Chapter 10
The Z-Transform

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10.0 Introduction

Similar to LT which is a more general transform comparing to CTFT. Now we introduce *z*-transform which is more general than DTFT.

z-transform is the discrete-time counterpart of the Laplace transform with noticeable differences.

10.1 The z-Transform

The response *y[n]* of the LTI system to a complex exponential input of the form z^n is

$$y[n] = H(z)z^n, \tag{10.1}$$

where

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}. \tag{10.2}$$

10.1 The z-Transform

The *z*-transform of a general discrete-time signal *x[n]* is defined as

$$X(z) \stackrel{\Delta}{=} \sum_{n=-\infty}^{+\infty} x[n] z^{-n}, \qquad (10.3)$$

雙邊z轉換定義

For convenience, the *z*-transform of *x[n]* will sometimes be denoted as $Z\{x[n]\}$ and the relationship between *x[n]* and its z-transform indicated as

$$x[n] \stackrel{Z}{\longleftrightarrow} X(z). \qquad (10.4)$$

z轉換關係符號

10.1 The z-Transform

To explore the relationships between FT and ZF, we express the complex variable *z* in **polar** form as

$$z = re^{j\omega}, \tag{10.5}$$

with *r* as the magnitude of *z* and *ω* as the angel of *z*. In terms of r and ω, eq. (10.3) becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n},$$

10.1 The z-Transform

or equivalently,

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} \left\{ x[n] r^{-n} \right\} e^{-j\omega n}. \qquad (10.6)$$

From eq. (10.6), we see that $X(re^{j\omega})$ is the Fourier transform of the sequence *x[n]* multiplied by a real exponential r^{-n}; that is,

$$X(re^{j\omega}) = F\left\{ x[n] r^{-n} \right\}. \qquad (10.7)$$

10.1 The z-Transform

For $r = 1$, or equivalently, $|z| = 1$, eq. (10.3) reduces to the Fourier transform; that is,

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) = F\{x[n]\}. \qquad (10.8)$$

z轉換與傅立葉轉換的關係

10.1 The z-Transform

TF can be seen as z-transform with $z=e^{j\omega}$.

$z = e^{j\omega}$ can be visualized as a circle with radius 1 centered at the origin. We call this circle “unit circle”

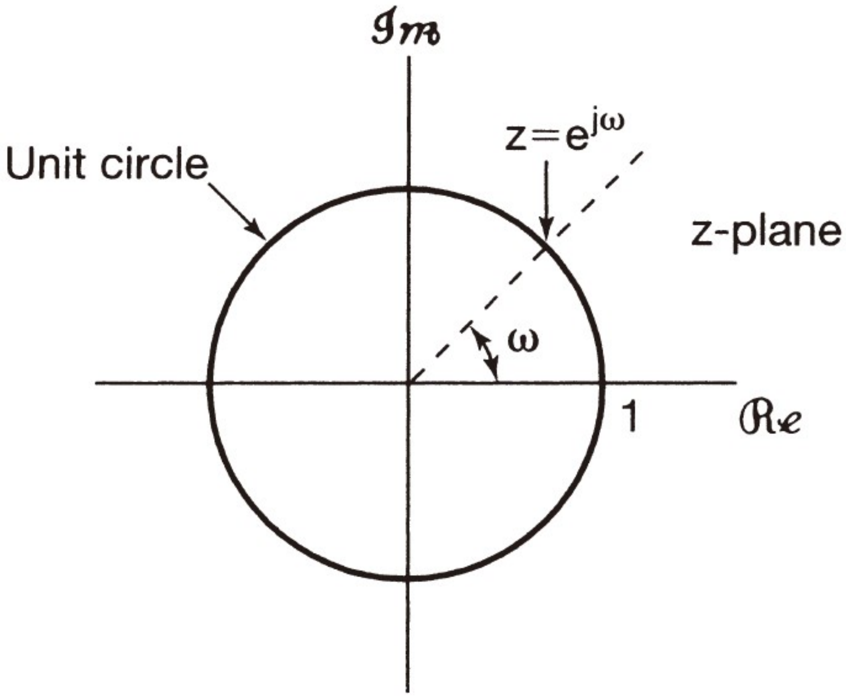


Figure 10.1 Complex *z*-plane. The *z*-transform reduces to the Fourier transform for values of *z* on the unit circle.

Example 10.1 $X(z) \overset{\Delta}{=} \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$ (10.3)

指數訊號的z轉換範例

Consider the signal $x[n] = a^n u[n]$. Then, from eq. (10.3), $X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$

For convergence of *X(z)*, we require that $\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$ (absolute summable).

Consequently, the region of convergence is the range of values of *z* for which $|z| > |a|$, or $\left|az^{-1}\right| < 1$ equivalently, .

Example 10.1

Then

可得X(z)及ROC如(10.9)式。

$$X(z)=\sum_{n=0}^{\infty}(az^{-1})^{n}=\frac{1}{1-az^{-1}}=\frac{z}{z-a},\qquad |z|>|a|. \tag{10.9}$$

Thus, the *z*-transform for this signal is well-defined for any value of *a*, with an ROC determined by the magnitude of a according to eq. (10.9). For example, for *a* = 1, *x[n]* is the unit step sequence with *z*-transform

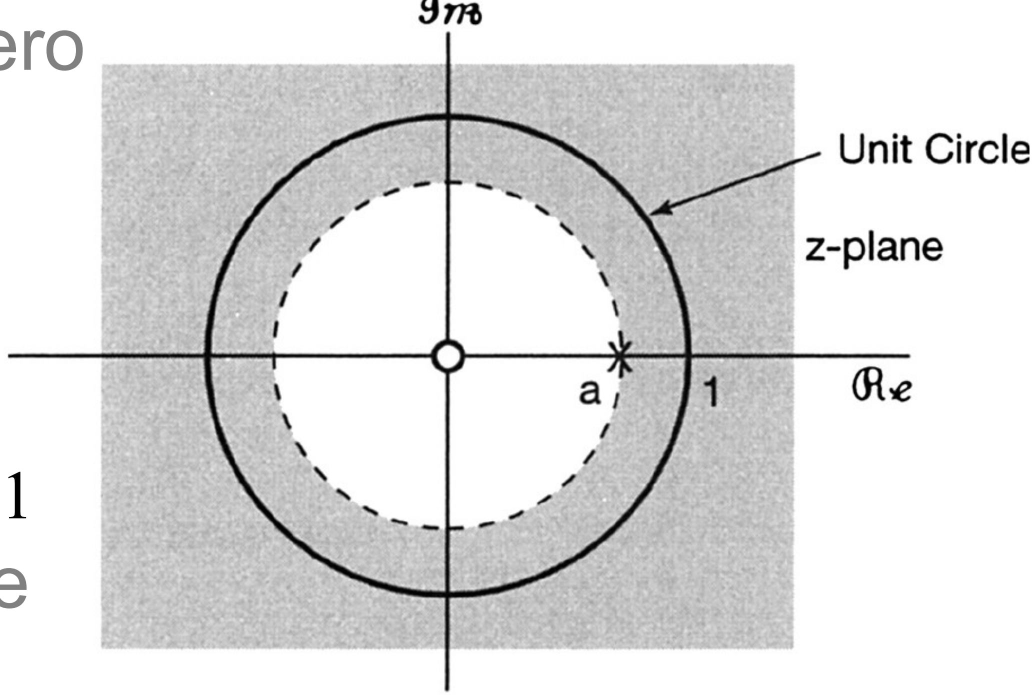
$$X(z)=\frac{1}{1-z^{-1}},\qquad |z|>1.$$

Example 10.1

We see that the z-transform in eq. (10.9) is a **rational** function. Consequently, just as with rational Laplace transforms, the z-transform can be characterized by its **zeros** (the roots of the numerator polynomial) and its **poles** (the roots of the denominator polynomial).

$$X(z)=\sum_{n=0}^{\infty}(az^{-1})^n=\frac{1}{1-az^{-1}}=\frac{z}{z-a}, \qquad |z|>|a|.$$

For this example, there is one zero, at z = 0, and one pole, at z = a. The pole-zero plot and the region of convergence for Example 10.1 are shown in Figure 10.2 for a value of a between 0 and 1. For $|a|>1$, the ROC does not include the unit circle, consistent with the fact that, for these values of a , the Fourier transform of $a^n u[n]$ does not converge.



Example 10.2

$$X(z) \stackrel{\Delta}{=} \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \quad (10.3)$$

Now let $x[n] = -a^n u[-n-1]$. Then

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \quad (10.10)$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

$$= 1 - \frac{1}{1 - a^{-1} z}, |a^{-1} z| < 1$$

Example 10.2

If $|a^{-1}z| < 1$, or equivalently, $|z| < |a|$, the sum in eq. (10.10) converges and

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|. \qquad (10.11)$$

可得*X(z)*及ROC如(10.11)式。

The pole-zero plot and region of convergence for this example are shown in Figure 10.3 for a value of a between 0 and 1.

Example 10.2

$$X(z) = \frac{z}{z-a}, \qquad |z| < |a|.$$

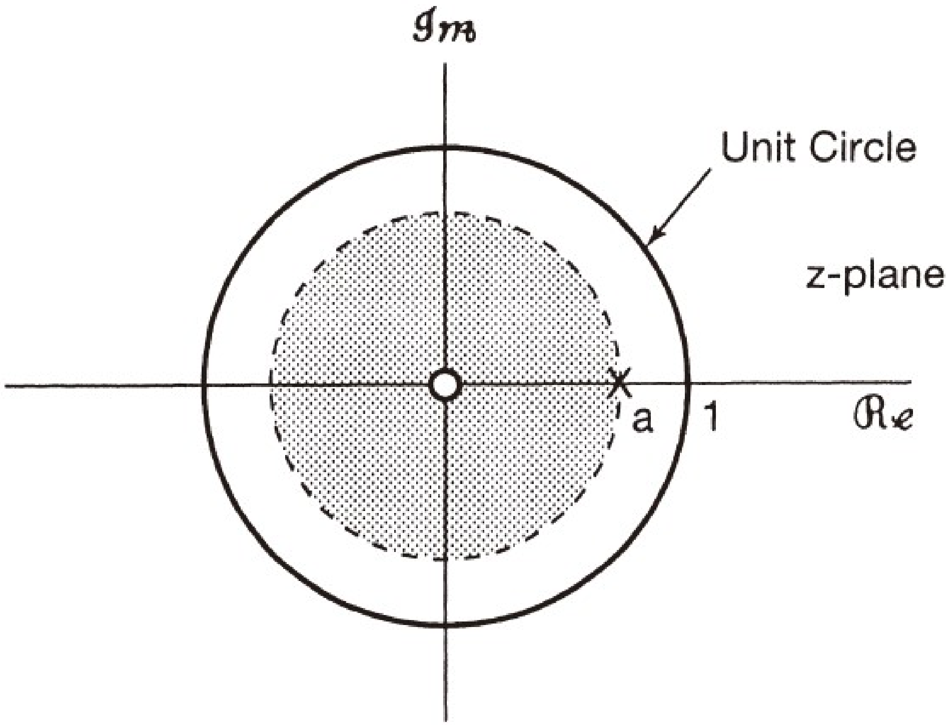
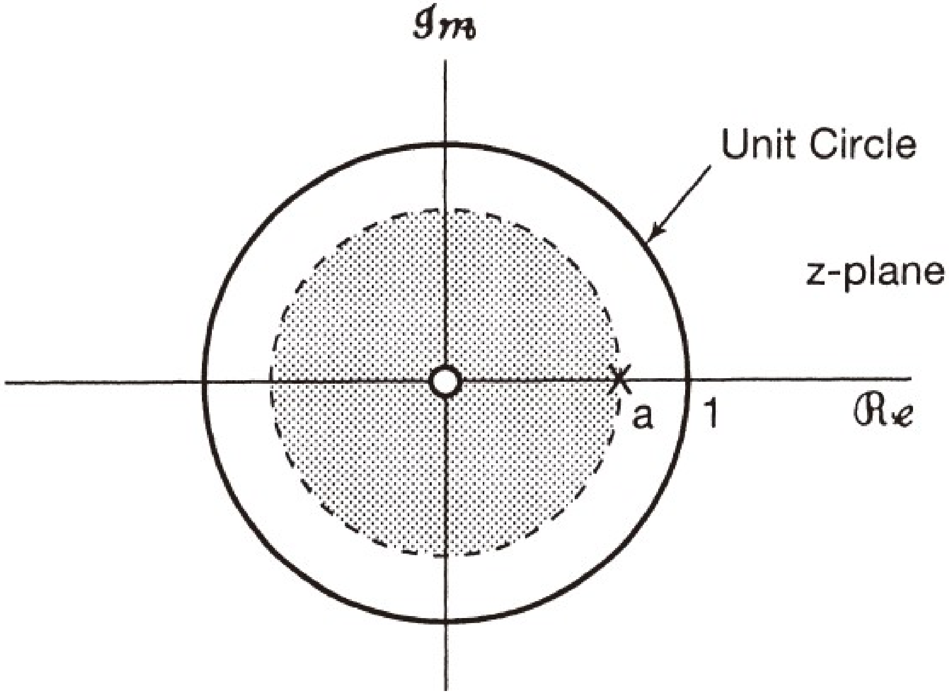


Figure 10.3 Pole-zero plot and region of convergence for Example 10.2 for $0 < a < 1$.

圖 10.3 顯示 ROC 爲 $|z|<|a|$ 的內側區域。

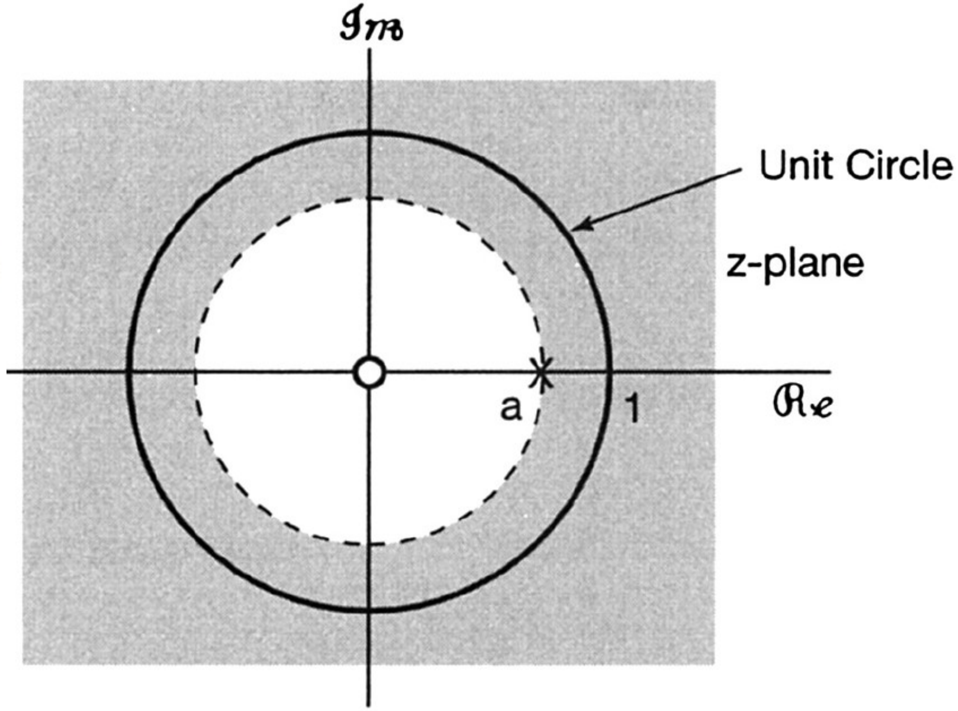
$$x[n] = -a^n u[-n-1]$$

$$X(z) = \frac{z}{z-a}, \qquad |z| < |a|.$$



$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{z}{z-a}, \qquad |z| > |a|.$$



differ only in their regions of convergence.

