Chapter 9
The Laplace Transform

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9.0 Introduction

As we will see, the Laplace and z-transforms have many of the properties that make Fourier analysis so useful. Especially for investigating the **stability** or **instability** of a (feedback) system, which cannot be done for FT.

3.2 The Response of LTI Systems to Complex Exponentials

- The importance of complex exponentials stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is,

continuous time: $e^{st} \to H(s)e^{st}$, (3.1)

discrete time: $z^n \to H(z)z^n$, (3.2)

where the complex amplitude factor H(s) or H(z) will in general be a function of the complex variable s or z.

- We use to set **$s=j\omega$, $z=e^{j\omega}$**

9.1 The Laplace Transform

response of a linear time-invariant system with impulse response *h(t)* to a complex exponential input of the form e^{st} is

$$y(t) = H(s)e^{st}, \tag{9.1}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt. \tag{9.2}$$

H(s) is the Laplace transform of the impulse response *h(t)*

9.1 The Laplace Transform

The Laplace transform of a general signal x(t) is defined as

$$X(s) \overset{\Delta}{=} \int_{-\infty}^{+\infty} x(t)e^{-st}dt, \qquad (9.3)$$

拉氏轉換定義

9.1 The Laplace Transform

For convenience, we will sometimes denote the Laplace transform in operator form as $L\{x(t)\}$ and denote the transform relationship between $x(t)$ and $X(s)$ as

$$x(t) \overset{L}{\longleftrightarrow} X(s). \tag{9.4}$$

When $s = j\omega$, eq. (9.3) becomes

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt, \tag{9.5}$$

which is FT.

$$X(s)\Big|_{s=j\omega} = F\{x(t)\}. \tag{9.6}$$

9.1 The Laplace Transform

$$X(s) \stackrel{\Delta}{=} \int_{-\infty}^{+\infty} x(t)e^{-st}dt \quad (9.3)$$

Another straightforward relationship to TF:
consider X(s) as specified in eq. (9.3) with s expressed as x = σ + jω, so that

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t}dt, \quad (9.7)$$

or

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} \left[x(t)e^{-\sigma t}\right] e^{-j\omega t}dt$$

$$= F\{x(t)e^{-\sigma t}\} \quad (9.8)$$

Note that α can be positive or negative.

Example 9.1

指數函數的拉氏轉換推導範例

Let the signal $x(t)=e^{-at}u(t)$. From Example 4.1, the Fourier transform $X(j\omega)$ converges for $a>0$ and is given by

$$X(j\omega)=\int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}dt=\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt=\frac{1}{j\omega+a}, \qquad a>0. \qquad (9.9)$$

From eq. (9.3), the Laplace transform is

$$X(s)=\int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt=\int_{0}^{\infty} e^{-(s+a)t}dt, \qquad (9.10)$$

Example 9.1 $X(j\omega)=\int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}dt=\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt=\frac{1}{j\omega+a}, \qquad a>0.$

or, with $s = \sigma + j\omega$,

$$X(\sigma+j\omega)=\int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt. \qquad (9.11)$$

By comparison with eq. (9.9) we recognize eq. (9.11) as the Fourier transform of $e^{-(\sigma+a)t}u(t)$, and thus,

$$X(\sigma+j\omega)=\frac{1}{(\sigma+a)+j\omega}, \qquad \sigma+a>0, \qquad (9.12)$$

Example 9.1 $$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \qquad \sigma + a > 0,$$

Or equivalently, since $s = \sigma + j\omega$ and $\sigma = \Re e\{s\}$,

$$X(s) = \frac{1}{s+a}, \qquad \Re e\{s\} > -a. \quad (9.13)$$

That is

$$e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+a}, \qquad \Re e\{s\} > -a. \quad (9.14)$$

For example, for a = 0, $x(t)$ is the unit step with Laplace transform $X(s)$ = 1/s, $\Re e\{s\} > 0$.

$\Re e\{s\} > -a$ *specifies the s that LT* converges

9.1 The Laplace Transform

If a is positive, then *X(s)* can be evaluated at $\sigma = 0$ to obtain

$$X(0 + j\omega) = \frac{1}{j\omega + a}. \qquad (9.15)$$

when $a \leqq 0$, $e^{-at}u(t)$ has valid LT but not FT.

