Homework No. 6 Solution

1. (Textbook 3.48(b) and (c))

(1) $x[n] = 2\sin\left(\frac{14\pi}{19}n\right) + \cos\left(\frac{10\pi}{19}n\right) + 1$ (7%)

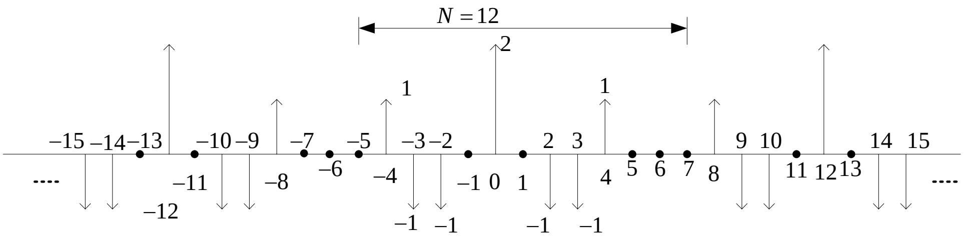
$$N = 19,\ \Omega_0 = \frac{2\pi}{19},$$

$$\begin{aligned} x[n] &= 2\sin\left(\frac{14\pi}{19}n\right) + \cos\left(\frac{10\pi}{19}n\right) + 1 \\ &= \frac{1}{j}\left(e^{j\frac{14\pi}{19}n} - e^{-j\frac{14\pi}{19}n}\right) + \frac{1}{2}\left(e^{j\frac{10\pi}{19}n} + e^{-j\frac{10\pi}{19}n}\right) + 1 \\ &= -je^{j(7)\frac{2\pi}{19}n} + je^{j(-7)\frac{2\pi}{19}n} + \frac{1}{2}e^{j(5)\frac{2\pi}{19}n} + \frac{1}{2}e^{j(-5)\frac{2\pi}{19}n} + e^{j(0)\frac{2\pi}{19}n} \end{aligned}$$

By inspection,

$$X[k] = \begin{cases} j, & k = -7 \\ 1/2, & k = -5 \\ 1, & k = 0 \\ 1/2, & k = 5 \\ -j, & k = 7 \\ 0, & \text{otherwise on } \{-9, -8, \ldots, 9\} \end{cases}$$

(2) $x[n] = \sum_{m=-\infty}^{\infty} (-1)^m \left(\delta[n-2m] + \delta[n+3m]\right)$ (8%)



$$N = 12,\ \Omega_0 = \frac{\pi}{6},$$

$$\begin{aligned} X[k] &= \frac{1}{12}\sum_{n=-5}^{6} x[n]e^{-jk\frac{\pi}{6}n} \\ &= \frac{1}{12}\left[e^{-j(-4)\frac{\pi}{6}k} - e^{-j(-3)\frac{\pi}{6}k} - e^{-j(-2)\frac{\pi}{6}k} + 2 - e^{-j(2)\frac{\pi}{6}k} - e^{-j(3)\frac{\pi}{6}k} + e^{-j(4)\frac{\pi}{6}k}\right] \\ &= \frac{1}{6}\left[\cos\left(\frac{2\pi}{3}k\right) - \cos\left(\frac{\pi}{2}k\right) - \cos\left(\frac{\pi}{3}k\right) + 1\right] \end{aligned}$$

2. (Textbook 3.49(a) and (c))

(1) $X[k] = a_k = \cos\left(\frac{8\pi}{21}k\right)$ (7%)