Example 10.4

Let us consider the signal

$$x[n] = \left(\frac{1}{3}\right)^n \sin(\frac{\pi}{4}n)u[n]$$

$$= \frac{1}{2j}(\frac{1}{3}e^{j\pi/4})^n u[n] - \frac{1}{2j}(\frac{1}{3}e^{-j\pi/4})^n u[n].$$

$$x[n] = a^n u[n]$$

Example 10.4

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|.$$

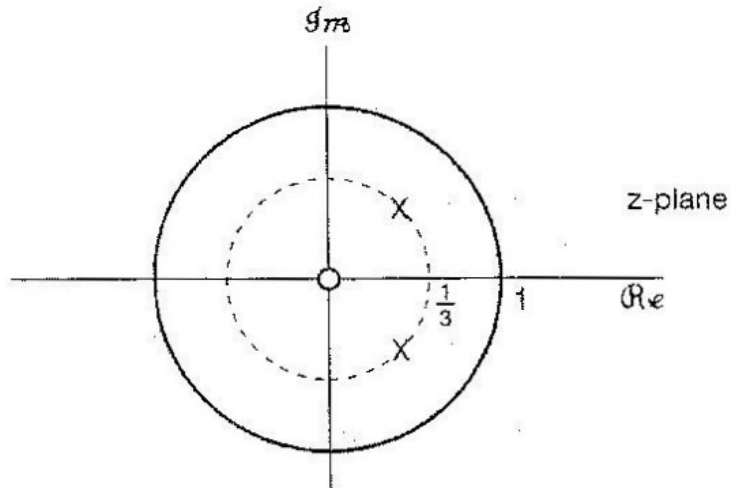
The z-transform of this signal is

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{2j} \left(\frac{1}{3} e^{j\pi/4} \right)^n u[n] - \frac{1}{2j} \left(\frac{1}{3} e^{-j\pi/4} \right)^n u[n] \right\} z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{j\pi/4} z^{-1} \right)^n - \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{-j\pi/4} z^{-1} \right)^n \qquad (10.19)$$

$$= \frac{1}{2j} \frac{1}{1 - \frac{1}{3} e^{j\pi/4} z^{-1}} - \frac{1}{2j} \frac{1}{1 - \frac{1}{3} e^{-j\pi/4} z^{-1}},$$

Example 10.4



or equivalently,

$$X(z) = \frac{\frac{1}{3\sqrt{2}} z}{(z - \frac{1}{3} e^{j\pi/4})(z - \frac{1}{3} e^{-j\pi/4})} \qquad (10.20)$$

For convergence of *X(z),* both sums in eq. (10.19) must converge, which requires that $\left|(1/3)e^{j\pi/4}z^{-1}\right| < 1$ and $\left|(1/3)e^{-j\pi/4}z^{-1}\right| < 1$, or equivalently, $|z| > 1/3$. The pole zero plot and ROC for this example are shown in Figure 10.5.

10.2 The Region of Convergence for The z-Transform

- Property 1: The ROC of *X(z)* consists of a ring in the z-plane centered about the origin.

  性質1：*X(z)*的ROC是由z平面上以原點為圓心的環狀區域。

  The ROC of the *z*-transform of *x[n]* consists of the values of $z=re^{j\omega}$ for which $x[n]r^{-n}$ is absolutely summable:

$$\sum_{n=-\infty}^{+\infty}\left|x[n]r^{-n}\right|=\sum_{n=-\infty}^{+\infty}\left|x[n]\right|r^{-n}<\infty. \qquad (10.21)$$

10.2 The Region of Convergence for The z-Transform

- Property 2: The ROC does not contain any poles.

  性質2：ROC不包含任何極點。

- Property 3: If *x[n]* is of finite duration, then the ROC is the entire *z*-plane, **except possibly z = 0 and/or z = ∞**.

  性質3：若*x[n]*為有限時間訊號，則ROC為整個*z*平面，但$z = 0$或$z = \infty$可能除外。

10.2 The Region of Convergence for The z-Transform

For x[n] with nonzero values from N_1 to N_2, where they are all finite, the *z*-transform is the sum of a finite number of terms; that is,

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}. \tag{10.22}$$

If $N_1<0$ and $N_2>0$, ROC does not include z=0 or z= ∞.

If $N_1>0$, ROC does include z=∞.

If $N_2<0$, ROC does include z=0.

Example 10.5

Consider the unit impulse signal *δ[n].* Its *z*-transform is given by

$$\delta[n] \xleftrightarrow{Z} \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = 1, \tag{10.23}$$

with an ROC consisting of the entire *z*-plane, including $z = 0$ and $z = \infty$. On the other hand, consider the delayed unit impulse *δ[n-1],* for which

$$\delta[n-1] \xleftrightarrow{Z} \sum_{n=-\infty}^{+\infty} \delta[n-1] z^{-n} = z^{-1}. \tag{10.24}$$

10.2 The Region of Convergence for The z-Transform

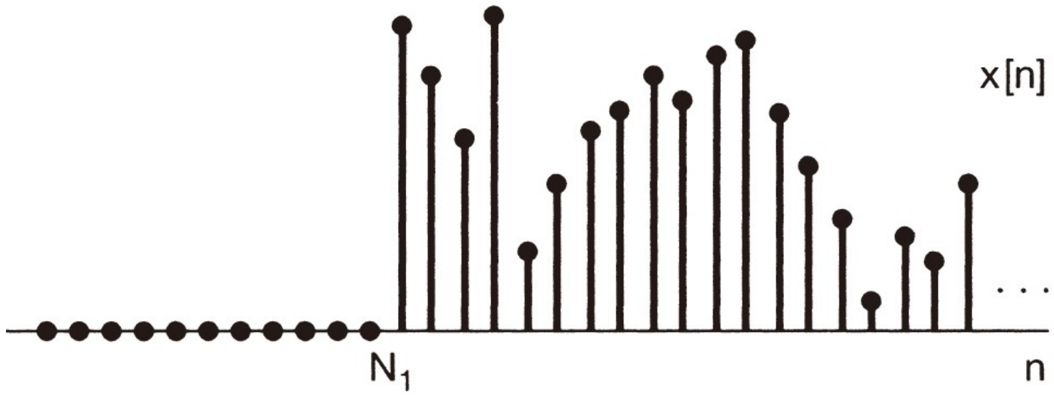
The ROC consists of the entire z-plane, including $z = \infty$ but excluding $z = 0$. Similarly, consider an impulse advanced in time, namely, $\delta[n+1]$. In this case,

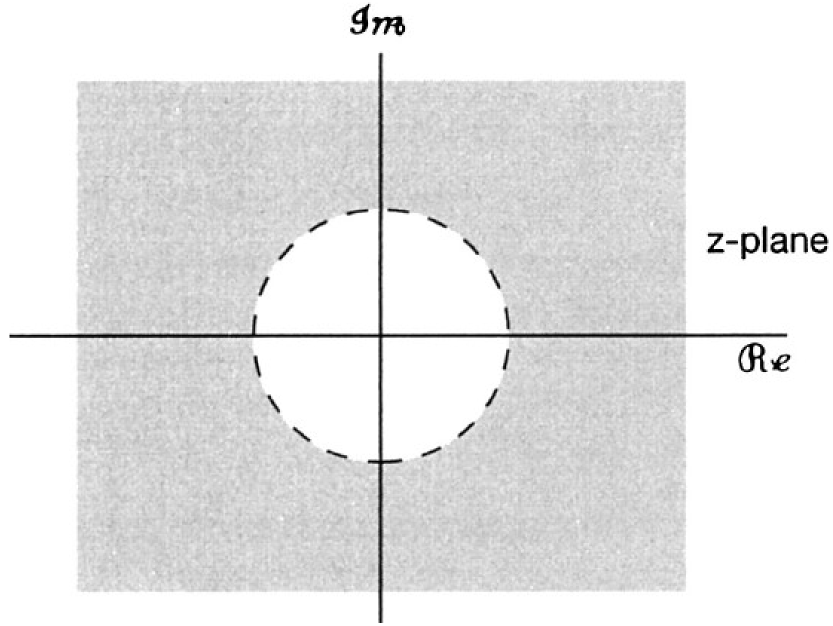
$$\delta[n+1] \xleftrightarrow{Z} \sum_{n=-\infty}^{+\infty} \delta[n+1] z^{-n} = z, \qquad (10.25)$$

the ROC include z=0, but there is a pole at infinity.

10.2 The Region of Convergence for The z-Transform

- Property 4: If *x[n]* is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of *z* for which $|z| > r_0$ will also be in the ROC.





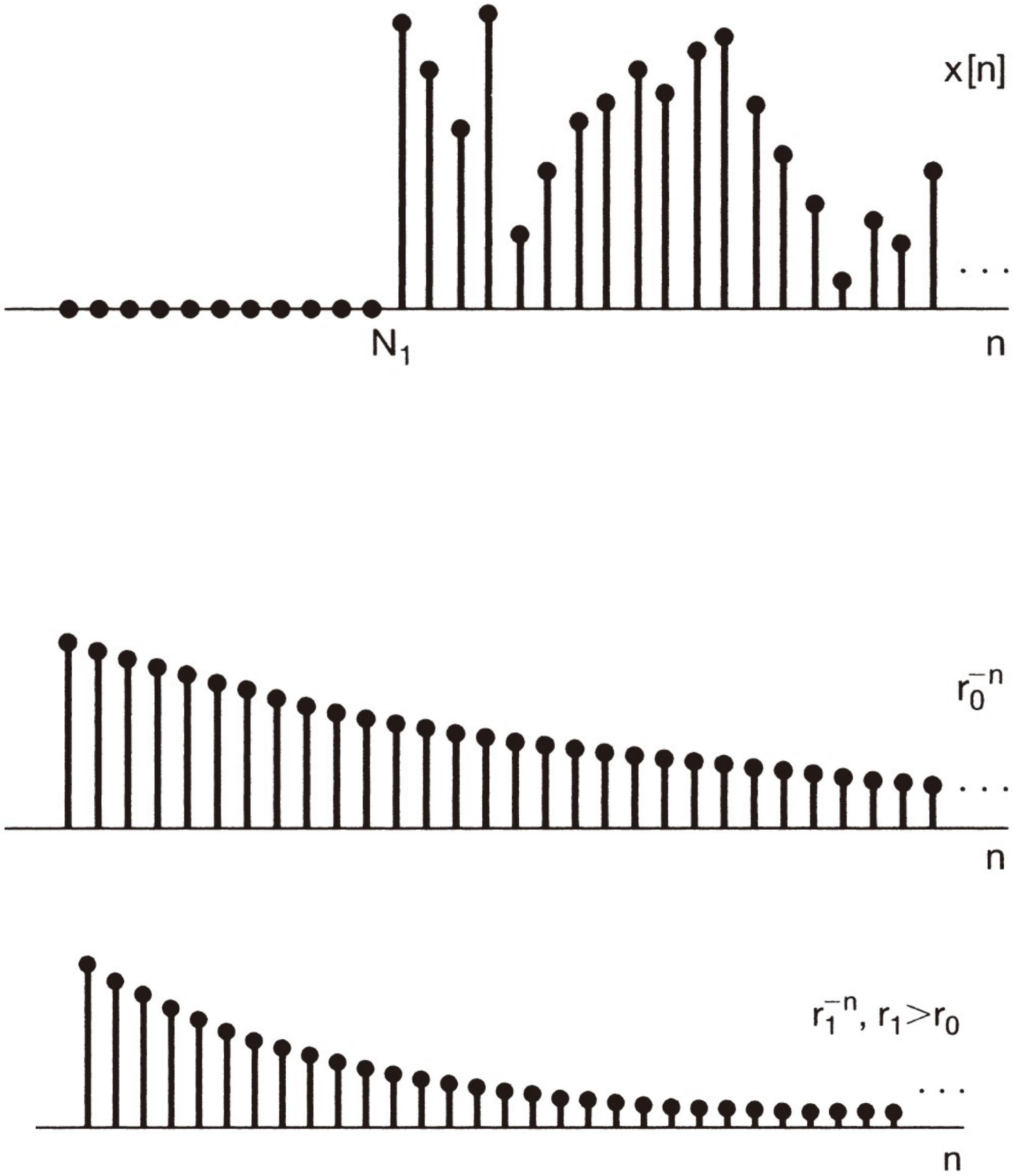


Figure 10.7 With $r_1 > r_0$, $x[n]r_1^{-n}$ decays faster with increasing n than does $x[n]r_0^{-n}$. Since $x[n] = 0, n < N_1$, this implies that if $x[n]r_0^{-n}$ is absolutely summable, then $x[n]r_1^{-n}$ will be also.

10.2 The Region of Convergence for The z-Transform

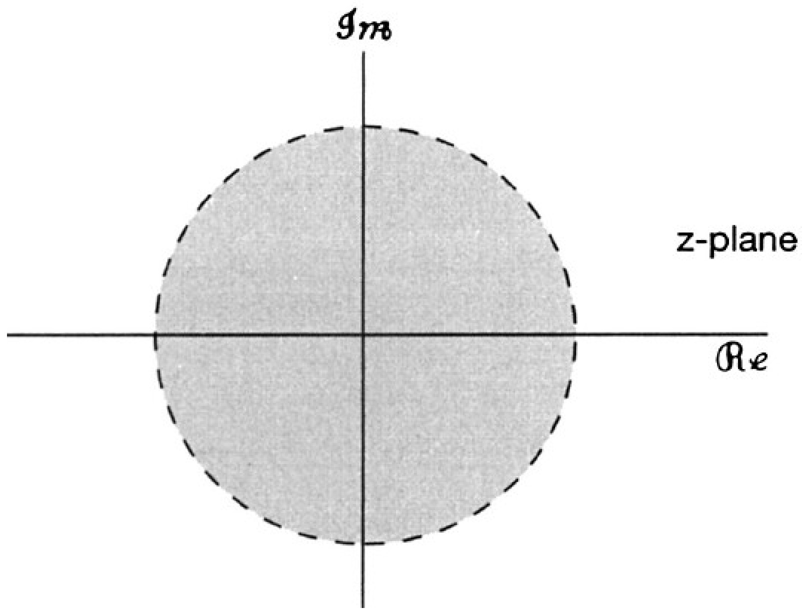
For right-sided sequences in general, eq. (10.3) takes the form

$$X(z) = \sum_{n=N_1}^{\infty} x[n]z^{-n}, \tag{10.26}$$

If $N_1<0$, ROC does not include ∞. If $N_1>=0$, ROC includes ∞.

10.2 The Region of Convergence for The z-Transform

- Property 5: If *x[n]* is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.