Example 9.2

$$X(s) \overset{\Delta}{=} \int_{-\infty}^{+\infty} x(t)e^{-st}dt \quad (9.3)$$

For comparison with Example 9.1, let us consider as a second example the signal

$$x(t) = -e^{-at}u(-t). \quad (9.16)$$

Then

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t)dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}dt = -\int_{0}^{\infty} e^{(s+a)\tau}d\tau \quad (9.17)$$

τ=-t

Example 9.2

$$X(s) = -\int_0^{\infty} e^{(s+a)\tau} d\tau$$

or

$$X(s) = \frac{1}{s+a}. \qquad (9.18)$$

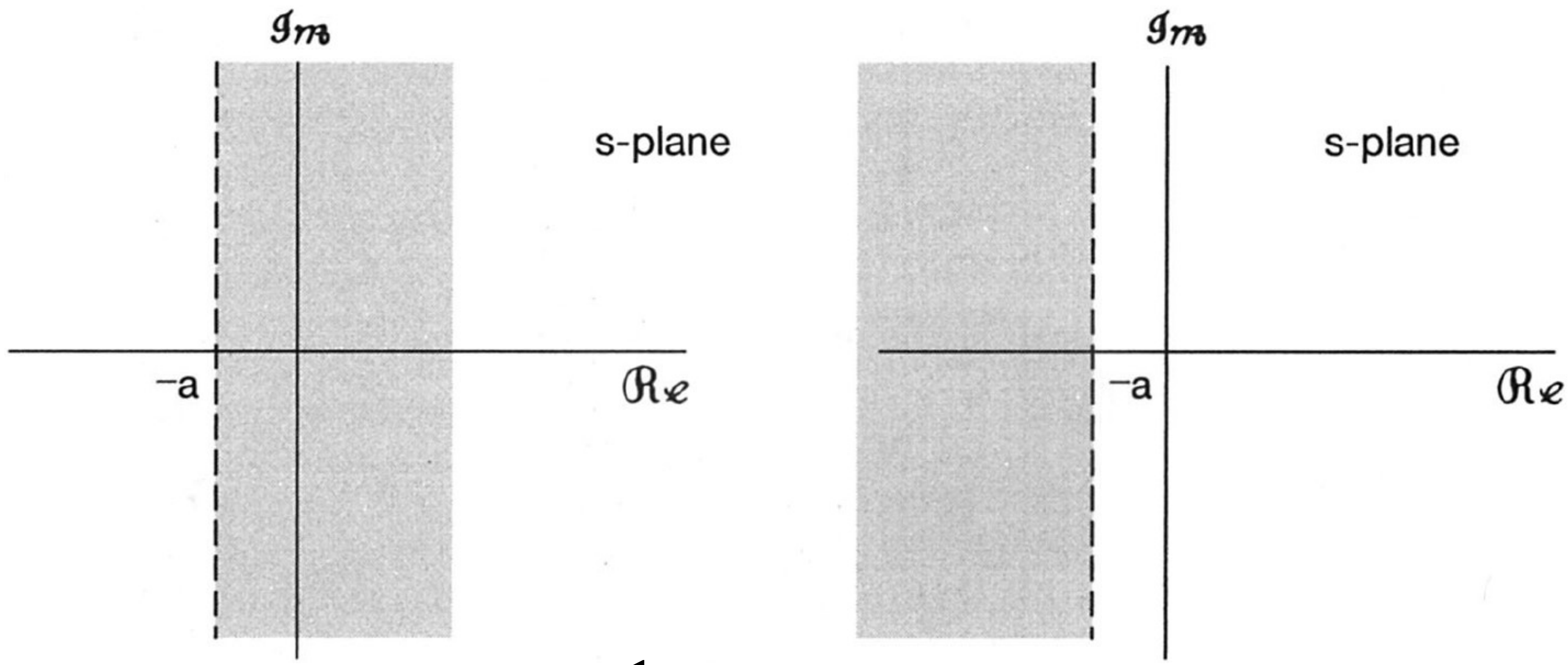
For convergence in this example, we require that $\Re e\{s+a\} < 0$, or $\Re e\{s\} < -a$; that is,

$$-e^{-at}u(-t) \xleftrightarrow{L} \boxed{\frac{1}{s+a}}, \qquad \boxed{\Re e\{s\} < -a.} \qquad (9.19)$$

$$e^{-at}u(t) \xleftrightarrow{L} \boxed{\frac{1}{s+a}}, \qquad \boxed{\Re e\{s\} > -a.} \qquad (9.14)$$

9.1 Region of Convergence (ROC)

$$e^{-at}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+a}, \qquad \Re e\{s\} > -a. \quad (9.14)$$



$$-e^{-at}u(-t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+a}, \qquad \Re e\{s\} < -a. \quad (9.19)$$

9.1 The Laplace Transform

We will have more to say about the ROC as we develop some insight into the properties of the Laplace transform.