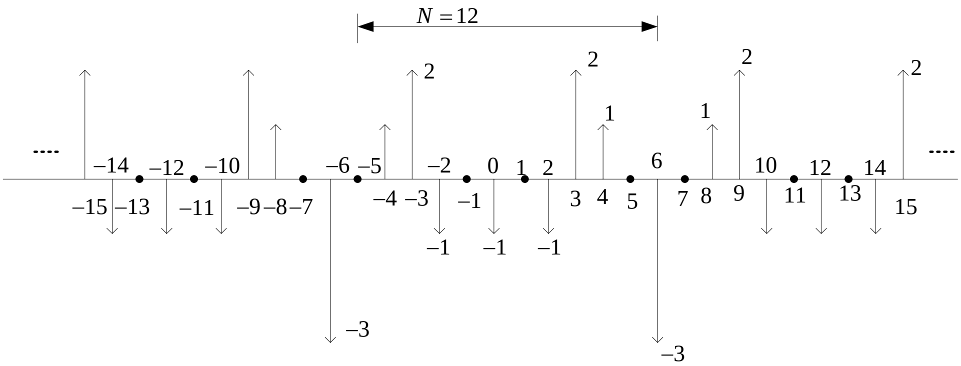
$$N = 21,\ \Omega_0 = \frac{2\pi}{21},$$

$$X[k] = a_k = \cos\left(\frac{8\pi}{21}k\right) = \frac{1}{2}\left[e^{j(4)\frac{2\pi}{21}} + e^{-j(4)\frac{2\pi}{21}}\right]$$

By inspection,

$$x[n] = \begin{cases} 21/2, & n = \pm 4 \\ 0, & \text{otherwise on } n \in \{-10, -9, \ldots, 10\} \end{cases}$$

(2) $X[k] = a_k = \sum_{m=-\infty}^{\infty} (-1)^m \left(\delta[k-2m] - 2\delta[k+3m]\right)$ (8%)



$$N = 12,\ \Omega_0 = \frac{\pi}{6},$$

$$x[n] = \sum_{k=-5}^{6} X[k] e^{jk\frac{\pi}{6}n}$$

$$= e^{j(-4)\frac{\pi}{6}n} + 2e^{j(-3)\frac{\pi}{6}n} - e^{j(-2)\frac{\pi}{6}n} - 1 - e^{j(2)\frac{\pi}{6}n} + 2e^{j(3)\frac{\pi}{6}n} + e^{j(4)\frac{\pi}{6}n} - 3e^{j(6)\frac{\pi}{6}n}$$

$$= 2\cos\left(\frac{2\pi}{3}n\right) + 4\cos\left(\frac{\pi}{2}n\right) - 2\cos\left(\frac{\pi}{3}\right) - 1 - 3(-1)^n$$

3. (Textbook 3.52(a) and (c))

(1) $x[n] = \left(\frac{3}{4}\right)^n u[n-4]$ (7%)

$$X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=\sum_{n=4}^{\infty}\left(\frac{3}{4}\right)^{n}e^{-j\Omega n}=\sum_{n=4}^{\infty}\left(\frac{3}{4}e^{-j\Omega}\right)^{n}=\frac{\left(\frac{3}{4}e^{-j\Omega}\right)^{4}}{1-\frac{3}{4}e^{-j\Omega}}$$

$$\left|X(\Omega)\right|=\frac{\left(\frac{3}{4}\right)^{4}}{\sqrt{\frac{25}{16}-\frac{3}{2}\cos(\Omega)}};\ \measuredangle X(\Omega)=-4\Omega+\tan^{-1}\left(\frac{3\sin(\Omega)}{4-3\cos(\Omega)}\right)$$

(2) $x[n]=\begin{cases}\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{N}n\right) & ,\ |n|\le N \\ 0 & ,\ \text{otherwise}\end{cases}$ (8%)

$$X(\Omega)=\sum_{n=-N}^{N}\left\{\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{N}n\right)\right\}e^{-j\Omega n}$$

$$=\frac{1}{2}\sum_{n=-N}^{N}\left\{1+\frac{1}{2}(e^{j\frac{\pi}{N}n}+e^{-j\frac{\pi}{N}n})\right\}e^{-j\Omega n}$$

$$=\frac{1}{2}\cdot\frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)}+\frac{1}{4}\cdot\frac{\sin\left(\frac{2N+1}{2}\left(\Omega-\frac{\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega-\frac{\pi}{N}\right)\right)}$$