10.2 The Region of Convergence for The z-Transform

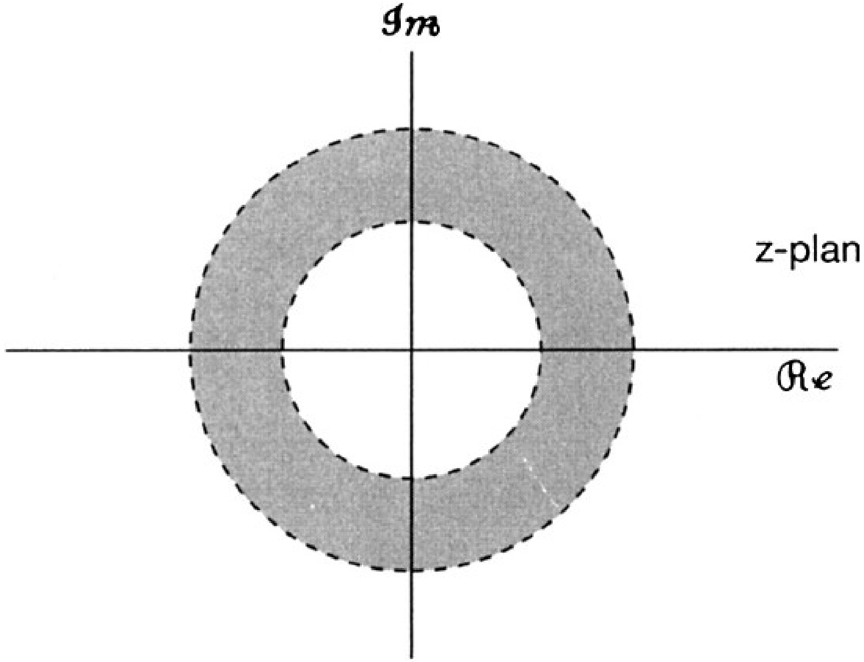
In general, for left-sided sequences, from eq. (10.3), the summation for the *z*-transform will be of the form

$$X(z) = \sum_{n=-\infty}^{N_2} x[n] z^{-n}, \qquad (10.27)$$

If $N_2 > 0$, ROC does not include 0. If $N_2 <= 0$, ROC includes 0.

10.2 The Region of Convergence for The z-Transform

- Property 6: If *x[n]* is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a **ring** in the *z*-plane that includes the circle $|z| = r_0$.



Example 10.6

Consider the signal

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, a > 0 \\ 0, & otherwise \end{cases}.$$

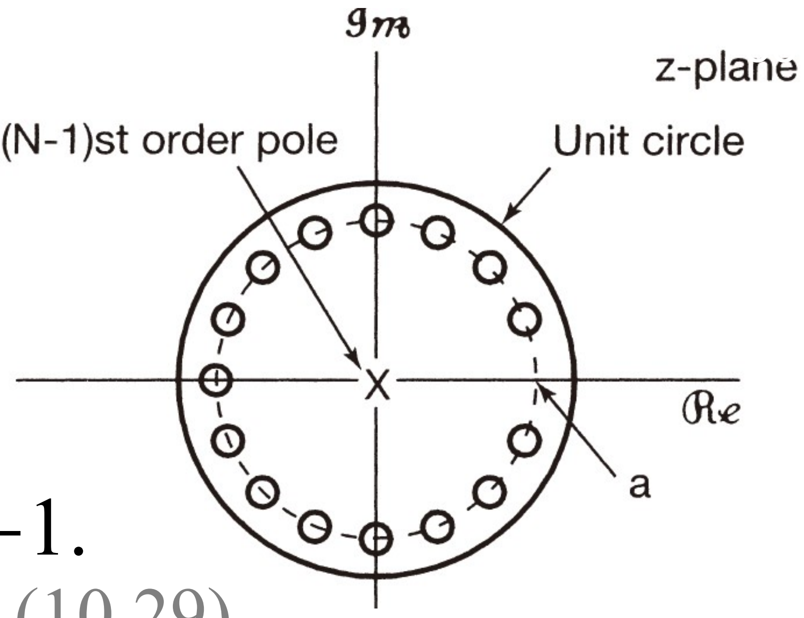
Then

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} \\ &= \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}. \end{aligned} \tag{10.28}$$

Example 10.6 $X(z)=\frac{1}{z^{N-1}}\frac{z^N-a^N}{z-a}$.(10.28)

Since *x[n]* is of finite length, it follows form Property 3 that the ROC includes the entire *z*-plane except possible the origin and /or infinity. In fact, from our discussion of Property 3, since *x[n]* is zero for $n<0$, the ROC will extend to infinity. However, since *x[n]* is nonzero for some positive values of *n*, the ROC will **not include the origin**. This is evident from eq. (10.28), from which we see that there is a pole of order $N-1$ at $z=0$.

Example 10.6



The N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N-1. \qquad (10.29)$$

The root for $k = 0$ cancels the pole at $z = a$. Consequently, there are no poles other than at the origin. The remaining zeros are at

$$X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}. \quad (10.28)$$

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 1, \ldots, N-1. \qquad (10.30)$$

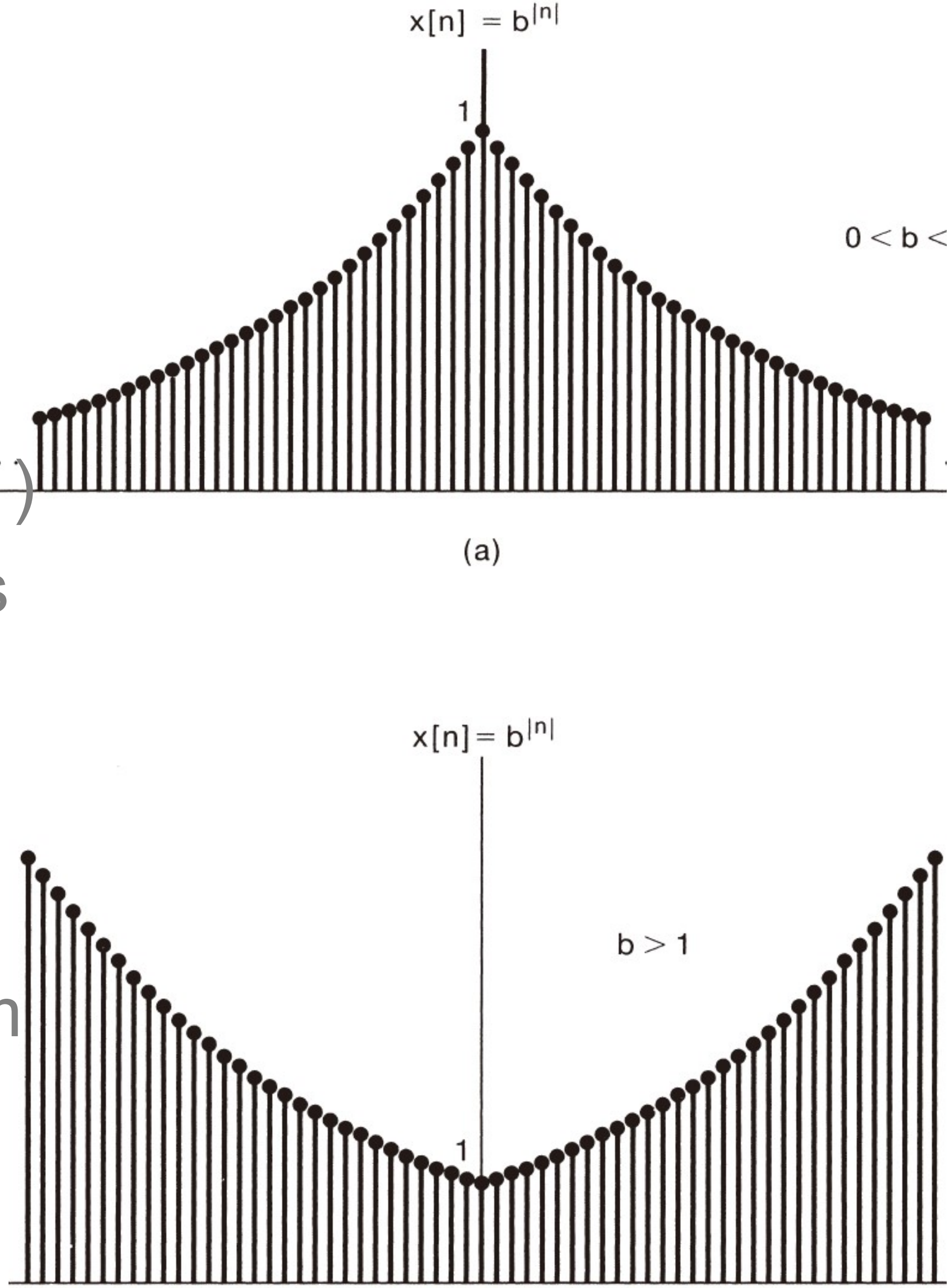
The pole-zero pattern is shown in Figure 10.9.

Example 10.7

Let

$$x[n] = b^{|n|}, b > 0. \qquad (10.31)$$

This two-sided sequence is illustrated in Figure 10.10, for both $b < 1$ and $b > 1$. The z-transform for the sequence can be obtained by expressing it as the sum of a right-sided and a left-sided sequence.



Example 10.7

We have

$$x[n] = b^n u[n] + b^{-n} u[-n-1]. \qquad (10.32)$$

From Example 10.1,

$$b^n u[n] \xleftrightarrow{Z} \frac{1}{1-bz^{-1}}, \qquad |z| > b, \qquad (10.33)$$

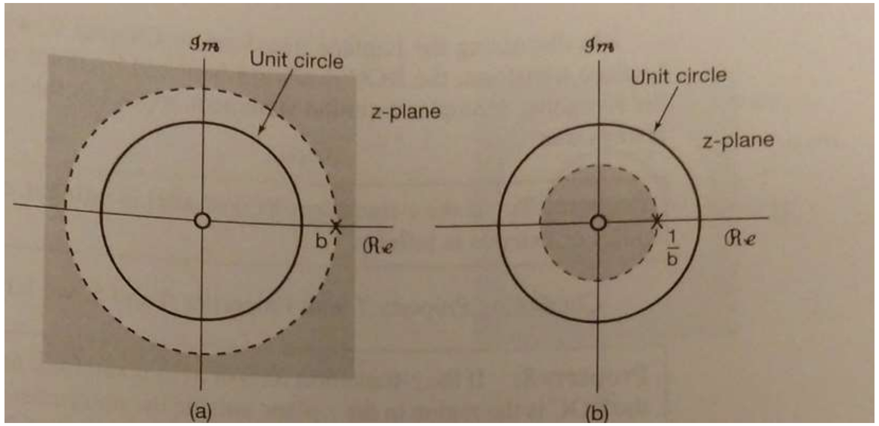
and from Example 10.2,

$$b^{-n} u[-n-1] \xleftrightarrow{Z} \frac{1}{1-b^{-1}z^{-1}}, \qquad |z| < \frac{1}{b}. \qquad (10.34)$$

Example 10.7

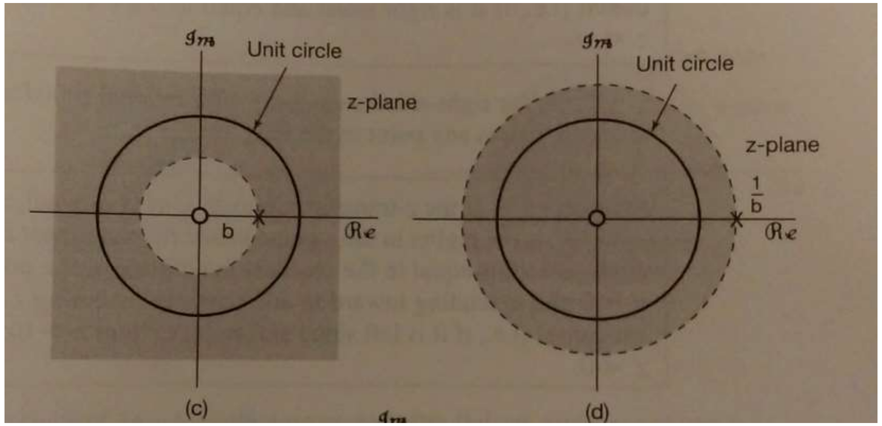
$$b^n u[n] \xleftrightarrow{Z} \frac{1}{1-bz^{-1}}, \qquad |z| > b,$$

b>1

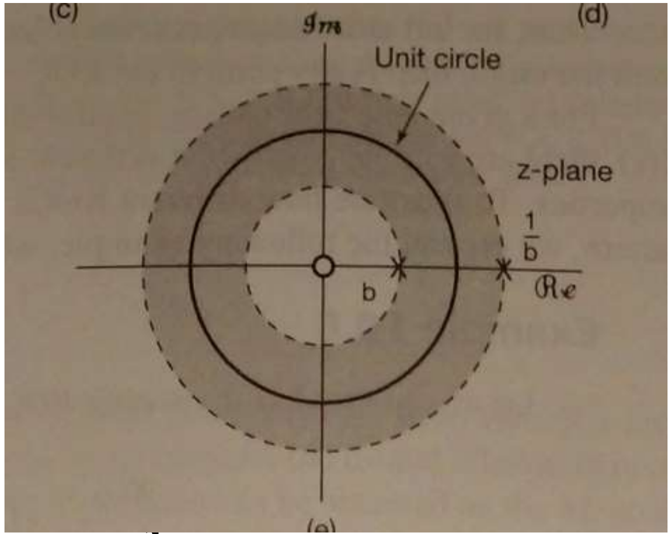


$$b^{-n} u[-n-1] \xleftrightarrow{Z} \frac{1}{1-b^{-1}z^{-1}}, |z| < \frac{1}{b}.$$

0<b<1



Example 10.7



$$X(z)=\frac{1}{1-bz^{-1}}-\frac{1}{1-b^{-1}z^{-1}}, \qquad b<|z|<\frac{1}{b}, \qquad (10.35)$$

or equivalently,

$$X(z)=\frac{b^2-1}{b}\frac{z}{(z-b)(z-b^{-1})}, \qquad b<|z|<\frac{1}{b}. \qquad (10.36)$$

The corresponding pole-zero pattern and ROC are shown in Figure 10.11(e).

10.2 The Region of Convergence for The z-Transform

- Property 7: If the *z*-transform *X(z)* of *x[n]* is **rational**, then its ROC is **bounded** by poles or extends to infinity.

性質7：若*x[n]*的*z*轉換*X(z)*為有理式，則其ROC的範圍受限於極點或是延伸至無限。

10.2 The Region of Convergence for The z-Transform

- Property 8: If the z-transform $X(z)$ of $x[n]$ is **rational**, and if $x[n]$ is right sided, then the ROC is the region in the z-plane outside the outermost pole—i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$. Furthermore, if $x[n]$ is **causal** (i.e., if it is right sided and equal to 0 for $n < 0$), then the ROC also includes $z = \infty$.

10.2 The Region of Convergence for The z-Transform

- Property 9: If the z-transform $X(z)$ of $x[n]$ is **rational**, and if $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost nonzero pole—i.e., inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z = 0$ and extending inward to and possibly including $z = 0$. In particular, if $x[n]$ is **anticausal** (i.e., if it is left sided and equal to 0 for $n > 0$),then the ROC also includes $z = 0$.