ROC為「收斂區域」的簡寫。

Example 9.4

In this example, we consider a signal that is the sum of a real and a complex exponential:

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t). \qquad (9.24)$$

Using Euler's relation, we can write

$$x(t) = \left[e^{-2t} + \frac{1}{2} e^{-(1-3j)t} + \frac{1}{2} e^{-(1+3j)t} \right] u(t),$$

Example 9.4

and the Laplace transform of *x(t)* then can be expressed as

$$X(s) = \int_{-\infty}^{\infty} e^{-2t} u(t) e^{-st} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1-3j)t} u(t) e^{-st} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1+3j)t} u(t) e^{-st} dt. \qquad (9.25)$$

Example 9.4

Each of the integrals in eq. (9.25) represents a Laplace transform of the type encountered in Example 9.1. It follows that

$$e^{-2t}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+2}, \qquad \Re e\{s\} > -2, \tag{9.26}$$

$$e^{-(1-3j)t}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+(1-3j)}, \qquad \Re e\{s\} > -1, \tag{9.27}$$

$$e^{-(1+3j)t}u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s+(1+3j)}, \qquad \Re e\{s\} > -1. \tag{9.28}$$

Example 9.4

$$\Re e\{s\} > -2, \qquad \Re e\{s\} > -1, \qquad \Re e\{s\} > -1.$$

For all three Laplace transforms to converge **simultaneously**, we must have . consequently, the Laplace transform of *x(t)* is

$$\frac{1}{s+2} + \frac{1}{2}\left(\frac{1}{s+(1-3j)}\right) + \frac{1}{2}\left(\frac{1}{s+(1+3j)}\right), \qquad \Re e\{s\} > -1, \tag{9.29}$$

or, with terms combined over a common denominator,

$$e^{-2t}u(t) + e^{-t}(\cos 3t)u(t) \stackrel{L}{\longleftrightarrow} \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \qquad \Re e\{s\} > -1. \tag{9.30}$$

9.1 The Laplace Transform

In each of the four preceding examples, the Laplace transform is rational, i.e., it is a ratio of polynomials in the complex variable s, so that

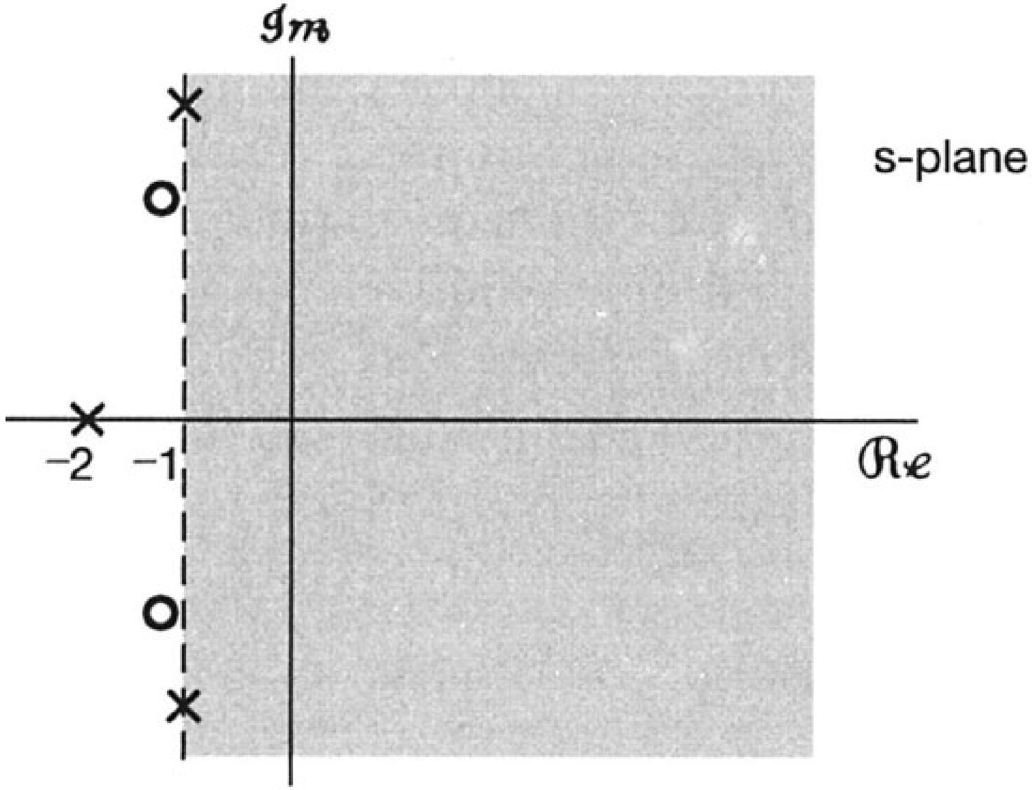
$$X(s) = \frac{N(s)}{D(s)}, \qquad (9.31)$$

As suggested by examples, X(s) will be rational whenever x(t) is a linear combination of real or complex exponentials. To specify, X(s) up to a scale factor, we just need the roots in N(s) and D(s).

9.1 The Laplace Transform

$$\frac{2s^2+5s+12}{(s^2+2s+10)(s+2)} = \frac{(s-(-5+j\sqrt{71})/4)(s-(-5-j\sqrt{71})/4)}{(s+(1-3j))(s+(1+3j))(s+2)} \quad \Re e\{s\} > -1.$$

Example 9.4, with the location of each root of the denominator polynomial in eq. (9.30) indicated with "×" and the location of the root of the numberator polynomial in eq. (9.30) indicated with "o."