$$+\frac{1}{4}\cdot\frac{\sin\left(\frac{2N+1}{2}\left(\Omega+\frac{\pi}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega+\frac{\pi}{N}\right)\right)}$$

$$\left(\begin{aligned}&\sum_{n=-N}^{N}e^{-j\Omega n}\underset{m=n+N}{=}\sum_{m=0}^{2N}e^{-j\Omega(m-N)}=e^{j\Omega N}\sum_{m=0}^{2N}e^{-j\Omega m}\\&=e^{j\Omega N}\frac{1-e^{-j\Omega(2N+1)}}{1-e^{-j\Omega}}=e^{j\Omega N}\frac{e^{-j\Omega\left(\frac{2N+1}{2}\right)}\left(e^{j\Omega\left(\frac{2N+1}{2}\right)}-e^{-j\Omega\left(\frac{2N+1}{2}\right)}\right)}{e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}\right)}\\&=e^{j\Omega N-j\Omega\left(\frac{2N+1}{2}\right)+j\frac{\Omega}{2}}\frac{\sin\left(\Omega\left(\frac{2N+1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)}=\frac{\sin\left(\Omega\left(\frac{2N+1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)}\end{aligned}\right)$$

4. (Textbook 3.53(a) and (b))

(1) $X(\Omega) = \cos(2\Omega) + j\sin(2\Omega)$ (7%)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j2\Omega}e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(2+n)}d\Omega$$

$$= \frac{1}{2\pi}\cdot\frac{e^{j\Omega(2+n)}}{j(2+n)}\Bigg|_{-\pi}^{\pi} = \frac{1}{2\pi}\cdot\frac{e^{j\pi(2+n)} - e^{-j\pi(2+n)}}{j(2+n)} = \frac{1}{2\pi}\cdot\frac{e^{j\pi n} - e^{-j\pi n}}{j(2+n)}$$

$$= \frac{1}{\pi}\cdot\frac{\sin(\pi n)}{(2+n)} = \begin{cases} \frac{1}{\pi}\cdot\frac{\partial\sin(\pi n)}{\partial n}\Big/\frac{\partial(2+n)}{\partial n}, & n=-2 \\ 0, & n\neq -2 \end{cases} \quad (\text{L}'\text{Hospital's rule})$$

$$= \begin{cases} \frac{1}{\pi}\cdot\frac{\pi\cos(\pi n)}{1}, & n=-2 \\ 0, & n\neq -2 \end{cases} = \begin{cases} 1, & n=-2 \\ 0, & n\neq -2 \end{cases} = \delta[n+2]$$

(2) $X(\Omega) = \sin(\Omega) + \cos\left(\frac{\Omega}{2}\right)$ (8%)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{j\Omega} - e^{-j\Omega}}{2j} + \frac{e^{j\Omega/2} + e^{-j\Omega/2}}{2}\right)e^{j\Omega n}d\Omega$$

$$= \frac{1}{2j}\delta[n+1] - \frac{1}{2j}\delta[n-1] + \frac{1}{2\pi}\cdot\frac{\cos(\pi n)}{n+0.5} - \frac{1}{2\pi}\cdot\frac{\cos(\pi n)}{n-0.5}$$

$$\left(\frac{1}{2\pi}\cdot\frac{1}{2}\int_{-\pi}^{\pi} e^{j\Omega/2}e^{j\Omega n}d\Omega = \frac{1}{2\pi}\cdot\frac{1}{2}\frac{e^{j(n+1/2)\Omega}}{j(n+0.5)}\Bigg|_{-\pi}^{\pi} = \frac{1}{2\pi}\cdot\frac{1}{2}\frac{je^{jn\pi} + je^{-jn\pi}}{j(n+0.5)} = \frac{1}{2\pi}\cdot\frac{\cos(\pi n)}{n+0.5}\right)$$

5. (Textbook 3.63(a), (b), (c), (d), and (f))

You are given $x[n] = n(3/4)^{|n|} \xleftrightarrow{DTFT} X(\Omega)$. Without evaluating $X(\Omega)$, find $y[n]$ if

(1) $Y(\Omega) = e^{-j4\Omega}X(\Omega)$ (4%)

$$y[n] = x[n-4] = (n-4)(3/4)^{|n-4|}$$

(2) $Y(\Omega) = \text{Re}\{X(\Omega)\}$ (4%)

Since $x[n]$ is real and odd, $X(\Omega)$ is pure imaginary, thus $y[n] = 0$.