Example 10.8

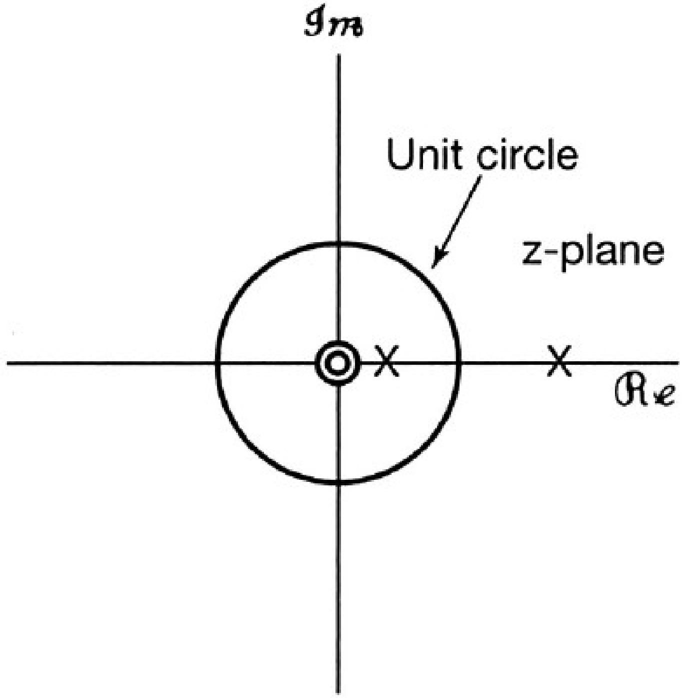
Let us consider all of the possible ROCs that can be connected with the function

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}. \qquad (10.37)$$

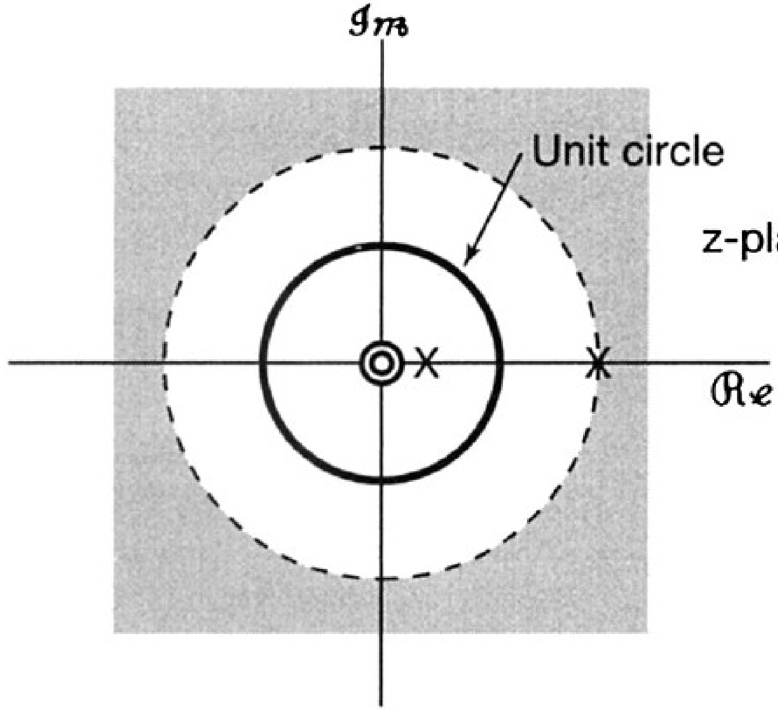
The associated pole-zero pattern is shown in Figure 10.12(a). Based on our discussion in this section, there are three possible ROCs that can be associated with this algebraic expression for the z-transform.

Example 10.8

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}.$$



(a)



(b)

Example 10.8

$$X(z)=\frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}.$$

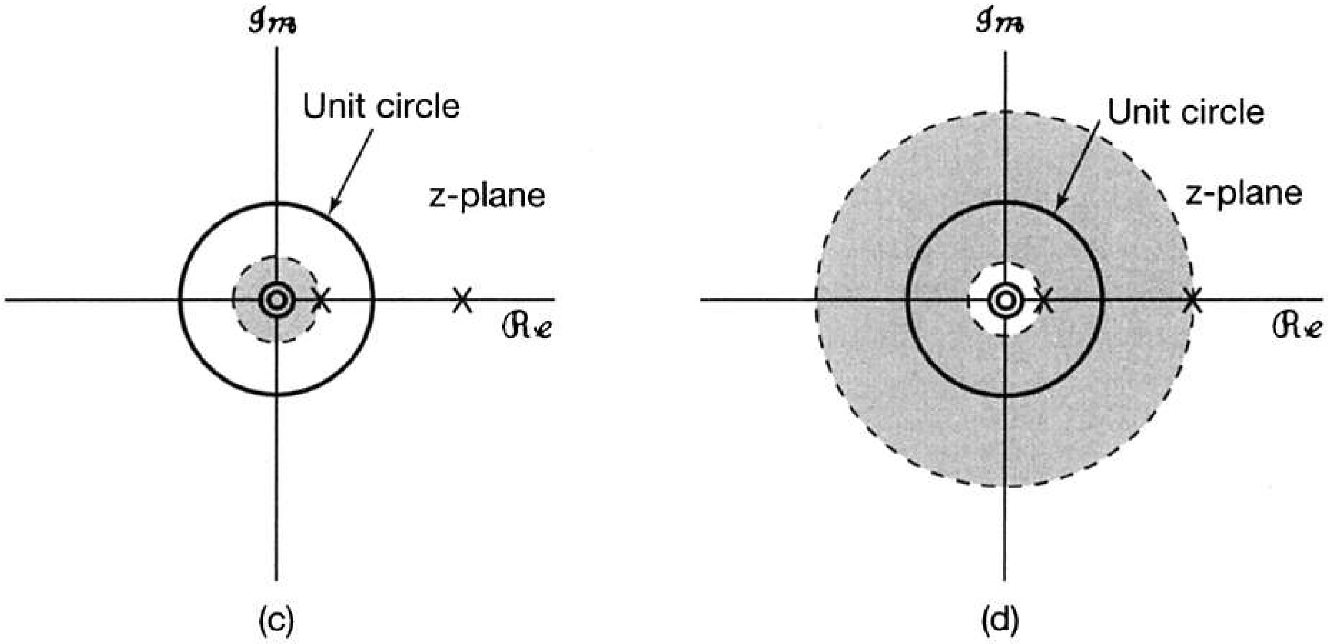


Figure 10.12 The three possible ROCs that can be connected with the expression for the z-transform in Example 10.8: (a) pole-zero pattern for $X(z)$; (b) pole-zero pattern and ROC if $x[n]$ is right sided; (c) pole-zero pattern and ROC if $x[n]$ is left sided; (d) pole-zero pattern and ROC if $x[n]$ is two sided. In each case, the zero at the origin is a second-order zero.

10.3 The Inverse z-Transform

This expression can be obtained on the basis of the interpretation, developed in Section 10.1, of the z-transform as the Fourier transform of an exponentially weighted sequence. Specifically, as expressed in eq. (10.7),

$$X(z = re^{j\omega}) = F\left\{x[n]r^{-n}\right\}, \qquad (10.38)$$

10.3 The Inverse z-Transform

for any value of r so that $z = re^{j\omega}$ is inside the ROC. Applying the inverse Fourier transform to both sides of eq. (10.38) yields

$$x[n]r^{-n} = F^{-1}\left\{X\left(re^{j\omega}\right)\right\},$$

or

$$x[n] = r^{n} F^{-1}\left[X\left(re^{j\omega}\right)\right]. \tag{10.39}$$

10.3 The Inverse z-Transform

Using the inverse Fourier transform expression in eq. (5.8), we have

$$x[n] = r^n \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega,$$

or moving the exponential factor r^n inside the integral and combining it with the term $e^{j\omega n}$,

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega. \qquad (10.40)$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega.$$

$dz = jre^{j\omega}d\omega \Rightarrow d\omega = 1/(jre^{j\omega})dz = (1/jz)\,dz$

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz, \qquad (10.41)$$

The value of r can be chosen as any value for which *X(z)* converges—i.e., any value such that the circular contour of integration $|z| = r$ is in the ROC.

The integration around a counterclockwise closed circular contour centered at the origin with radius r, and |z|=r must be in ROC.

10.3 The Inverse z-Transform

There are, however, a number of alternative procedures for obtaining a sequence from its *z*-transform.

反z轉換亦有一些替代的方法，如部分分式展開法等等。

Example 10.9

Consider the *z*-transform

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})},\qquad |z|>\frac{1}{3}. \tag{10.42}$$

There are two poles, one at z = 1/3 and one at z = ¼, and the ROC lies outside the outermost pole. That is, the ROC consists of all points with magnitude greater than that of the pole with the larger magnitude, namely the pole at z = 1/3. From Property 4 in Section 10.2, we then know that the inverse transform is a **right**-sided sequence.

Example 10.9

As described in the appendix, *X(z)* can be expanded by the method of **partial fractions**. For this example, the partial-fraction expansion, expressed in polynomials in z^{-1}, is

$$X(z) = \frac{1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}}. \qquad (10.43)$$

Example 10.9

Thus, *x[n]* is the sum of two terms, one with *z*-transform $1/\left[1-(1/4)z^{-1}\right]$ and the other with *z*-trandform $2/\left[1-(1/3)z^{-1}\right]$. In order to determine the inverse *z*-transform of each of these individual terms, we must specify the ROC associated with each. Since the ROC for *X(z)* is outside the outermost pole, the ROC for each individual term in eq. (10.43) must also be outside the pole associated with that term.

Example 10.9

$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|.$$

That is, the ROC for each term consists of all points with magnitude greater than the magnitude of the corresponding pole. Thus,

$$x[n] = x_1[n] + x_2[n], \tag{10.44}$$

where

$$x_1[n] \overset{Z}{\longleftrightarrow} \frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}, \tag{10.45}$$

$$x_2[n] \overset{Z}{\longleftrightarrow} \frac{2}{1 - \frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}. \tag{10.46}$$

Example 10.9

From Example 10.1, we can identify by inspection that

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n] \qquad (10.47)$$

and

$$x_2[n] = 2\left(\frac{1}{3}\right)^n u[n], \qquad (10.48)$$

and thus,

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]. \qquad (10.49)$$

10.3 The Inverse z-Transform

The inverse transform of each term can then be obtained by inspection. In particular, suppose that the partial-fraction expansion of $X(z)$ is of the form

$$X(z)=\sum_{i=1}^{m}\frac{A_i}{1-a_i z^{-1}}, \qquad (10.55)$$

若將$X(z)$展開成(10.55)式的部分分式型式，則可個別求其反z轉換。

10.3 The Inverse z-Transform

So that the inverse transform of *X(z)* equals the sum of the inverse transforms of the individual terms in the equation. If the ROC of *X(z)* is **outside** the pole at $z = a_i$, the inverse transform of the corresponding term in eq. (10.55) is $A_i a_i^n u[n]$. On the other hand, if the ROC of *X(z)* is **inside** the pole at $z = a_i$, the inverse transform of this term is $-A_i a_i^n u[-n-1]$

若*X(z)*的單項的ROC在極點 $z = a_i$ 外側，在其反*z*轉換，為 $A_i a_i^n u[n]$；若*X(z)*的單項的ROC在極點 $z = a_i$ 內側，則其反*z*轉換為 $-A_i a_i^n u[-n-1]$。

Example 10.13

Consider
$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|.$$

This expression can be expanded in a power series by long division:

$$\begin{array}{r l}
 & 1 + az^{-1} + a^2 z^{-2} + \cdots \\
\cline{2-2}
1 - az^{-1} \,\big) & 1 \\
 & 1 - az^{-1} \\
\cline{2-2}
 & \quad az^{-1} \\
 & \quad az^{-1} - a^2 z^{-2} \\
\cline{2-2}
 & \qquad\qquad a^2 z^{-2}
\end{array}$$

Example 10.13

$$X(z) \stackrel{\Delta}{=} \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \quad (10.3)$$

or

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \ldots. \qquad (10.58)$$

The series expansion of eq. (10.58) converges, since $|z| > |a|$, or equivalently, $|az^{-1}| < 1$ Comparing this equation with the definition of the *z*-transform in equation (10.3), we see, by matching terms in powers of z, that *x[n]* = 0, *n* < 0; *x[0]* = 1; *x[1]* = *a*; *x[2]* = a^2; and in general, $x[n] = a^n u[n]$ which is consistent with Example 10.1.

Example 10.13

If, instead, the ROC of *X(z)* is specified as $|z| < |a|$ or, equivalently, $|az^{-1}| > 1$, then the power-series expansion for $1/(1 - az^{-1})$ in eq. (10.58) does not converge. However, we can obtain a convergent power series by long division again.

$$\begin{array}{rl} & -a^{-1}z - a^{-2}z^{2} - \cdots \\ -az^{-1} + 1 & \overline{\big)\; 1 \qquad\qquad\qquad} \\ & 1 - a^{-1}z \\ \cline{2-2} & a^{-1}z \end{array}$$

Example 10.13

$$X(z) \stackrel{\Delta}{=} \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \quad (10.3)$$

or

$$\frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^{2} - \ldots \qquad (10.59)$$

In this case, then, $x[n] = 0$, $n \geqq 0$; and $x[-1] = -a^{-1}$, $x[-2] = -a^{-2}$, …; that is, $x[n] = -a^{n}u[-n-1]$. This is consistent with Example 10.2.

10.4.1 First-Order Systems

The impulse response of a first-order causal discrete-time system is of the general form

$$h[n] = a^n u[n], \qquad (10.64)$$

and from Example 10.1, its z-transform is

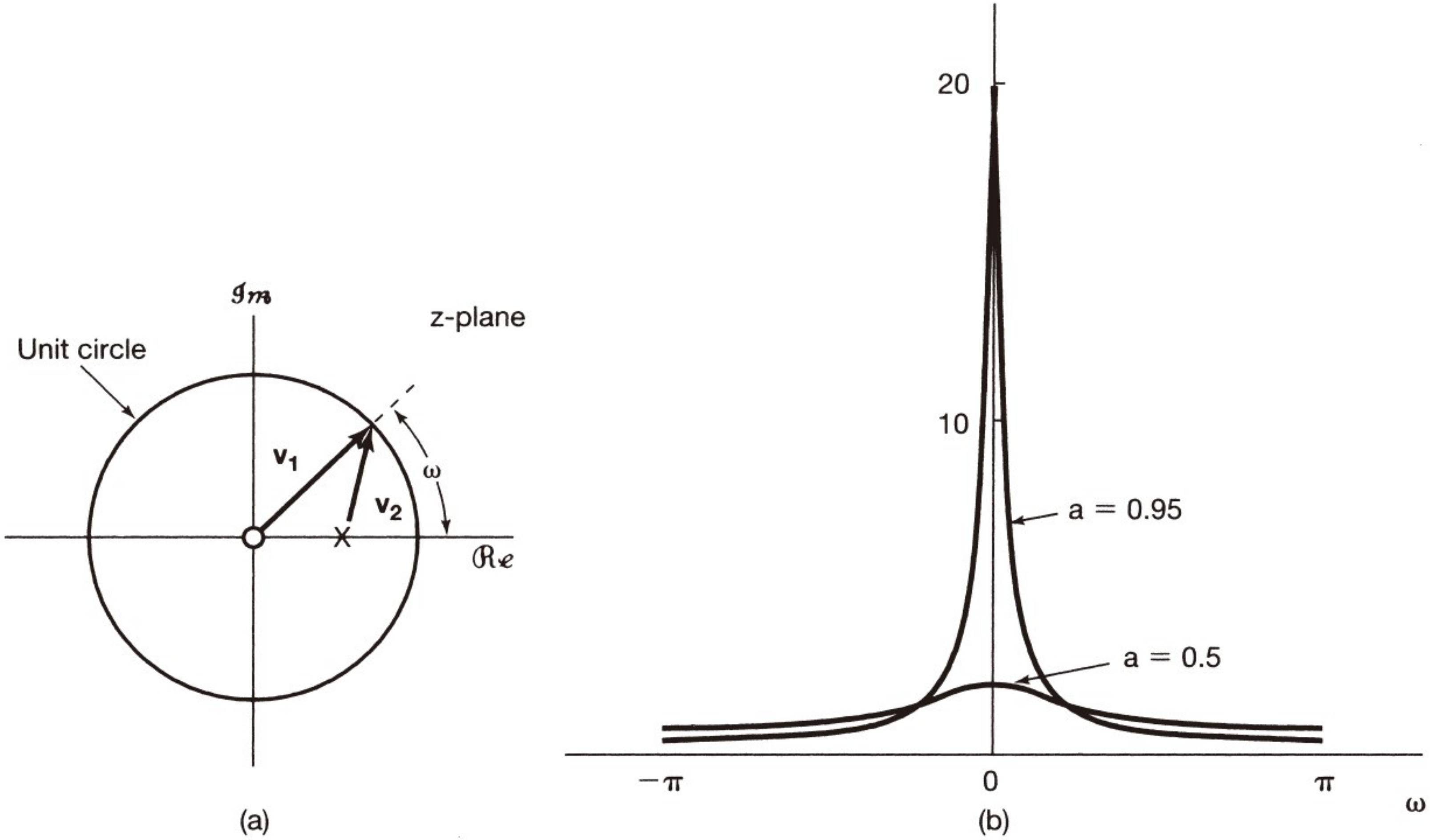
$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|. \qquad (10.65)$$

10.4.1 First-Order Systems

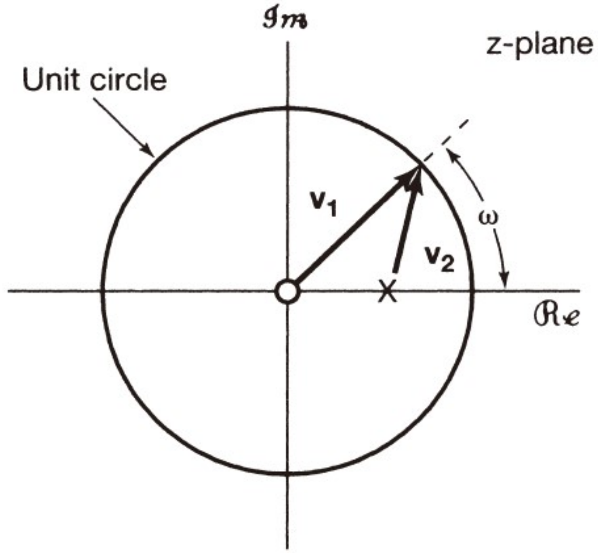
For $|a|<1$, the ROC includes the unit circle, and consequently, the Fourier transform of *h[n]* converges and is equal to *H(z)* for $z=e^{j\omega}$.

