1.

<Sol.>

- Homogeneous solution

$$r^2-\frac{1}{4}r-\frac{1}{8}=0 \Rightarrow r=\frac{1}{2},-\frac{1}{4}$$

$$y^h[n]=c_1(\frac{1}{2})^n+c_2(-\frac{1}{4})^n$$

- Particular solution

(a) $x[n]=(\frac{1}{8})^n u[n]$

$$y_p[n]=p(\frac{1}{8})^n u[n]$$

$$p(\frac{1}{8})^n-\frac{1}{4}p(\frac{1}{8})^{n-1}-\frac{1}{8}p(\frac{1}{8})^{n-2}=(\frac{1}{8})^n+(\frac{1}{8})^{n-1} \Rightarrow p=-1$$

$$\therefore y_p[n]=-(\frac{1}{8})^n u[n]$$

$$y[n]=c_1(\frac{1}{2})^n+c_2(-\frac{1}{4})^n-(\frac{1}{8})^n u[n]$$

From $y[-1]=1,\ y[-2]=0$

$$\Rightarrow \mathrm{y}[0]=\frac{5}{4},\ \mathrm{y}[1]=\frac{25}{16}$$

$$\Rightarrow \begin{cases} y[0]=c_1+c_2-1=\frac{5}{4} \\ y[1]=\frac{1}{2}c_1-\frac{1}{4}c_2-\frac{1}{8}=\frac{25}{16} \end{cases} \Rightarrow \begin{cases} c_1+c_2=\frac{9}{4} \\ \frac{1}{2}c_1-\frac{1}{4}c_2=\frac{27}{16} \end{cases} \Rightarrow \begin{cases} c_1=3 \\ c_2=-\frac{3}{4} \end{cases}$$

$$y[n]=3(\frac{1}{2})^n-\frac{3}{4}(-\frac{1}{4})^n-(\frac{1}{8})^n u[n]$$

(b) $x[n]=e^{j\frac{\pi}{4}n}u[n]$

$$y^p[n]=pe^{j\frac{\pi}{4}n}u[n]$$

$$pe^{j\frac{\pi}{4}n}-\frac{1}{4}pe^{j\frac{\pi}{4}(n-1)}-\frac{1}{8}pe^{j\frac{\pi}{4}(n-2)}=e^{j\frac{\pi}{4}n}+e^{j\frac{\pi}{4}(n-1)}$$

$$p=\frac{1+e^{-j\frac{\pi}{4}}}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}}$$

$$y^p[n] = -\frac{1+e^{-j\frac{\pi}{4}}}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}}e^{j\frac{\pi}{4}n}u[n]$$

$$y[n] = c_1(\frac{1}{2})^n + c_2(-\frac{1}{4})^n - \frac{1+e^{-j\frac{\pi}{4}}}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}}e^{j\frac{\pi}{4}n}u[n]$$

From $y[-1]=1,\ y[-2]=0$

$\Rightarrow \mathrm{y}[0]=\frac{5}{4},\ \mathrm{y}[1]=\frac{23}{16}+e^{j\frac{\pi}{4}},$ we set $\mathrm{K}=1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}$

$$\Rightarrow \begin{cases} y[0]=\frac{5}{4}=c_1+c_2-\left(1+e^{-j\frac{\pi}{4}}\right)\mathrm{K}^{-1} \\ y[1]=\frac{23}{16}+e^{j\frac{\pi}{4}}=\frac{1}{2}c_1-\frac{1}{4}c_2-\left(1+e^{j\frac{\pi}{4}}\right)\mathrm{K}^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} c_1=\frac{7}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}+\left(\frac{5}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}+\frac{1}{3}e^{-j\frac{\pi}{4}}\right)\mathrm{K}^{-1} \\ c_2=-\frac{13}{12}-\frac{4}{3}e^{j\frac{\pi}{4}}-\left(\frac{2}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}-\frac{2}{3}e^{-j\frac{\pi}{4}}\right)\mathrm{K}^{-1} \end{cases}$$