9.1 The Laplace Transform

For rational Laplace transforms, the roots of the numerator polynomial are commonly referred to as the **zeros** of *X(s),* since, for those values of *s*, *X(s)* = 0. The roots of the denominator are referred to as the **poles** of X(s), since X(s)=infinite for those values of s. The plot on the s-plane is referred to as the **pole-zero** plot of X(s).

Note that the pole-zero plot and the ROC will specify the time domain signal up to a scale factor.

9.1 The Laplace Transform

Zeros and Poles at infinite:

In general, if the order of the denominator exceeds the order of the numerator by *k*, *X(s)* will have *k* zeros at infinity. Similarly, if the order of the numerator exceeds the order of the denominator by k, *X(s)* will have k poles at infinity.

Example 9.5

Let
$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t). \qquad (9.32)$$

The Laplace transform of the second and third terms on the right-hand side of eq. (9.32) can be evaluated from Example 9.1. The Laplace transform of the unit impulse can be evaluated directly as

$$L\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)e^{-st}dt = 1, \qquad (9.33)$$

Example 9.5

Which is valid for any value of s. That is, the ROC of *L{δ(t)}* is the entire s-plane. Using this result, together with the Laplace transforms of the other two terms in eq. (9.32), we obtain

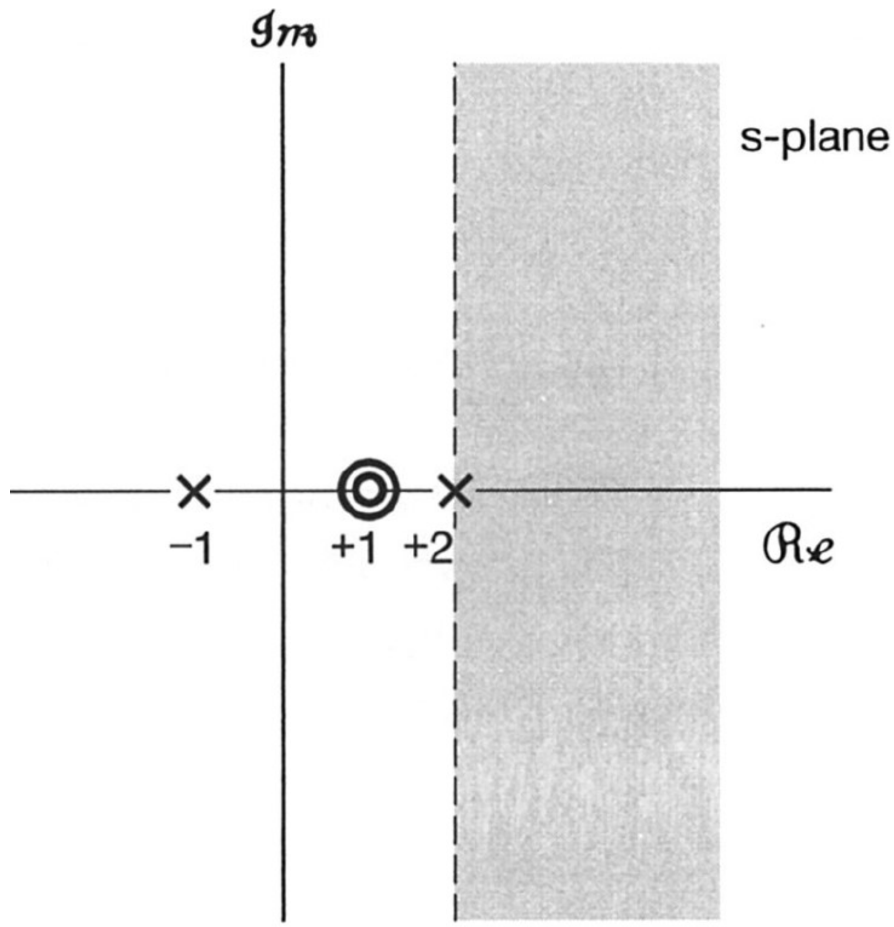
$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}, \qquad \Re d\{s\} > 2, \tag{9.34}$$

or

$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)}, \qquad \Re e\{s\} > 2, \tag{9.35}$$