(3) $Y(\Omega) = \frac{d}{d\Omega}X(\Omega)$ (4%)

$$y[n] = -jnx[n] = -jn^2(3/4)^{|n|}$$

(4) $Y(\Omega) = X(\Omega) \circledast X(\Omega - \pi/2)$ (4%)

$$y[n] = 2\pi x[n]\left(e^{j\frac{\pi}{2}n} x[n]\right) = 2\pi n^2 (3/4)^{2|n|} e^{j\frac{\pi}{2}n}$$

(5) $Y(\Omega) = X(\Omega) + X(-\Omega)$ (4%)

$$y[n] = x[n] + x[-n] = n(3/4)^{|n|} - n(3/4)^{|n|} = 0$$

6.

(1) (Textbook 3.61(c))Use the tables of transforms and properties to find the inverse DTFTs of the following signals: (10%)

$$X(\Omega) = \cos(4\Omega)\frac{\sin\left(\frac{3}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

$$A(\Omega) = \frac{\sin\left(\frac{3}{2}\Omega\right)}{\sin\left(\frac{\Omega}{2}\right)} \xleftrightarrow{F} a[n] = \begin{cases} 1, & |n| \le 1 \\ 0, \text{otherwise} \end{cases}$$

(See Problem 3(2) or lecture note of Chapter 4, pp.10)

$$X(\Omega) = \cos(4\Omega)A(\Omega) \xleftrightarrow{F} x[n] = \frac{1}{2}a[n+4] + \frac{1}{2}a[n-4]$$

$$x[n] = \begin{cases} 1/2, |n+4| \le 1,\ |n-4| \le 1 \\ 0, \qquad \text{otherwise} \end{cases} = \begin{cases} 1/2, 3 \le |n| \le 5 \\ 0,\ \text{otherwise} \end{cases}$$

(2) (Textbook 3.68(c)) Determine the frequency response and the impulse response for the system described by the following difference equations: (10%)

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{3}{4}x[n-1]$$

（抱歉，題目有打字錯誤，教科書的 difference equation 應為

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n] - \frac{3}{4}x[n-1]$$

所以用這兩種算的都可以。）

$$\left(1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}\right)Y(\Omega)=\left(1-\frac{3}{4}e^{-j\Omega}\right)X(\Omega)$$

$$H(\Omega)=\frac{Y(\Omega)}{X(\Omega)}=\frac{1-\frac{3}{4}e^{-j\Omega}}{1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}}=\frac{-1/3}{1-\frac{1}{2}e^{-j\Omega}}+\frac{4/3}{1+\frac{1}{4}e^{-j\Omega}}$$

$$=\left(-\frac{1}{3}\left(\frac{1}{2}\right)+\frac{4}{3}\left(-\frac{1}{4}\right)\right)u[n]$$

or

$$\left(1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}\right)Y(\Omega)=\left(3-\frac{3}{4}e^{-j\Omega}\right)X(\Omega)$$

$$H(\Omega)=\frac{Y(\Omega)}{X(\Omega)}=\frac{3-\frac{3}{4}e^{-j\Omega}}{1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-j2\Omega}}=\frac{1}{1-\frac{1}{2}e^{-j\Omega}}+\frac{2}{1+\frac{1}{4}e^{-j\Omega}}$$

$$=\left(\left(\frac{1}{2}\right)+2\left(-\frac{1}{4}\right)\right)u[n]$$