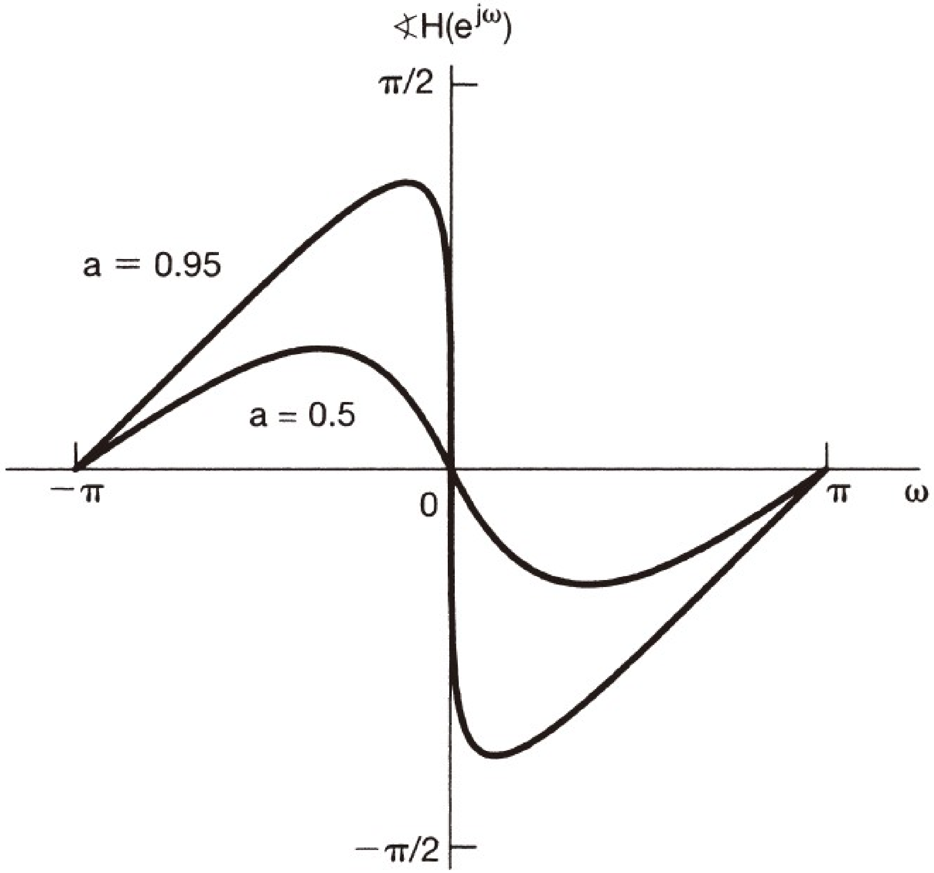
$$H(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}. \tag{10.66}$$

Note first that, as illustrated in Figure 10.13, the magnitude of the peak of $H(e^{j\omega})$ at ω = 0 decreases as $|a|$ decreases toward 0.



10.4.1 First-Order Systems





(c)

圖 10.3 爲 (10.64) 至 (10.66) 式的一階系統的極零點圖及其頻率響應圖。由極零點圖可知在單位圓上以 $\omega = 0$ 時所得的 $\mathbf{v}_1$ 與 $\mathbf{v}_2$ 的大小比值最大。不同的 a 值可使 $\omega = 0$ 時的頻率響應程度大不相同。

Figure 10.13 (a) Pole and zero vectors for the geometric determination of the frequency response for a first-order system for a value of a between 0 and 1; (b) magnitude of the frequency response for $a = 0.95$ and $a = 0.5$; (c) phase of the frequency response for $a = 0.95$ and $a = 0.5$.

10.4.2 Second-Order Systems

with impulse response and frequency response given in eqs. (6.64) and (6.60), which we respectively repeat here as

$$h[n] = r^n \frac{\sin(n+1)\theta}{\sin\theta} u[n] \tag{10.67}$$

and

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}, \tag{10.68}$$

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}},$$

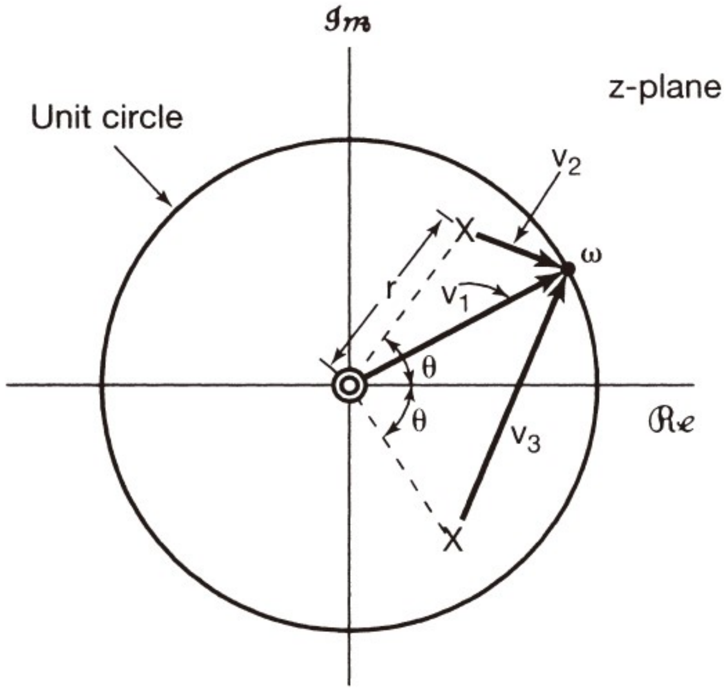
Where $0 < r < 1$ and $0 \leqq \theta \leqq \pi$. Since $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$, we can infer from eq. (10.68) that the system function, corresponding to the *z*-transform of the system impulse response, is

$$H(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}. \qquad (10.69)$$

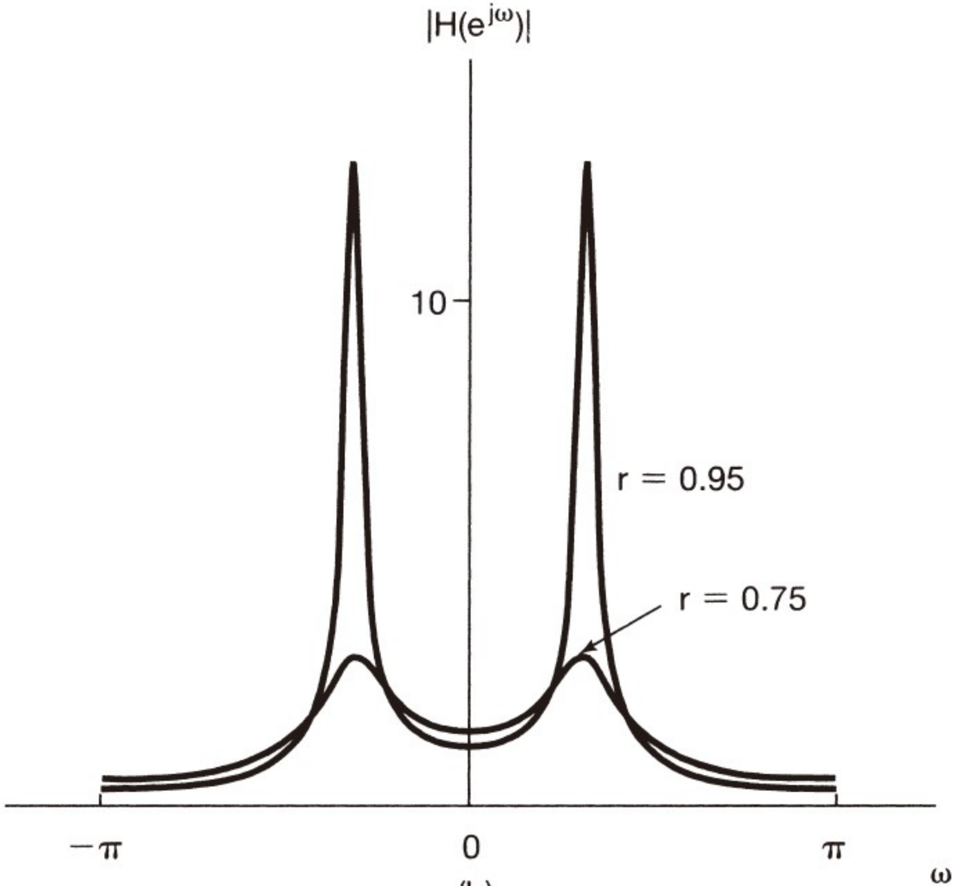
The poles of *H(z)* are located at

$$z_1 = re^{j\theta}, \qquad z_2 = re^{-j\theta}, \qquad (10.70)$$

When r approach 1, the frequency response peak more sharply. Corresponding to r decreasing, the impulse response decays more rapidly and the step response settles more quickly.



(a)



(b)

10.4.2 Second-Order Systems

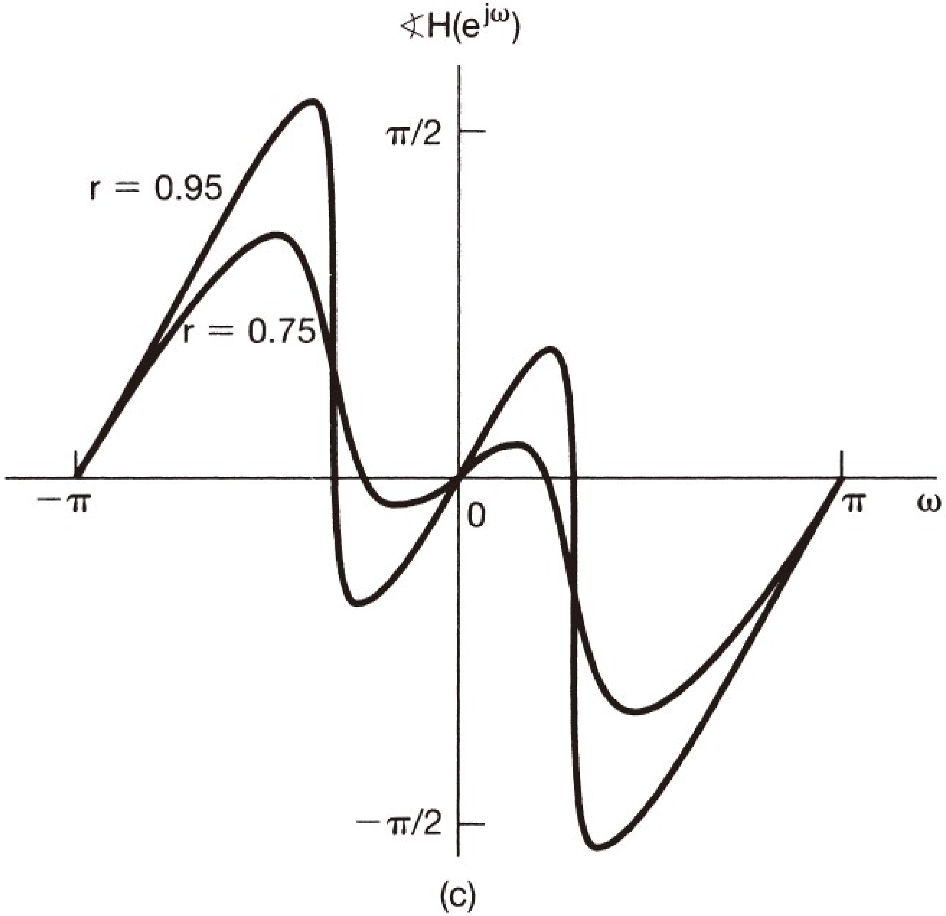


Figure 10.14 (a) Zero vector $\mathbf{v}_1$ and pole vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ used in the geometric calculation of the frequency responses for a second-order system; (b) magnitude of the frequency response corresponding to the reciprocal of the product of the lengths of the pole vectors for $r = 0.95$ and $r = 0.75$; (c) phase of the frequency response for $r = 0.95$ and $r = 0.75$.

10.5.1 Linearity

If

$$x_1[n] \xleftrightarrow{Z} X_1(z), \qquad with\, ROC = R_1,$$

and

$$x_2[n] \xleftrightarrow{Z} X_2(z), \qquad with\, ROC = R_2,$$

then

$ax_1[n]+bx_2[n] \xleftrightarrow{Z} aX_1(z)+bX_2(z),$ with ROC **containing** $R_1 \cap R_2$.

(10.71)

線性性質（即重疊原理）

10.5.2 Time Shifting

If

$$x[n] \xleftrightarrow{Z} X(z), \qquad with\, ROC = R,$$

then

$$x[n-n_0] \xleftrightarrow{Z} z^{-n_0} X(z), \qquad (10.72)$$

with ROC = R, except for the possible addition or deletion of the origin or infinity.