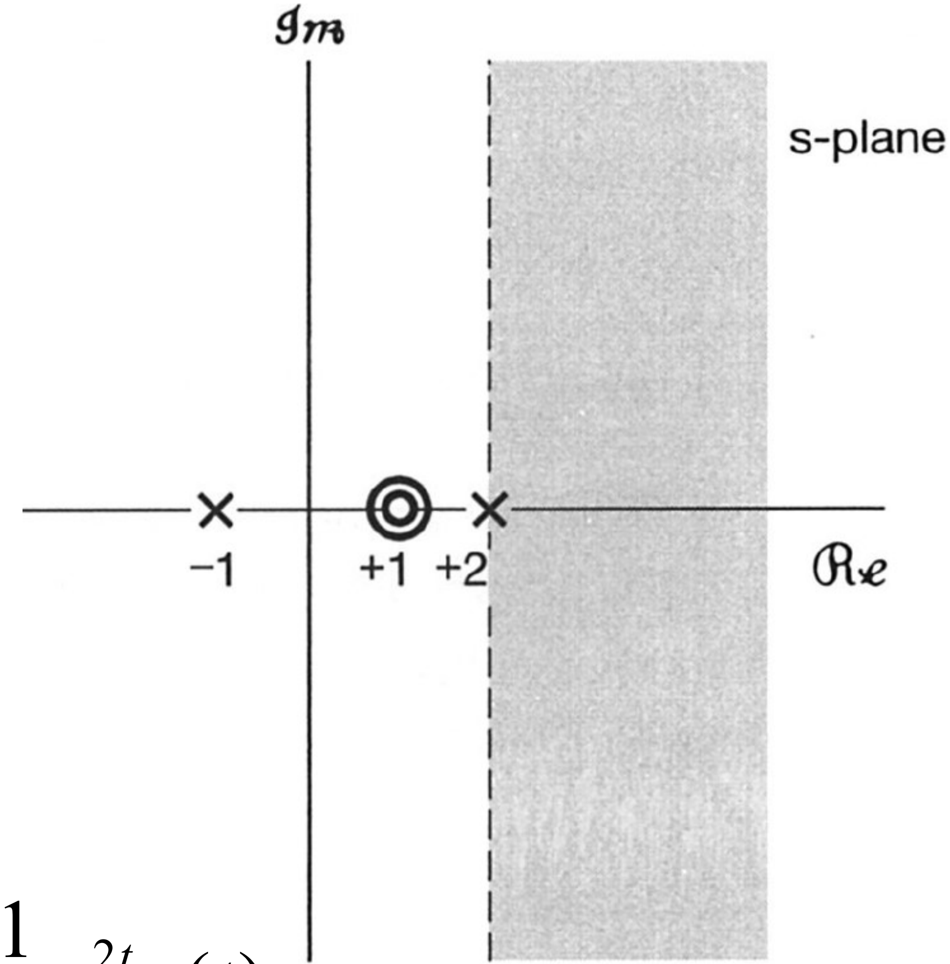
Example 9.5

where the ROC is the set of values of s for which the Laplace transforms of all three terms in *x(t)* converge. The pole-zero plot for this example is shown in Figure 9.3, together with the ROC. Also, since the degrees of the numerator and denominator of *X(s)* are equal, *X(s)* has neither poles nor zeros at infinity.



9.1 The Laplace Transform

However, if the ROC of the Laplace transform does not include the *jω-axis*, (i.e., if $\Re e\{s\}=0$), then the Fourier transform does not converge.



$$x(t)=\delta(t)-\frac{4}{3}e^{-t}u(t)+\frac{1}{3}e^{2t}u(t).$$

9.2 The Region of Convergence For Laplace Transforms

- Preperty 1: The ROC of *X(s)* consists of strips parallel to the $j\omega$-axis in the s-plane.

  性質1：*X(s)*的ROC是由*s*平面上平行於$j\omega$的帶狀區域所組成。

That is, the ROC of the Laplace transform of *x(t)* consists of those values of s for which $x(t)e^{-\sigma t}$ is absolutely integrable:

$$\int_{-\infty}^{+\infty} \left| x(t)e^{-\sigma t} \right| dt = \int_{-\infty}^{+\infty} \left| x(t) \right| e^{-\sigma t} dt < \infty. \qquad (9.36)$$

9.2 The Region of Convergence For Laplace Transforms

- Property 2: For rational Laplace transforms, the ROC does not contain any poles.

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt \le \int_{-\infty}^{+\infty} \left|x(t)e^{-st}\right|dt$$

$$X(s) = \infty \quad for \quad poles$$

- Property 3: If *x(t)* is of finite duration and is absolutely integrable, then the ROC is the entire *s*-plane.

  性質3： 若*x(t)*為有限時間訊號且絕對可積分，則ROC為整個*s*平面。

9.2 The Region of Convergence For Laplace Transforms

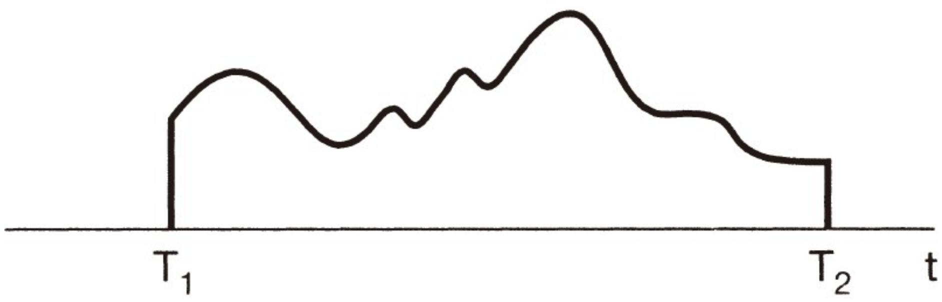


Figure 9.4 Finite-duration signal.

