Homework No. 7 Solution

1. (Textbook 6.27(a) and (d)) Determine the **bilateral** Laplace transform and ROC for the following signals:

(1) $x(t)=e^{-t}u(t+2)$ (7%)

$$X(s)=\int_{-\infty}^{\infty} e^{-t}u(t+2)e^{-st}dt$$
$$=\int_{-2}^{\infty} e^{-t}e^{-st}dt=\int_{-2}^{\infty} e^{-t(1+s)}dt$$
$$=\frac{e^{2(1+s)}}{1+s},\ \operatorname{Re}\{s+1\}>0 \Rightarrow \text{ROC}:\operatorname{Re}\{s\}>-1$$

(2) $x(t)=\sin(t)u(t)$ (8%)

$$X(s)=\int_{0}^{\infty}\frac{1}{2j}\left(e^{jt}-e^{-jt}\right)e^{-st}dt$$
$$=\int_{0}^{\infty}\frac{1}{2j}e^{t(j-s)}dt-\int_{0}^{\infty}\frac{1}{2j}e^{-t(j+s)}dt$$
$$=\frac{1}{2j}\left(\frac{-1}{j-s}-\frac{1}{j+s}\right)=\frac{1}{1+s^2}$$
$$\operatorname{Re}\{j-s\}<0 \text{ and } \operatorname{Re}\{j+s\}>0 \Rightarrow \text{ROC}:\operatorname{Re}\{s\}>0$$

2. (Textbook 6.42(a) and (d)) Use the tables of transforms and properties to determine the time signals that correspond to the following **bilateral** Laplace transforms:

(1) $X(s)=e^{5s}\dfrac{1}{s+1}$ with ROC $\operatorname{Re}\{s\}<-2$ (7%)

題目打錯，此小題送分！以下為正確題目的解：

$X(s)=e^{5s}\dfrac{1}{s+2}$ with ROC $\operatorname{Re}\{s\}<-2$

$$\underset{\text{left-sided}}{A(s)=\frac{1}{s+2}}\xleftrightarrow{\mathcal{L}}a(t)=-e^{-2t}u(-t)$$
$$X(s)=e^{5s}A(s)\xleftrightarrow{\mathcal{L}}x(t)=a(t+5)=-e^{-2(t+5)}u(-t-5)$$

(2) $X(s)=s^{-2}\dfrac{d}{ds}\left(\dfrac{e^{-3s}}{s}\right)$ with ROC $\operatorname{Re}\{s\}>0$ (8%)

$$A(s)=\frac{1}{s}\overset{\mathcal{L}}{\longleftrightarrow}a(t)=u(t)$$

right-sided

$$B(s)=e^{-3s}A(s)\overset{\mathcal{L}}{\longleftrightarrow}b(t)=a(t-3)=u(t-3)$$

$$C(s)=\frac{d}{ds}B(s)\overset{\mathcal{L}}{\longleftrightarrow}c(t)=-tb(t)=-tu(t-3)$$

$$D(s)=\frac{1}{s}C(s)\overset{\mathcal{L}}{\longleftrightarrow}\begin{array}{l}d(t)=\int_{-\infty}^{t}c(\tau)d\tau=-\int_{3}^{t}\tau d\tau=-\frac{1}{2}(t^2-9)u(t-3)\\(\because u(t)*c(t),\ 3\le t<\infty)\end{array}$$

$$x(t)=\int_{-\infty}^{t}d(\tau)d\tau=-\frac{1}{2}\int_{-\infty}^{t}(\tau^2-9)u(\tau-3)d\tau$$

$$X(s)=\frac{1}{s}D(s)\overset{\mathcal{L}}{\longleftrightarrow}\quad=-\frac{1}{2}\int_{3}^{t}(\tau^2-9)d\tau$$

$$=\left[-\frac{1}{6}(t^3-27)+\frac{9}{2}(t-3)\right]u(t-3)$$

3. (Textbook 6.43(a)) Use the method of partial fractions to determine the time signals corresponding to the following **bilateral** Laplace transform:

$$X(s)=\frac{-s-4}{s^2+3s+2}=\frac{-3}{s+1}+\frac{2}{s+2}$$

(1) With ROC $\mathrm{Re}\{s\}<-2$ (5%)