時間移位性質

10.5.3 **Scaling** in the z-Domain

If

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad with\ ROC = R,$$

then

$$z_0^n x[n] \overset{Z}{\longleftrightarrow} X\left(\frac{z}{z_0}\right), \qquad with\ ROC = |z_0|R, \qquad (10.73)$$

刻度變換性質

10.5.3 Scaling in the z-Domain

An important special case of eq. (10.73) is when $z_0 = e^{j\omega_0}$. In this case, $|z_0|R = R$ and

$$e^{j\omega_0 n}x[n] \stackrel{Z}{\longleftrightarrow} X(e^{-j\omega_0}z). \qquad (10.74)$$

$$e^{j\omega_0 n}x[n] \xleftrightarrow{Z} X(e^{-j\omega_0}z).$$

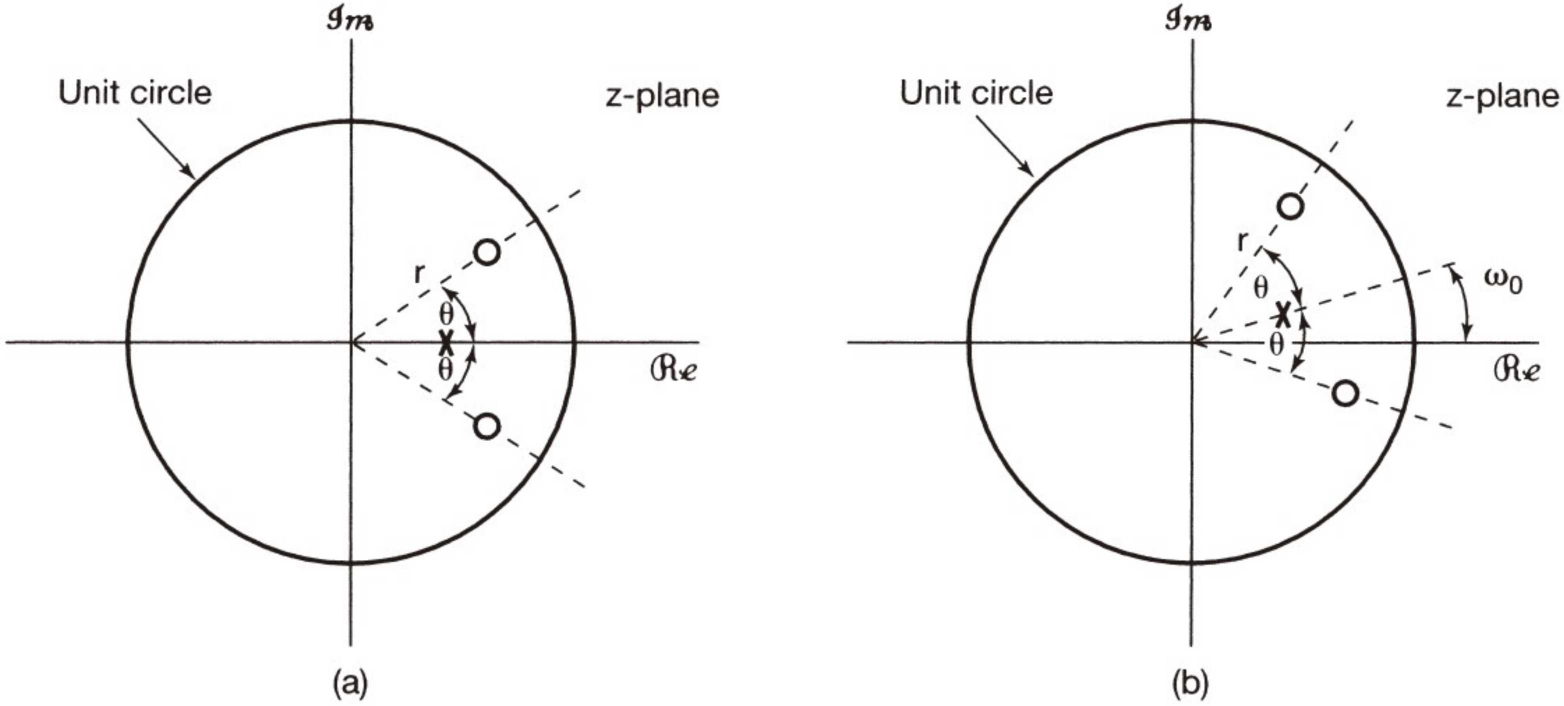


Figure 10.15 Effect on the pole-zero plot of time-domain multiplication by a complex exponential sequence $e^{j\omega_0 n}$: (a) pole-zero pattern for the z-transform for a signal $x[n]$; (b) pole-zero pattern for the z-transform of $x[n]e^{j\omega_0 n}$.

10.5.4 Time Reversal

If

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad with\ ROC = R,$$

then

$$x[-n] \overset{Z}{\longleftrightarrow} X\left(\frac{1}{z}\right), \qquad with\ ROC = \frac{1}{R}. \tag{10.75}$$

時間倒轉性質

10.5.5 Time Expansion

Specifically, the sequence $x_{(k)}[n]$, introduced in Section 5.3.7 and defined as

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases} \quad (10.76)$$

10.5.5 Time Expansion

has $k-1$ zeros inserted between successive values of the original signal. In this case, if

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad with\ ROC = R,$$

then

$$x_{(k)}[n] \overset{Z}{\longleftrightarrow} X(z^k), \qquad with\ ROC = R^{1/k}, \tag{10.77}$$

時間延展性質

10.5.5 Time Expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

The interpretation of this result follows from the power-series form of the z-transform, from which we see that the coefficient of the term z^{-n} equals the value of the signal at time n. That is, with

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n},$$

it follows that

$$X(z^k) = \sum_{n=-\infty}^{+\infty} x[n](z^k)^{-n} = \sum_{n=-\infty}^{+\infty} x[n]z^{-kn} = \sum_{m=\mathrm{int}\, mul\ \ of\ \ k} x[m/k]z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} x_k[m]z^{-m} = Z\{x_k[m]\} \qquad (10.78)$$

10.5.6 Conjugation

If

$$x[n] \xleftrightarrow{Z} X(z), \qquad with\ ROC = R, \tag{10.79}$$

then

$$x^*[n] \xleftrightarrow{Z} X^*(z^*), \qquad with\ ROC = R. \tag{10.80}$$

共軛性質

Consequently, if *x[n]* is real, we can conclude from eq. (10.80) that

$$X(z) = X^*(z^*).$$

10.5.7 The Convolution Property

If

$$x_1[n] \xleftrightarrow{Z} X_1(z), \qquad with\ ROC = R_1,$$

and

$$x_2[n] \xleftrightarrow{Z} X_2(z), \qquad with\ ROC = R_2,$$

then

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z), \quad \text{with ROC containing} R_1 \cap R_2. \tag{10.81}$$

迴旋運算性質

Example 10.15

Consider an LTI system for which

$$y[n] = h[n] * x[n], \tag{10.82}$$

where

$$h[n] = \delta[n] - \delta[n-1].$$

Note that

$$\delta[n] - \delta[n-1] \xleftrightarrow{Z} 1 - z^{-1}, \tag{10.83}$$

Example 10.15

with ROC equal to the entire z-plane except the origin. Also, the z-transform in eq. (10.83) has a zero at $z = 1$. From eq. (10.81), we see that if

$$x[n] \xleftrightarrow{Z} X(z), \qquad with\ ROC = R,$$

then

$$y[n] \xleftrightarrow{Z} (1 - z^{-1})X(z), \tag{10.84}$$

Example 10.15

with ROC equal to R, with the possible deletion of $z = 0$ and/or addition of $z = 1$.

Note that for this system

$$y[n] = \left[\delta[n] - \delta[n-]\right] * x[n] = x[n] - x[n-1].$$

Example 10.15

That is, $y[n]$ is the first difference of the sequence $x[n]$. Since the first-difference operation is commonly thought of as a discrete-time counterpart to differentiation, eq. (10.83) can be thought of as the z-transform counterpart of the Laplace transform **differentiation** property presented in Section 9.5.7.

10.5.8 Differentiation in The Z-Domain

If

$$x[n] \xleftrightarrow{Z} X(z), \qquad with\ ROC = R,$$

Then

$$nx[n] \xleftrightarrow{Z} -z\frac{dX(z)}{dz}, \qquad with\ ROC = R. \tag{10.87}$$

z域微分性質

Example 10.17

If

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|, \tag{10.88}$$

Then

$$nx[n] \stackrel{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}}, \qquad |z| > |a|. \tag{10.89}$$

By differentiating, we have converted the z-transform to a **rational** expression. The inverse z-transform of the right-hand side of eq. (10.89) can be obtained by using Example 10.1 together with the **time-shifting** property, eq. (10.72), set forth in Section 10.5.2.

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad with\ ROC = R.$$

Specifically, from Example 10.1 and the linearity property,

$$a(-a)^n u[n] \overset{Z}{\longleftrightarrow} \frac{a}{1+az^{-1}}, \qquad |z| > |a|. \tag{10.90}$$

Combining this with the time-shifting property yields

$$a(-a)^{n-1} u[n-1] \overset{Z}{\longleftrightarrow} \frac{az^{-1}}{1+az^{-1}}, \qquad |z| > |a|.$$

Consequently,

$$x[n] = \frac{-(-a)^n}{n} u[n-1]. \tag{10.91}$$

10.5.9 The Initial-Value Theorem

If $x[n] = 0$, $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z). \tag{10.95}$$

初值定理

This property follows by considering the limit of each term individually in the expression for the z-transform, with $x[n]$ zero for $n < 0$. With this constraint,

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

Example 10.19

The initial-value theorem can also be useful in checking the **correctness** of the *z*-transform calculation for a signal. For example, consider the signal *x[n]* in Example 10.3. From eq. (10.12), we see that *x[0]* = 1. Also, from eq. (10.14),

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})} = 1,$$

which is consistent with the initial-value theorem.

表10.1為常用的z轉換性質。

10.5.10 Summary of Properties

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \to \infty} X(z)$			

In Table 10.1, we summarize the properties of the z-transform.

10.6 Some Common z-Transform Pairs

In Table 10.2, we have listed a number of useful z-transform pairs.

表10.2為常用的z轉換對（常用的訊號及其z轉換）。

10.6 Some Common z-Transform Pairs

TABLE 10.2 SOME COMMON *z*-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos \omega_0 n]u[n]$	$\dfrac{1 - [\cos \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin \omega_0 n]u[n]$	$\dfrac{[\sin \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\dfrac{1 - [r\cos \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\dfrac{[r\sin \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$

10.7 Analysis and Characterization of LTI Systems Using z-Transforms

The z-transform plays a particularly important role in the analysis and representation of discrete-time LTI systems. From the **convolution** property presented in Section 10.5.7,

$$Y(z) = H(z)X(z), \qquad (10.96)$$

LTI系統的輸入、輸出及脈衝響應的z轉換滿足（10.96）式的關係。

10.7.1 Causality

As we saw in Property 8 in Section 10.2, for a **causal** system the power series

$$H(z)=\sum_{n=0}^{\infty} h[n]z^{-n}$$

does not include any positive power of z->ROC include infinity.

A discrete-time LTI system is **causal** if and only if the ROC of its system function is the **exterior of a circle, including infinity**.

一離散時間LTI系統為因果的，若且唯若系統函數的ROC為某個圓的外側，包含無限大處。

10.7.1 Causality

A discrete-time LTI system with **rational** system function *H(z)* is causal if and only if: (a) the ROC is the exterior of a circle outside the outermost pole; and (b) with *H(z)* expressed as a ratio of polynomials in *z*, **the order of the numerator cannot be greater than the order of the denominator (i.e., no pole at infinity).**

一個具有有理式系統函數*H(z)*的離散時間LTI系統為因果的，若且唯若：(a) ROC為最外側極點之外的某個圓的外側；且(b) *H(z)*表成多項式比值後，分子次數不大於分母次數。

Example 10.21

Consider a system with system function

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 - 2z^{-1}}, \qquad |z| > 2 \tag{10.97}$$

Since the ROC for this system function is the exterior of a circle outside the outermost pole, we know that the impulse response is **right-sided**. To determine if the system is causal, we then need only check the other condition required for causality, namely that *H(z),* when expressed as a ratio of polynomials in *z*, has **numerator degree no larger than the denominator**. For this example,

Example 10.21

$$H(z)=\frac{2-\frac{5}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}=\frac{2z^2-\frac{5}{2}z}{z^2-\frac{5}{2}z+1}, \qquad (10.98)$$

so that the numerator and denominator of *H(z)* are both of degree two, and consequently we can conclude that the system is **causal**. This can also be verified by calculating the **inverse transform** of *H(z).* In particular, using transform pair 5 in Table 10.2, we find that the impulse response of this system is

$$h[n]=\left[\left(\frac{1}{2}\right)^n+2^n\right]u[n]. \qquad (10.99)$$

Since *h[n]* = 0 for *n*<0, we can confirm that the system is causal.

10.7.2 Stability

An LTI system is **stable** if and only if the impulse response is absolutely summable.

=> if and only if the ROC of its system function *H(z)* includes the unit circle, $|z|=1$.

一個LTI系統為穩定，若且唯若*H(z)*的ROC包含單位圓 。

Example 10.22

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \qquad |z| > 2 \quad (10.97)$$

$$h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right]u[n] \quad (10.99)$$

Consider again the system function in eq. (10.97). Since the associated ROC is the region $|z| > 2$, which does not include the unit circle, the system is not stable. This can also be seen by noting that the impulse response in eq. (10.99) is not absolutely summable. If, however, we consider a system whose system function has the same algebraic expression as in eq. (10.97) but whose ROC is $1/2 < |z| < 2$, then the ROC dies contain the unit circle, so that the corresponding system is **noncausal** but **stable**.

Example 10.22

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \quad 0.5 < |z| < 2$$

In this case, using transform pairs 5 and 6 from Table 10.2, we find that the corresponding impulse response is

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1], \qquad (10.100)$$

which is absolutely summable.

Example 10.22

Also, for the third possible choice of ROC associated with the algebraic expression for *H(z)* in eq. (10.97), namely, $|z| < 1/2$, the corresponding system is **neither causal** (since the ROC is not outside the outermost pole) **nor stable** (since the ROC does not include the unit circle). This can also be seen from the impulse response, which (using transform pair 6 in Table 10.2) is

$$h[n] = -\left[\left(\frac{1}{2}\right)^n + 2^n\right]u[-n-1].$$

10.7.2 Stability

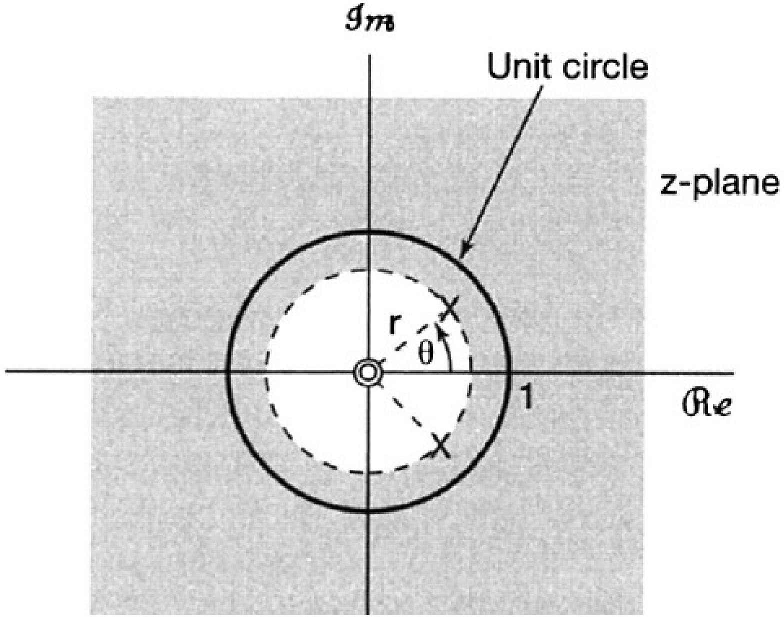
A causal LTI system with rational system function *H(z)* is stable if and only if all of the poles of *H(z)* lie **inside the unit circle**—i.e., they must all have magnitude smaller than 1.