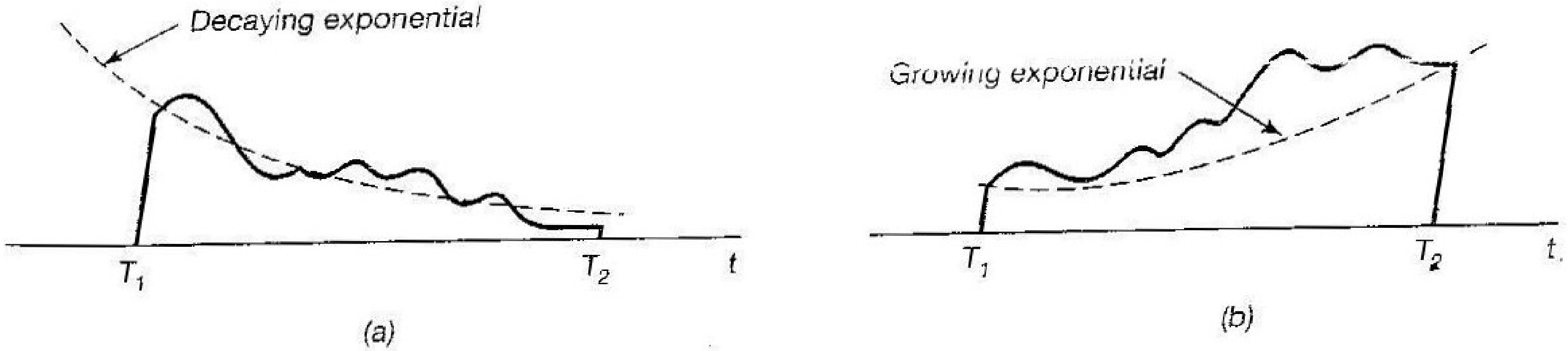


Figure 9.5 *(a) Finite-duration signal of Figure 9.4 multiplied by a decaying exponential; (b) finite-duration signal of Figure 9.4 multiplied by a growing exponential.*

9.2 The Region of Convergence For Laplace Transforms

A more formal verification of Property 3 is as follows: Suppose that *x(t)* is absolutely integrable, so that

$$\int_{T_1}^{T_2} |x(t)| dt < \infty. \qquad (9.37)$$

For $s = \sigma + j\omega$ to be in the ROC, we require that $x(t)e^{-\sigma t}$ be absolutely integrable, i.e.,

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < \infty. \qquad (9.38)$$

9.2 The Region of Convergence For Laplace Transforms

Eq. (9.37) verifies that s is in the ROC when $\Re e\{s\} = \sigma = 0$. For $\sigma > 0$, the maximum value of $e^{-\sigma t}$ over the interval on which *x(t)* is nonzero is $e^{-\sigma T_1}$, and thus we can write

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)| dt. \quad (9.39)$$

9.2 The Region of Convergence For Laplace Transforms

Since the right-hand side of eq.(9.39) is bounded, so is the left-hand side; therefore, the *s*-plane for $\Re e\{s\} > 0$ must also be in the ROC. By a similar argument, if $\sigma < 0$, then

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)| dt, \quad (9.40)$$

Example 9.6

Let

$$x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & otherwise \end{cases}. \qquad (9.41)$$

Then

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]. \qquad (9.42)$$

It seems s=-a has a pole.

Example 9.6

In fact, however, in the algebraic expression in eq. (9.42), both numerator and denominator are zero at *s* = -*a*, and thus, to determine *X(s)* at *s* = -*a*, we can use $L'hôpital's$ rule to obtain

$$\lim_{s \to -a} X(s) = \lim_{s \to -a} \left[\frac{\frac{d}{ds}(1 - e^{-(s+a)T})}{\frac{d}{ds}(s+a)} \right] = \lim_{s \to -a} Te^{-aT}e^{-sT},$$

so that

$$X(-a) = T. \tag{9.43}$$

9.2 The Region of Convergence For Laplace Transforms

$$X(s) \stackrel{\Delta}{=} \int_{-\infty}^{+\infty} x(t) e^{-st} dt \quad (9.3)$$

- Property 4: If *x(t)* is right sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then all values of *s* for which $\Re e\{s\} > \sigma_0$ will also be in the ROC.