Left-sided: $x(t)=(3e^{-t}-2e^{-2t})u(-t)$

(2) With ROC $\mathrm{Re}\{s\}>-1$ (5%)

Right-sided: $x(t)=(-3e^{-t}+2e^{-2t})u(t)$

(3) With ROC $-2<\mathrm{Re}\{s\}<-1$ (5%)

Two-sided: $x(t)=3e^{-t}u(-t)+2e^{-2t}u(t)$

4. (Textbook 6.45(a) and 6.46(b))

(1) A system has the indicated transfer function $H(s)$. Determine the impulse response, assuming (a) that the system is causal and (b) that the system is stable. (10%)

$$H(s)=\frac{2s^2+2s-2}{s^2-1}=2+\frac{1}{s+1}+\frac{1}{s-1}$$

(a) Causal system

$$h(t)=2\delta(t)+(e^{-t}+e^{t})u(t)$$

(b) Stable system

$$h(t)=2\delta(t)+e^{-t}u(t)-e^{t}u(-t)$$

(2) A stable system has the indicated input $x(t)$ and output $y(t)$. Use Laplace transforms to determine the transfer function and impulse response of the system. (10%)

$$x(t)=e^{-2t}u(t),\ y(t)=-2e^{-t}u(t)+2e^{-3t}u(t)$$

$$X(s)=\frac{1}{s+2}$$

$$Y(s)=\frac{-2}{s+1}+\frac{2}{s+3}$$

$$H(s)=Y(s)/X(s)=\frac{-4(s+2)}{(s+1)(s+3)}=\frac{-2}{s+1}+\frac{-2}{s+3}$$

$$h(t)=\left(-2e^{-t}-2e^{-3t}\right)u(t)$$

5. (Textbook 6.28(f) and (g)) Determine the **unilateral** Laplace transform of the following signals, using the defining equation:

(1) $x(t)=u(t)-u(t-2)$ (7%)

$$X(s)=\int_{0^-}^{2}e^{-st}dt=\frac{1-e^{-2s}}{s}$$

(2) $x(t)=\begin{cases}\sin(\pi t), & 0<t<1\\ 0, & \text{otherwise}\end{cases}$ (8%)

$$X(s)=\int_{0^-}^{1}\frac{1}{2j}\left(e^{j\pi t}-e^{-j\pi t}\right)e^{-st}dt=\frac{\pi\left(1+e^{-s}\right)}{s^2+\pi^2}$$

6. (Textbook 6.32(b), (c), (d), and (f)) Given the transform pair $x(t)\xleftrightarrow{\mathcal{L}}\frac{2s}{s^2+2}$, where $x(t)=0$ for $t<0$, determine the Laplace transform of the following time signals: (20%)

(1) $x(t-2)\xleftrightarrow{\mathcal{L}}e^{-2s}X(s)=e^{-2s}\frac{2s}{s^2+2}$

(2) $x(t)*\frac{d}{dt}x(t)$

$$b(t)=\frac{d}{dt}x(t)\xleftrightarrow{\mathcal{L}}B(s)=sX(s)$$

$$y(t)=x(t)*b(t)\xleftrightarrow{\mathcal{L}}Y(s)=X(s)B(s)=sX^2(s)=s\left(\frac{2s}{s^2+2}\right)^2$$

(3) $e^{-t}x(t) \xleftrightarrow{\mathcal{L}} X(s+1) = \dfrac{2(s+1)}{(s+1)^2+2}$

(4) $\int_0^t x(3\tau)\,d\tau \xleftrightarrow{\mathcal{L}} Y(s) = \dfrac{X(s/3)}{3s} = \dfrac{2}{s^2+18}$