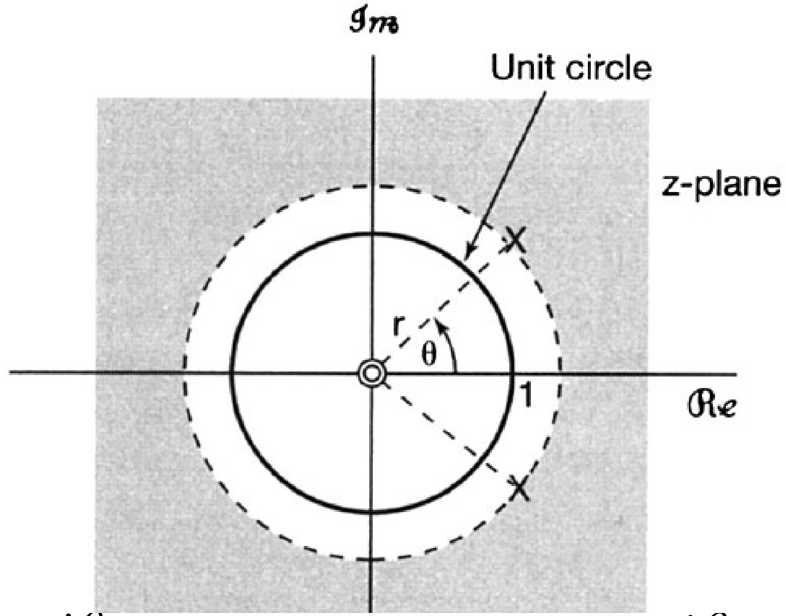
一個具有有理系統函數*H(z)*的因果LTI系統為穩定，若且唯若*H(z)*的所有極點均位於單位圓內。亦即，所有的極點的大小均小於1。

Example 10.24

The system function for a second-order system with complex poles was given in eq. (10.69), specifically,

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}}, \qquad (10.101)$$





With poles located at $z_1 = re^{j\theta}$ and $z_2 = re^{-j\theta}$. Assuming causality, we see that the ROC is outside the outermost pole (i.e., $|z| > |r|$). The pole-zero plot and ROC for this system are shown in Figure 10.16 for r < 1 and r > 1. For r < 1, the poles are inside the unit circle, the ROC includes the unit circle, and therefore, the system is stable. For r > 1, the poles are outside the unit circle, the ROC does not include the unit circle, and the system is unstable.

Example 10.25

Consider an LTI system for which the input *x[n]* and output *y[n]* satisfy the linear constant-coefficient difference equation

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]. \qquad (10.102)$$

Example 10.25 $y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1].$

Applying the z-transform to both sides of eq. (10.102), and using the **linearity** property set forth in Section 10.5.1 and the **time-shifting** property presented in Section 10.5.2, we obtain

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z),$$

or

$$Y(z)=X(z)\left[\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right]. \qquad (10.103)$$

Example 10.25

From eq. (10.96), then,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}. \tag{10.104}$$

This provides the **algebraic** expression for *H(z),* but **not the region of convergence**. In fact, there are two distinct impulse responses that are consistent with the difference equation (10.102), one right sided and the other left sided. Correspondingly, there are two different choices for the ROC associated with the algebraic expression (10.104).

Example 10.25

One, $|z| > 1/2$, is associated with the assumption that h[n] is **right** sided, and the other $|z| < 1/2$, is associated with the assumption that h[n] is **left** sided.

Consider first the choice of ROC equal to $|z| > 1/2$. Writing

$$H(z) = \left(1 + \frac{1}{3}z^{-1}\right)\frac{1}{1 - \frac{1}{2}z^{-1}},$$

Example 10.25

$$H(z) = \left(1 + \frac{1}{3} z^{-1}\right) \frac{1}{1 - \frac{1}{2} z^{-1}},$$

we can use transform pair 5 in Table 10.2, together with the linearity and time-shifting properties, to find the corresponding impulse response

$$h[n] = \left(\frac{1}{2}\right)^{n} u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

Example 10.25

For the other choice of ROC, namely, $|z| < 1/2$, we can use transform pair 6 in Table 10.2 and the linearity and time-shifting properties, yielding

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

In this case, the system is **anticausal** (*h[n]* = 0 for *n* > 0) and unstable.

10.7.2 Stability

In particular, consider an LTI system for which the input and output satisfy a linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{m} b_k x[n-k]. \qquad (10.105)$$

*N*階線性常係數差分方程式（*N*階LTI系統）

10.7.2 Stability

Then taking *z*-transforms of both sides of eq. (10.105) and using the linearity and time-shifting properties, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

or

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k},$$

10.7.2 Stability

So that

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \qquad (10.106)$$

系統函數與差分方程系數的關係式

Example 10.26

Suppose that we are given the following information about an LTI system:

1. If the input to the system is $x_1[n]=(1/6)^n u[n]$, then the output is

$$y_1[n]=\left[a\left(\frac{1}{2}\right)^n+10\left(\frac{1}{3}\right)^n\right]u[n],$$

Example 10.26

2. If $x_1[n]=(-1)^n$, then the output is $y_2[n]=\frac{7}{4}(-1)^n$. As we now show, from these two pieces of information, we con determine the system function *H(z)* for this system, including the value of the number *a*, and can also immediately deduce a number of other properties of the system.

The *z*-transforms of the signals specified in the first piece of information are

$$X_1(z)=\frac{1}{1-\frac{1}{6}z^{-1}}, \qquad |z|>\frac{1}{6}, \tag{10.107}$$

Example 10.26

$$X_1(z) = \frac{1}{1-\frac{1}{6}z^{-1}}, \qquad |z| > \frac{1}{6},$$

$$Y_1(z) = \frac{a}{1-\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{(a+10)-(5+\frac{a}{3})z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}, \qquad |z| > \frac{1}{2}. \tag{10.108}$$

From eq. (10.96), it follows that the algebraic expression for the system function is

$$H(z) = \frac{Y_1(z)}{X_1(z)} = \frac{\left[(a+10)-(5+\frac{a}{3})z^{-1}\right]\left[1-\frac{1}{6}z^{-1}\right]}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}. \tag{10.109}$$

Example 10.26 $y[n] = H(z)z^n,$

Furthermore, we know that the response to $x_2[n] = (-1)^n$ must equal $(-1)^n$ multiplied by the system function *H(z)* evaluated at *z* = -1. Thus from the second piece of information given, we see that

$$\frac{7}{4} = H(-1) = \frac{\left[(a+10)+5+\frac{a}{3}\right]\left[\frac{7}{6}\right]}{(\frac{3}{2})(\frac{4}{3})}. \qquad (10.110)$$

$$y_2[n] = \frac{7}{4}(-1)^n$$

Example 10.26

Solving eq. (10.110), we find that a = -9, so that

$$H(z)=\frac{(1-2z^{-1})(1-\frac{1}{6}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}, \tag{10.111}$$

or

$$H(z)=\frac{1-\frac{13}{6}z^{-1}+\frac{1}{3}z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}, \tag{10.112}$$

Example 10.26

$$H(z)=\frac{(1-2z^{-1})(1-\frac{1}{6}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})},$$

$$X_1(z)=\frac{1}{1-\frac{1}{6}z^{-1}}, \qquad |z|>\frac{1}{6},$$

or, finally,

$$H(z)=\frac{z^2-\frac{13}{6}z+\frac{1}{3}}{z^2-\frac{5}{6}z+\frac{1}{6}}.$$

Also, from the convolution property, we know that the ROC of $Y_1(z)$ must include at least the intersections of the ROCs of $X_1(z)$ and $H(z)$. Examining the three possible ROCS for $H(z)$ (namely, $|z|<1/3, 1/3<|z|<1/2$, and $|z|>1/2$), we find that the only choice that is consistent with the ROCs of $X_1(z)$ and $Y_1(z)$ is $|z|>1/2$.

Example 10.26

$$H(z) = \frac{1 - \frac{13}{6} z^{-1} + \frac{1}{3} z^{-2}}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}},$$

Since the ROC for the system includes the unit circle, we know that the system is **stable**. Furthermore, from eq. (10.113) with *H(z)* viewed as a ratio of polynomials in z, the order of the numerator **does not** exceed that of the denominator, and thus we can conclude that the LTI system is **causal**. Also, using eqs. (10.112) and (10.106), we can write the difference equation that, together with the condition of **initial rest**, characterizes the system:

$$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n] - \frac{13}{6} x[n-1] + \frac{1}{3} x[n-2].$$

10.8.1 System Functions for Interconnections of LTI Systems

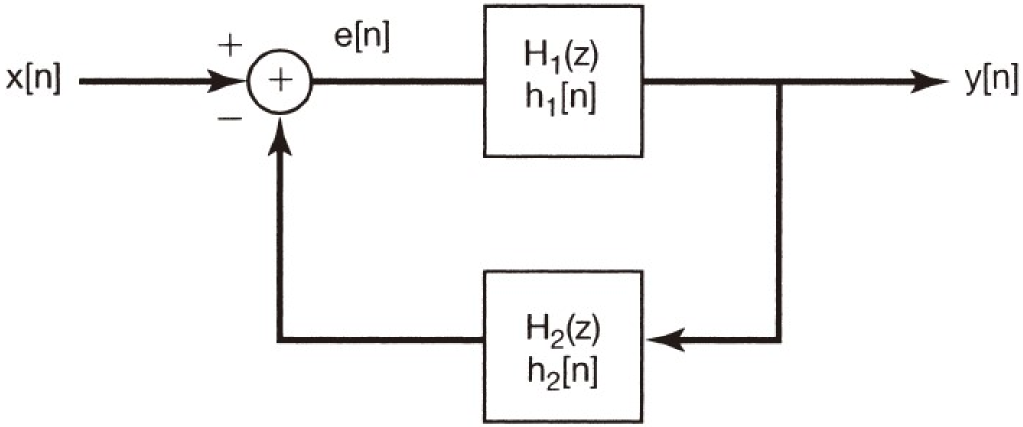


Figure 10.17 Feedback interconnection of two systems.

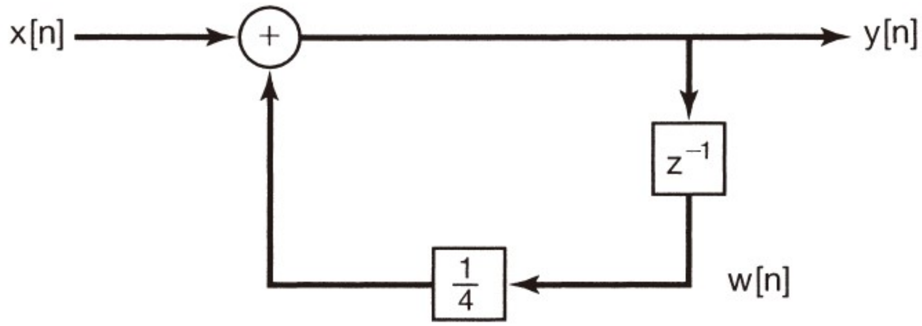
The specific equations for the interconnection of Figure 10.17 exactly parallel eqs. (9.159)—(9.163), with the final result that the overall system function for the feedback system of Figure 10.17 is

$$\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}. \qquad (10.115)$$

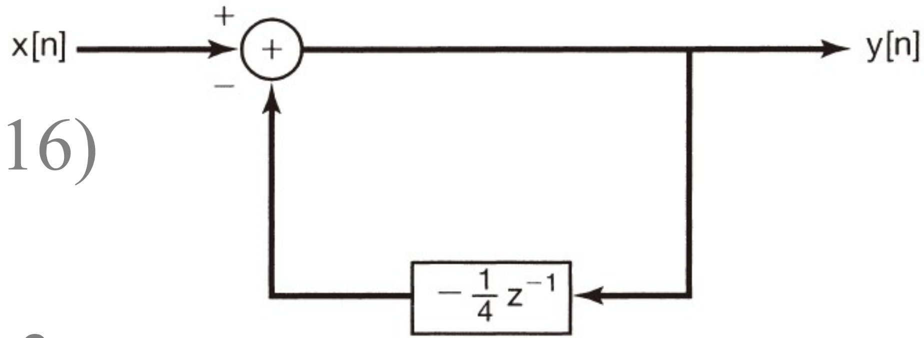
Example 10.28

Consider the causal LTI system with system function

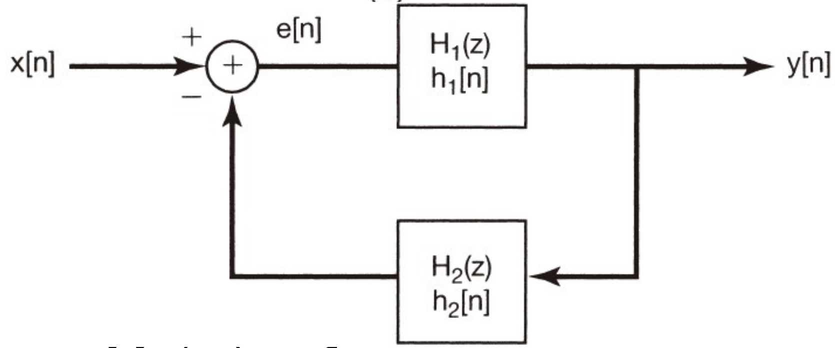
$$H(z) = \frac{1}{1-\frac{1}{4}z^{-1}}. \qquad (10.116)$$



(a)



(b)



Using the results in Section 10.7.3, we find that this system can also be described by the difference equation

$$y[n] - \frac{1}{4}y[n-1] = x[n],$$

$$H_1(z) = 1$$

$$H_2(z) = -1/4z^{-1}$$

Example 10.31

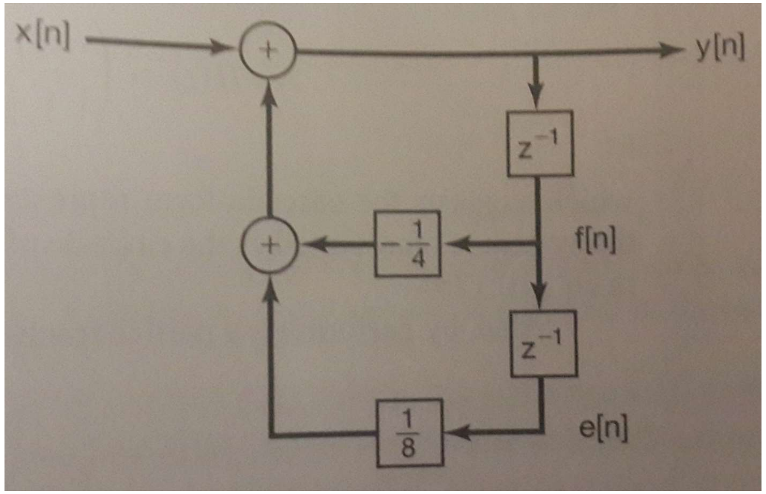
Finally, consider the system function

$$H(z) = \frac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}. \tag{10.121}$$

Writing

$$H(z) = \left(\frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \right) \left(1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2} \right) \tag{10.122}$$

