{5}{2}t}u(-t)$. Because $h(t)$ has nonzero value when $t \leq 0$, this kind of system is noncausal.

(b.m) (10%)

$$\begin{aligned}
\frac{d^3}{dt^3}y(t) - 3\frac{d}{dt}y(t) - 2y(t) &= 3\frac{d^2}{dt^2}x(t) + 8\frac{d}{dt}x(t) - 10x(t) \\
((j\omega)^3 - 3j\omega - 2)Y(\omega) &= (3(j\omega)^2 + 8j\omega - 10)X(\omega)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow H(\omega) &= \frac{Y(\omega)}{X(\omega)} = \frac{-3\omega^2 + 8j\omega - 10}{-j\omega^3 - 3j\omega - 2} \\
&= \frac{-3\omega^2 + 8j\omega - 10}{(j\omega+1)^2(j\omega-2)} \\
&= \frac{A}{(j\omega+1)^2} + \frac{B}{j\omega+1} + \frac{C}{j\omega-2}
\end{aligned}$$

$$\begin{aligned}
A &= \{\frac{A}{(j\omega+1)^2}(j\omega+1)^2 + \frac{B}{j\omega+1}(j\omega+1)^2 + \frac{C}{j\omega-2}(j\omega+1)^2\}|_{\omega=j} \\
&= \{(j\omega+1)^2 H(\omega)\}|_{\omega=j} \\
&= \{(j\omega+1)^2 \times \frac{-3\omega^2 + 8j\omega - 10}{(j\omega+1)^2(j\omega-2)}\}|_{\omega=j} \\
&= 5
\end{aligned}$$

$$\begin{aligned}
B &= \frac{1}{j}\frac{d}{d\omega}\{\frac{A}{(j\omega+1)^2}(j\omega+1)^2 + \frac{B}{j\omega+1}(j\omega+1)^2 + \frac{C}{j\omega-2}(j\omega+1)^2\}|_{\omega=j} \\
&= \{\frac{1}{j}\frac{d}{d\omega}(j\omega+1)^2 H(\omega)\}|_{\omega=j} \\
&= \{\frac{1}{j}\frac{d}{d\omega}(\frac{-3\omega^2 + 8j\omega - 10}{(j\omega+1)^2(j\omega-2)})\}|_{\omega=j} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
C &= \{(j\omega-2)H(\omega)\}|_{\omega=-2j} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow H(\omega) &= \frac{5}{(j\omega+1)^2} + \frac{1}{j\omega+1} + \frac{2}{j\omega-2} \\
\Rightarrow h(t) &= 5te^{-t}u(t) + e^{-t}u(t) - 2e^{2t}u(-t)
\end{aligned}$$

Because $h(t)$ has nonzero value when $t < 0$, this kind of system is noncausal.

3.75 (d) (10%)

$$x(t) = \frac{\sin(\pi t)}{\pi t} \overset{FT}{\longleftrightarrow} X(j\omega) = \begin{cases} 1 & |\omega| \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \pi \int_{-\infty}^{\infty} \left(\frac{\sin(\pi t)}{\pi t} \right)^2 dt &= \frac{1}{2} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega \\ &= \frac{1}{2} \int_{-\pi}^{\pi} 1 \, d\omega \\ &= \pi \end{aligned}$$

Problem 1. (a) (10%)

$$\begin{aligned} x(t) &= e^{-3t}u(t), h(t) = e^{-2t}u(t) \\ \Rightarrow X(\omega) &= \frac{1}{j\omega + 3} \\ \Rightarrow H(\omega) &= \frac{1}{j\omega + 2} \\ \Rightarrow Y(\omega) &= X(\omega)H(\omega) \\ &= \frac{1}{(j\omega + 3)(j\omega + 2)} \\ &= \frac{-1}{j\omega + 3} + \frac{1}{j\omega + 1} \\ \Rightarrow y(t) &= (e^{-2t} - e^{-3t})u(t) \end{aligned}$$

(b) (10%)

$$\begin{aligned} x(t) &= e^{-4t}u(t), h(t) = e^{-4t}u(t) \\ \Rightarrow X(\omega) &= \frac{1}{j\omega + 4} \\ \Rightarrow H(\omega) &= \frac{1}{j\omega + 4} \\ \Rightarrow Y(\omega) &= X(\omega)H(\omega) \\ &= \frac{1}{(j\omega + 4)^2} \\ \Rightarrow y(t) &= te^{-4t}u(t) \end{aligned}$$