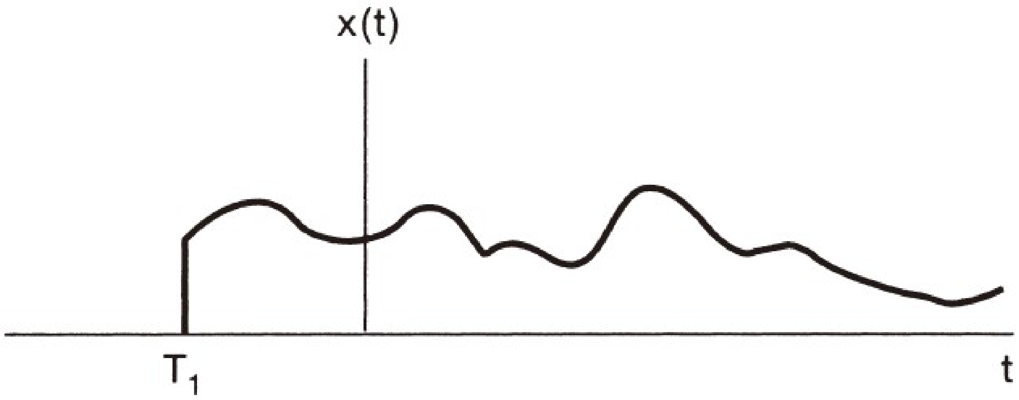


Figure 9.6 Right-sided signal.

$x(t)=0$ for $t<T_1$

9.2 The Region of Convergence For Laplace Transforms

It is possible that such signal has no value of s for LT to converge.

One example is the signal $x(t) = e^{t^2} u(t)$. However, suppose that the Laplace transform converges for some value of *σ*, which we denote by σ_0 . Then

$$\int |x(t)| e^{-\sigma_0 t} dt < \infty, \qquad (9.44)$$

or equivalently, since *x(t)* is right sided,

$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty. \qquad (9.45)$$

Then if $\sigma_1 > \sigma_0$, it must also be true that $x(t)e^{-\sigma_1 t}$ is absolutely integrable, since $e^{-\sigma_1 t}$ decays faster than $e^{-\sigma_0 t}$ as $t \to +\infty$, as illustrated in Figure 9.7. Formally, we can say that with $\sigma_1 > \sigma_0$,

$$\int_{T_1}^{\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t} dt \qquad (9.46)$$

$$\leq e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} dt.$$

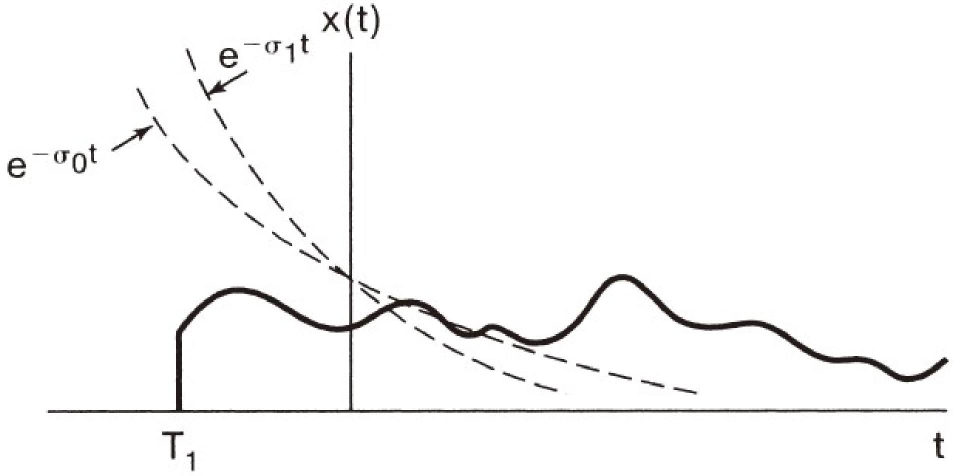


Figure 9.7 If $x(t)$ is right sided and $x(t)e^{-\sigma_0 t}$ is absolutely integrable, then $x(t)e^{-\sigma_1 t}$, $\sigma_1 > \sigma_0$, will also be absolutely integrable.

9.2 The Region of Convergence For Laplace Transforms

- Property 5: If *x(t)* is left sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then all values of *s* for which $\Re e\{s\} < \sigma_0$ will also be in the ROC.

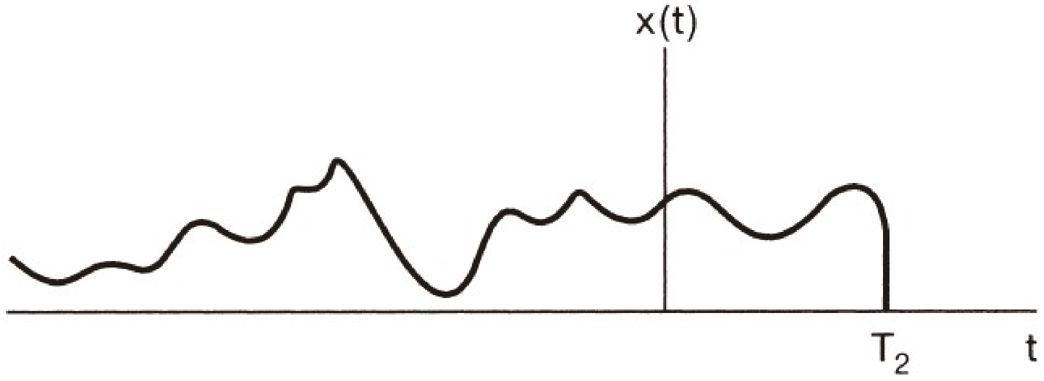
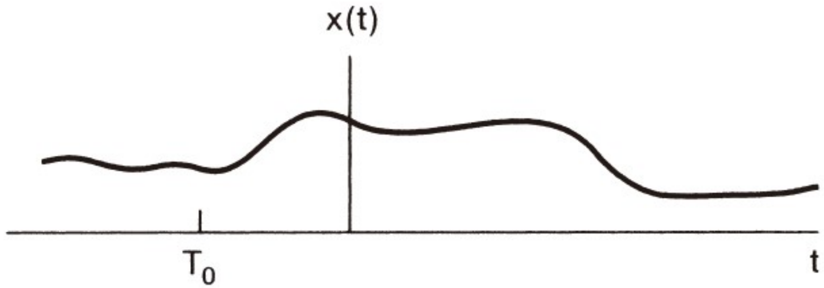