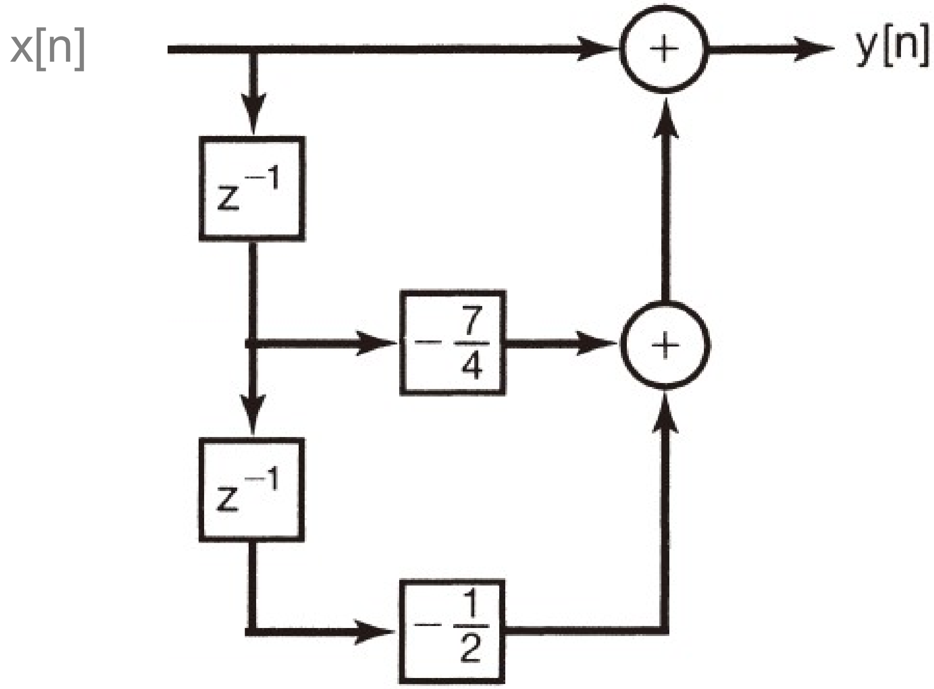
$$\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} \Rightarrow y[n]+\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n]$$

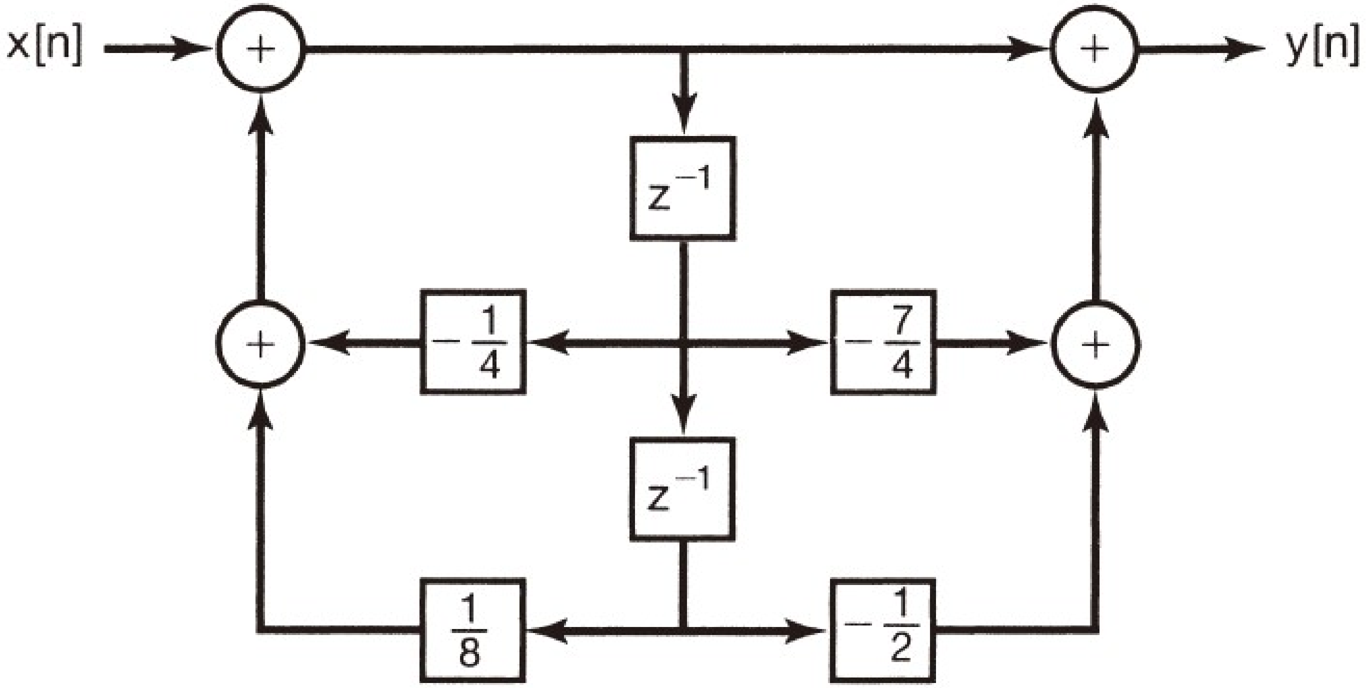


suggests representing the system as the cascade of the system in Figure 10.20(a) and the system with system function

$$1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2} \Rightarrow$$

$$y[n]=x[n]-\frac{7}{4}x[n-1]-\frac{1}{2}x[n-2]$$





The result is the direct-form block diagram shown in Figure 10.21, the details of the construction of which are examined in Problem 10.38. The coefficients in the direct-form representation can be determined by inspection form the coefficients in the system function of eq. (10.121).

Example 10.31

We can also write $H(z)$ in the forms

$$H(z) = \left(\frac{1 + \frac{1}{4} z^{-1}}{1 + \frac{1}{2} z^{-1}} \right) \left(\frac{1 - 2z^{-1}}{1 - \frac{1}{4} z^{-1}} \right) \qquad (10.123)$$

and

$$H(z) = 4 + \frac{5/3}{1 + \frac{1}{2} z^{-1}} - \frac{14/3}{1 - \frac{1}{4} z^{-1}}. \qquad (10.124)$$

Eq. (10.123) suggests a **cascade**-form representation, while eq. (10.124) leads to a **parallel**-form block diagram. These are also considered in Problem 10.38.

10.9 The Unilateral z-Transform

The unilateral *z*-transform of a sequence *x[n]* is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}. \qquad (10.125)$$

單邊*z*轉換的定義

As in previous chapters, we adopt a convenient shorthand notation for s signal and its unilateral *z*-transform:

$$x[n] \xleftrightarrow{UZ} X(z) = UZ\{x[n]\} \qquad (10.126)$$

訊號與其*z*轉換對應的符號

Example 10.33

$$x[n-n_0] \xleftrightarrow{Z} z^{-n_0}X(z),$$

Let

$$x[n] = a^{n+1}u[n+1]. \tag{10.129}$$

In this case the unilateral and bilateral transforms are not equal, since $x[-1] = 1 \neq 0$. The bilateral transform is obtained form Example 10.1 and the **time-shifting** property set forth in Section 10.5.2. Specifically,

$$X(z) = \frac{z}{1+az^{-1}}, \qquad |z| > |a| \tag{10.130}$$

Example 10.33

$$X(z) = \frac{z}{1 + az^{-1}}, \qquad |z| > |a|$$

In contrast, the unilateral transform is

$$\mathrm{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{n+1} z^{-n},$$

or

$$\mathrm{X}(z) = \frac{a}{1 - az^{-1}}, \qquad |z| > |a|. \qquad (10.132)$$

10.9 The Unilateral z-Transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

10.3 shows an alternative way to obtain inverse transform. For instance, in Example 10.13 we performed long division on the bilateral transform

$$X(z) = \frac{1}{1 - az^{-1}} \tag{10.134}$$

$$\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad a^2z^{-2}
\end{array}$$

$$\begin{array}{r|r}
 & -a^{-1}z - a^{-2}z^2 - \cdots \\
\hline
-az^{-1} + 1 & 1 \\
 & \underline{1 - a^{-1}z} \\
 & a^{-1}z
\end{array}$$

10.9 The Unilateral z-Transform

In two ways, corresponding to the two possible ROCs for *X(z)*. Only one of these choices, namely, that corresponding to the ROC |z| > |a|, led to a series expansion **without positive** powers of *z*, i.e.,

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \ldots, \qquad (10.135)$$

series expansion **without positive** powers of z =>
Not every rational function of z can be an unilaternal z-transform.

10.9 The Unilateral z-Transform

In particular, if we consider a rational function of z written as a ratio of polynomials in z (not in z^{-1}), i.e.,

$$\frac{p(z)}{q(z)}, \qquad (10.136)$$

the degree of the numerator must be no bigger than the degree of the denominator.

Example 10.35

$$x[n] = a^{n+1}u[n+1]. \qquad X(z) = \frac{z}{1+az^{-1}}, \qquad |z| > |a|$$

A simple example illustrating the preceding point is given by the rational function in eq. (10.130), which we can write as a ratio of polynomials in z:

$$\frac{z^2}{z-a}. \tag{10.137}$$

Example 10.35

There are two possible bilateral transforms that can be associated with this function, namely those corresponding to the two possible ROCs, $|z| < |a|$ and $|z| > |a|$. The choice $|z| > |a|$ corresponds to a right-sided sequence, but not to a signal that is zero for all n < 0, since its inverse transform, which is given by eq. (10.129), is nonzero for n = -1.

Example 10.35 $\quad \dfrac{p(z)}{q(z)},$

More generally, if we associate eq. (10.136) with the bilateral transform with the ROC that is the exterior of the circle with radius given by the magnitude of the largest root of *q(z),* then the inverse transform will certainly be right sided. However, **for it to be zero for all n < 0, it must also be the case that degree(*p(z)*) $\leqq$ degree(*q(z)*).**

10.9.2 Properties of The Unilateral z-Transform

TABLE 10.3 PROPERTIES OF THE UNILATERAL z-TRANSFORM

Property	Signal	Unilateral z-Transform
—	$x[n]$	$\mathcal{X}(z)$
—	$x_1[n]$	$\mathcal{X}_1(z)$
—	$x_2[n]$	$\mathcal{X}_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$a\mathcal{X}_1(z) + b\mathcal{X}_2(z)$
Time delay	$x[n-1]$	$z^{-1}\mathcal{X}(z) + x[-1]$
Time advance	$x[n+1]$	$z\mathcal{X}(z) - zx[0]$
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$ $z_0^n x[n]$ $a^n x[n]$	$\mathcal{X}(e^{-j\omega_0}z)$ $\mathcal{X}(z/z_0)$ $\mathcal{X}(a^{-1}z)$
Time expansion	$x_k[n] = \begin{cases} x[m], & n = mk \\ 0, & n \neq mk \end{cases}$ for any m	$\mathcal{X}(z^k)$
Conjugation	$x^*[n]$	$\mathcal{X}^*(z^*)$
Convolution (assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$)	$x_1[n] * x_2[n]$	$\mathcal{X}_1(z)\mathcal{X}_2(z)$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})\mathcal{X}(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{1}{1 - z^{-1}}\mathcal{X}(z)$
Differentiation in the z-domain	$nx[n]$	$-z\frac{d\mathcal{X}(z)}{dz}$

Initial Value Theorem

$$x[0] = \lim_{z\to\infty} \mathcal{X}(z)$$

表10.3列出單邊z轉換常用的性質，未列出ROC及因其區域必在某一個圓之外。

10.9.2 Properties of The Unilateral z-Transform

Let us examine the difference in the convolution property first. Table 10.3 states that if $x_1[n]=x_2[n]=0$ for all n < 0, then

$$x_1[n]*x_2[n] \xleftrightarrow{UZ} \mathrm{X}_1(z)\mathrm{X}_2(z). \qquad (10.138)$$

(10.138)式可應用在因果LTI系統上。

Example 10.36

Consider the causal LTI system described by the difference equation

$$y[n]+3y[n-1]=x[n], \tag{10.140}$$

together with the condition of initial rest. The system function for this system is

$$\mathrm{H}(z)=\frac{1}{1+3z^{-1}}. \tag{10.141}$$

Example 10.36

Suppose that the input to the system is $x[n] = \alpha u[n]$, where α is a given constant. In this case, the unilateral (and bilateral) z-transform of the output $y[n]$ is

$$y(z) = \mathrm{H}(z)\mathrm{X}(z) = \frac{\alpha}{(1+3z^{-1})(1-z^{-1})}$$

$$= \frac{(3/4)\alpha}{1+3z^{-1}} + \frac{(1/4)\alpha}{1-z^{-1}}. \qquad (10.142)$$

Applying Example 10.32 to each term of eq. (10.142) yields

$$y[n] = \alpha\left[\frac{1}{4} + \left(\frac{3}{4}\right)(-3)^n\right]u[n]. \qquad (10.143)$$

10.9.2 Properties of The Unilateral z-Transform

An important point to note here is that the convolution property for unilateral z-transforms applies only if the **signals** $x_1[n]$ **and** $x_2[n]$ **in eq. (10.138) are both identically zero for** $n < 0$.

單邊z轉換的迴旋運算性質只適用於 $x_1[n]$ 及 $x_2[n]$ 在$n < 0$時均為0。

10.9.2 Properties of The Unilateral z-Transform

To develop the **shifting** property for the unilateral transform, consider the signal

$$y[n] = x[n-1]. \tag{10.144}$$

Then

$$
\begin{aligned}
Y(z) &= \sum_{n=0}^{\infty} x[n-1] z^{-n} \\
&= x[-1] + \sum_{n=1}^{\infty} x[n-1] z^{-n} \\
&= x[-1] + \sum_{n=0}^{\infty} x[n] z^{-(n+1)},
\end{aligned}
$$

10.9.2 Properties of The Unilateral z-Transform

$$Y(z) = \sum_{n=0}^{\infty} x[n-1]z^{-n}$$

$$= x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

$$= x[-1] + \sum_{n=0}^{\infty} x[n]z^{-(n+1)},$$

or

$$Y(z) = x[-1] + z^{-1}\sum_{n=0}^{\infty} x[n]z^{-n}, \qquad (10.145)$$

so that

$$Y(z) = x[-1] + z^{-1}X(z). \qquad (10.146)$$

10.9.2 Properties of The Unilateral z-Transform

$$Y(z) = x[-1] + z^{-1}X(z).$$

$$y[n] = x[n-1].$$

By repeated application of eq. (10.146), the unilateral transform of

$$w[n] = y[n-1] = x[n-2] \qquad (10.147)$$

is

$$W(z) = y[-1] + z^{-1}Y(z) = x[-2] + z^{-1}(x[-1] + z^{-1}X(z))$$

$$= x[-2] + x[-1]z^{-1} + z^{-2}X(z). \qquad (10.148)$$

10.9.2 Properties of The Unilateral z-Transform

There is also a **time advance** property for unilateral transforms that relates the transform of an advanced version of *x[n]* to *X(z).* Specifically, as shown in Problem 10.60,

$$x[n+1] \xleftrightarrow{UZ} zX(z) - zx[0]. \qquad (10.149)$$

Example 10.37

$$y[n]+3y[n-1]=x[n] \quad (10.140)$$

Consider again the difference equation (10.140) with $x[n] = \alpha u[n]$ and with the initial condition

$$y[-1]=\beta. \qquad (10.150)$$

利用單邊z轉換求解差分方程的範例。

已知輸入*x[n]*及輸出的初始條件*y[-1]*。

Example 10.37

$$y[n]+3y[n-1]=x[n] \quad (10.140)$$

$$\mathrm{Y}(z)=x[-1]+z^{-1}\mathrm{X}(z).$$

Applying the unilateral transform to both sides of eq. (10.140) and using the linearity and time delay properties, we obtain

$$\mathrm{Y}(z)+3\beta+3z^{-1}\mathrm{Y}(z)=\frac{\alpha}{1-z^{-1}}. \quad (10.151)$$

先對差分方程取*z*轉換並代入初始條件。

Solving for *Y(z)* yields

$$\mathrm{Y}(z)=-\frac{3\beta}{1+3z^{-1}}+\frac{\alpha}{(1+3z^{-1})(1-z^{-1})}. \quad (10.152)$$

求出輸出的*z*轉換*y(z)*。

10.10 Summary

- Z-Transform
  - ROC & properties
  - Zero-Pole
  - Properties
  - Unilateral Transformation