Homework No. 3 Solution

1. Determine the homogeneous and particular solutions for the system described by the following differential equation for the given inputs and initial conditions:

$$\frac{d^2}{dt^2}y(t)+4y(t)=3\frac{d}{dt}x(t),\quad y(0^-)=-1,\ \frac{d}{dt}y(t)\bigg|_{t=0^-}=1$$

(5%) Homogeneous solution

$$r^2+4=0\Rightarrow r=\pm j2$$
$$y^{(h)}(t)=c_1e^{j2t}+c_2e^{-j2t}$$

(a) (10%) $x(t)=t$

$$y^{(p)}(t)=p_1t+p_2$$
$$4p_1t+4p_2=3\Rightarrow p_1=0,\ p_2=\frac{3}{4}$$
$$\therefore y^{(p)}(t)=\frac{3}{4}$$

$$y(t)=y^{(h)}(t)+y^{(p)}(t)=c_1e^{j2t}+c_2e^{-j2t}+\frac{3}{4}$$

$$\left.\begin{aligned}y(0)&=c_1+c_2+\frac{3}{4}=-1\Rightarrow c_1+c_2=-\frac{7}{4}\\ y'(0)&=j2c_1-j2c_2=1\Rightarrow c_1-c_2=\frac{1}{2j}=-\frac{j}{2}\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}c_1&=-\frac{7}{8}-\frac{j}{4}\\ c_2&=-\frac{7}{8}+\frac{j}{4}=c_1^*\end{aligned}\right.$$

$$\begin{aligned}y^{(h)}(t)&=c_1e^{j2t}+c_1^*e^{-j2t}=c_1e^{j2t}+\left(c_1e^{j2t}\right)^*=2\,\mathrm{Re}\left\{c_1e^{j2t}\right\}\\ &=2\,\mathrm{Re}\left\{\left(-\frac{7}{8}-\frac{j}{4}\right)\left(\cos(2t)+j\sin(2t)\right)\right\}=2\left(-\frac{7}{8}\cos(2t)+\frac{1}{4}\sin(2t)\right)\\ &=-\frac{7}{4}\cos(2t)+\frac{1}{2}\sin(2t)\end{aligned}$$

$$\therefore y(t)=-\frac{7}{4}\cos(2t)+\frac{1}{2}\sin(2t)+\frac{3}{4}$$

(b) (15%) $x(t)=e^{-t}$

$$y^{(p)}(t)=pe^{-t}$$
$$pe^{-t}+4pe^{-t}=-3e^{-t}\Rightarrow p=-\frac{3}{5}$$
$$\therefore y^{(p)}(t)=-\frac{3}{5}e^{-t}$$

$$y(t) = y^{(h)}(t) + y^{(p)}(t) = c_1 e^{j2t} + c_2 e^{-j2t} - \frac{3}{5}e^{-t}$$

$$\left.\begin{array}{l} y(0) = c_1 + c_2 - \frac{3}{5} = -1 \Rightarrow c_1 + c_2 = -\frac{2}{5} \\ y'(0) = j2c_1 - j2c_2 + \frac{3}{5} = 1 \Rightarrow c_1 - c_2 = -\frac{j}{5} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c_1 = -\frac{1}{5} - \frac{j}{10} \\ c_2 = -\frac{1}{5} + \frac{j}{10} = c_1^* \end{array}\right.$$

$$y^{(h)}(t) = 2\,\mathrm{Re}\left\{c_1 e^{j2t}\right\} = -\frac{2}{5}\cos t + \frac{1}{5}\sin t$$

$$\therefore y(t) = -\frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t) - \frac{3}{5}e^{-t}$$

(c) (20%) $x(t) = \cos(t) + \sin(t)$

$$y^{(p)}(t) = p_1 \cos(t) + p_2 \sin(t)$$
$$y'^{(p)}(t) = -p_1 \sin(t) + p_2 \cos(t),\ y''^{(p)}(t) = -p_1 \cos(t) - p_2 \sin(t)$$
$$-p_1 \cos(t) - p_2 \sin(t) + 4p_1 \cos(t) + 4p_2 \sin(t) = -3\sin(t) + 3\cos(t)$$
$$3p_1 \cos(t) + 3p_2 \sin(t) = 3\cos(t) - 3\sin(t)$$
$$p_1 = 1,\ p_2 = -1$$
$$\therefore y^{(p)}(t) = \cos(t) - \sin(t)$$