$$y[n]=\left(\frac{7}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}+\frac{\left(\frac{5}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}+\frac{1}{3}e^{-j\frac{\pi}{4}}\right)}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}}\right)(\frac{1}{2})^n+\left(-\frac{13}{12}-\frac{4}{3}e^{j\frac{\pi}{4}}-\frac{\left(\frac{2}{3}+\frac{4}{3}e^{j\frac{\pi}{4}}-\frac{2}{3}e^{-j\frac{\pi}{4}}\right)}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}}\right)(-\frac{1}{4})^n-\frac{1+e^{-j\frac{\pi}{4}}}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}-\frac{1}{8}e^{-j\frac{\pi}{2}}}e^{j\frac{\pi}{4}n}u[n]$$

2.

<Sol.>

(a)

- Homogeneous solution

$$r^2+5r+6=0 \Rightarrow r=-2,-3$$
$$y^h(t)=c_1e^{-2t}+c_2e^{-3t}$$

- Particular solution

$$y^p(t)=p_1e^{-t}+p_2te^{-2t}$$
$$\frac{dy^p(t)}{dt}=-p_1e^{-t}+p_2e^{-2t}-2p_2te^{-2t}$$
$$\frac{d^2y^p(t)}{dt^2}=p_1e^{-t}-4p_2e^{-2t}+4p_2te^{-2t}$$

$$[p_1e^{-t}-4p_2e^{-2t}+4p_2te^{-2t}]+5[-p_1e^{-t}+p_2e^{-2t}-2p_2te^{-2t}]+6[p_1e^{-t}+p_2te^{-2t}]=e^{-t}+e^{-2t}$$

$$\begin{cases} p_1=\frac{1}{2} \\ p_2=1 \end{cases}$$

$$\Rightarrow y(t)=c_1e^{-2t}+c_2e^{-3t}+\frac{1}{2}e^{-t}+te^{-2t};t>0$$

From $y(t)\big|_{t=0^-}=1$ and $\left.\frac{dy(t)}{dt}\right|_{t=0^-}=0$.

$$\begin{cases} c_1=1 \\ c_2=\frac{-1}{2} \end{cases}$$

$$\Rightarrow y(t)=\left(e^{-2t}-\frac{1}{2}e^{-3t}+\frac{1}{2}e^{-t}+te^{-2t}\right)u(t)$$

(b) natural response

$$y^{(n)}(t)=c_3e^{-2t}+c_4e^{-3t}$$

$$\frac{d}{dt}y^{(n)}(t)=-2c_3e^{-2t}-3c_4e^{-3t}$$

$$\rightarrow \begin{cases} c_3+c_4=1 \\ -2c_3-3c_4=0 \end{cases} \rightarrow \begin{cases} c_3=3 \\ c_4=-2 \end{cases} \rightarrow y^{(n)}(t)=3e^{-2t}-2e^{-3t}$$

(c) forced response

$$y^{(f)}(t) = c_5 e^{-2t} + c_6 e^{-3t} + \frac{1}{2} e^{-t} + t e^{-2t}$$

$$\frac{d}{dt} y^{(f)}(t) = -2c_5 e^{-2t} - 3c_6 e^{-3t} - \frac{1}{2} e^{-t} + e^{-2t} - 2t e^{-2t}$$

$$\rightarrow \begin{cases} c_5 + c_6 + \frac{1}{2} = 0 \\ -2c_5 - 3c_6 + \frac{1}{2} = 0 \end{cases} \rightarrow \begin{cases} c_5 = -2 \\ c_6 = \frac{3}{2} \end{cases} \rightarrow y^{(f)}(t) = -2e^{-2t} + \frac{3}{2} e^{-3t} + \frac{1}{2} e^{-t} + t e^{-2t}$$