$$y(t) = y^{(h)}(t) + y^{(p)}(t) = c_1 e^{j2t} + c_2 e^{-j2t} + \cos(t) - \sin(t)$$

$$\left.\begin{array}{l} y(0) = c_1 + c_2 + 1 = -1 \Rightarrow c_1 + c_2 = -2 \\ y'(0) = j2c_1 - j2c_2 - 1 = 1 \Rightarrow c_1 - c_2 = -j \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c_1 = -1 - \frac{j}{2} \\ c_2 = -1 + \frac{j}{2} = c_1^* \end{array}\right.$$

$$y^{(h)}(t) = 2\,\mathrm{Re}\left\{c_1 e^{j2t}\right\} = -2\cos t + \sin t$$

$$\therefore y(t) = -2\cos(2t) + \sin(2t) + \cos(t) - \sin(t)$$

2. Determine the homogeneous and particular solutions for the system described by the following difference equation for the given inputs and initial conditions:

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1],\quad y[-1] = 1,\ y[-2] = 0$$

(5%) Homogeneous solution

$$r^2 - \frac{1}{4}r - \frac{1}{8} = 0 \Rightarrow r = \frac{1}{2},\ -\frac{1}{4}$$
$$y^{(h)}[n] = c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{4}\right)^n$$

(a) (15%) $x[n] = nu[n]$

$$y^{(p)}[n]=(p_1 n+p_2)u[n]$$

$$p_1 n+p_2-\frac{1}{4}\left(p_1(n-1)+p_2\right)-\frac{1}{8}\left(p_1(n-2)+p_2\right)=n+n-1$$

$$\frac{5}{8}p_1 n+\frac{1}{2}p_1+\frac{5}{8}p_2=2n-1 \Rightarrow p_1=\frac{16}{5},\ p_2=-\frac{104}{25}$$

$$\therefore y^{(p)}[n]=\left(\frac{16}{5}n-\frac{104}{25}\right)u[n]$$

$$y[n]=c_1\left(\frac{1}{2}\right)^n+c_2\left(-\frac{1}{4}\right)^n+\left(\frac{16}{5}n-\frac{104}{25}\right)u[n]$$

$$y[0]-\frac{1}{4}y[-1]-\frac{1}{8}y[-2]=x[0]+x[-1] \Rightarrow y[0]-\frac{1}{4}=0 \Rightarrow y[0]=\frac{1}{4}$$

$$y[1]-\frac{1}{4}y[0]-\frac{1}{8}y[-1]=x[1]+x[0] \Rightarrow y[1]-\frac{1}{16}-\frac{1}{8}=1 \Rightarrow y[1]=\frac{19}{16}$$

$$\left.\begin{array}{l} y[0]=c_1+c_2-\frac{104}{25}=\frac{1}{4} \Rightarrow c_1+c_2=\frac{441}{100} \\ y[1]=\frac{1}{2}c_1-\frac{1}{4}c_2+\frac{16}{5}-\frac{104}{25}=\frac{19}{16} \Rightarrow \frac{1}{2}c_1-\frac{1}{4}c_2=\frac{859}{400} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c_1=\frac{13}{3} \\ c_2=\frac{23}{300} \end{array}\right.$$

$$\therefore y[n]=\frac{13}{3}\left(\frac{1}{2}\right)^n+\frac{23}{300}\left(-\frac{1}{4}\right)^n+\left(\frac{16}{5}n-\frac{104}{25}\right)u[n]$$

(b) (10%) $x[n]=\left(\frac{1}{8}\right)^n u[n]$

$$y^{(p)}[n]=p\left(\frac{1}{8}\right)^n u[n]$$

$$p\left(\frac{1}{8}\right)^n-\frac{1}{4}p\left(\frac{1}{8}\right)^{n-1}-\frac{1}{8}p\left(\frac{1}{8}\right)^{n-2}=\left(\frac{1}{8}\right)^n+\left(\frac{1}{8}\right)^{n-1}$$

$$p-\frac{1}{4}8p-\frac{1}{8}64p=1+8 \Rightarrow p=-1$$

$$\therefore y^{(p)}[n]=-\left(\frac{1}{8}\right)^n u[n]$$

$$y[n]=c_1\left(\frac{1}{2}\right)^n+c_2\left(-\frac{1}{4}\right)^n-\left(\frac{1}{8}\right)^n u[n]$$

$$y[0]-\frac{1}{4}y[-1]-\frac{1}{8}y[-2]=x[0]+x[-1] \Rightarrow y[0]-\frac{1}{4}=1 \Rightarrow y[0]=\frac{5}{4}$$

$$y[1]-\frac{1}{4}y[0]-\frac{1}{8}y[-1]=x[1]+x[0] \Rightarrow y[1]-\frac{5}{16}-\frac{1}{8}=\frac{1}{8}+1 \Rightarrow y[1]=\frac{25}{16}$$

$$\left.\begin{aligned} y[0] &= c_1 + c_2 - 1 = \frac{5}{4} \Rightarrow c_1 + c_2 = \frac{9}{4} \\ y[1] &= \frac{1}{2}c_1 - \frac{1}{4}c_2 - \frac{1}{8} = \frac{25}{16} \Rightarrow \frac{1}{2}c_1 - \frac{1}{4}c_2 = \frac{27}{16} \end{aligned}\right\} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -\frac{3}{4} \end{cases}$$

$$\therefore y[n] = 3\left(\frac{1}{2}\right)^n - \frac{3}{4}\left(-\frac{1}{4}\right)^n - \left(\frac{1}{8}\right)^n u[n]$$

(c) (20%) $x[n] = e^{j\frac{\pi}{4}n}u[n]$

$$y^{(p)}[n] = pe^{j\frac{\pi}{4}n}u[n]$$

$$pe^{j\frac{\pi}{4}n} - \frac{1}{4}pe^{j\frac{\pi}{4}(n-1)} - \frac{1}{8}pe^{j\frac{\pi}{4}(n-2)} = e^{j\frac{\pi}{4}n} + e^{j\frac{\pi}{4}(n-1)}$$

$$p - \frac{1}{4}pe^{-j\frac{\pi}{4}} - \frac{1}{8}pe^{-j\frac{\pi}{2}} = 1 + e^{-j\frac{\pi}{4}} \Rightarrow p = \frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}}$$

$$\therefore y^{(p)}[n] = \frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}}e^{j\frac{\pi}{4}n}u[n]$$

$$y[n] = c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{4}\right)^n + \frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}}e^{j\frac{\pi}{4}n}u[n]$$

$$y[0] - \frac{1}{4}y[-1] - \frac{1}{8}y[-2] = x[0] + x[-1] \Rightarrow y[0] - \frac{1}{4} = 1 \Rightarrow y[0] = \frac{5}{4}$$

$$y[1] - \frac{1}{4}y[0] - \frac{1}{8}y[-1] = x[1] + x[0] \Rightarrow y[1] - \frac{5}{16} - \frac{1}{8} = e^{j\frac{\pi}{4}} + 1 \Rightarrow y[1] = e^{j\frac{\pi}{4}} + \frac{23}{16}$$

$$y[0] = c_1 + c_2 + \frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}} = \frac{5}{4}$$

$$\Rightarrow c_1 + c_2 = \frac{5}{4} - \frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}}$$

$$y[1] = \frac{1}{2}c_1 - \frac{1}{4}c_2 + \frac{e^{j\frac{\pi}{4}} + 1}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}} = e^{j\frac{\pi}{4}} + \frac{23}{16}$$

$$\Rightarrow \frac{1}{2}c_1 - \frac{1}{4}c_2 = e^{j\frac{\pi}{4}} + \frac{23}{16} - \frac{e^{j\frac{\pi}{4}} + 1}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}}$$

$$\begin{cases} c_1 = \frac{7}{3} + \frac{4}{3}e^{j\frac{\pi}{4}} - \frac{5 + 4e^{j\frac{\pi}{4}} + e^{-j\frac{\pi}{4}}}{3\left(1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}\right)} \\ c_2 = -\frac{13}{12} - \frac{4}{3}e^{j\frac{\pi}{4}} + \frac{2 - 2e^{-j\frac{\pi}{4}} + 4e^{j\frac{\pi}{4}}}{3\left(1 - \frac{1}{4}e^{-j\frac{\pi}{4}} - \frac{1}{8}e^{-j\frac{\pi}{2}}\right)} \end{cases}$$