Figure 9.8 Left-sided signal.

$x(t)=0$ for $t>T_2$

9.2 The Region of Convergence For Laplace Transforms

- Property 6: If *x(t)* is two sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the *s*-plane that includes the line $\Re e\{s\} = \sigma_0$.



(a)

9.2 The Region of Convergence For Laplace Transforms

$$x(t) = x_L(t) + x_R(t), \quad split \quad at \quad T_0$$

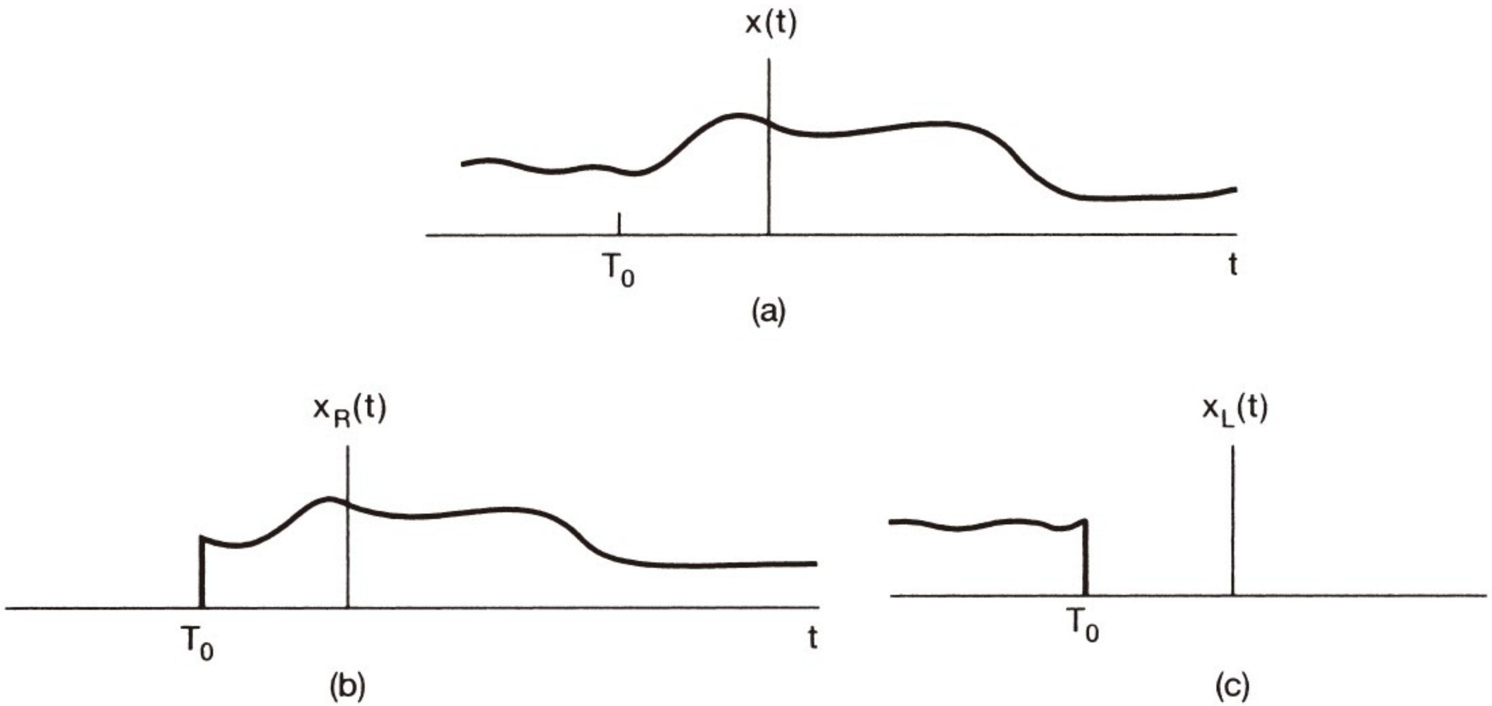
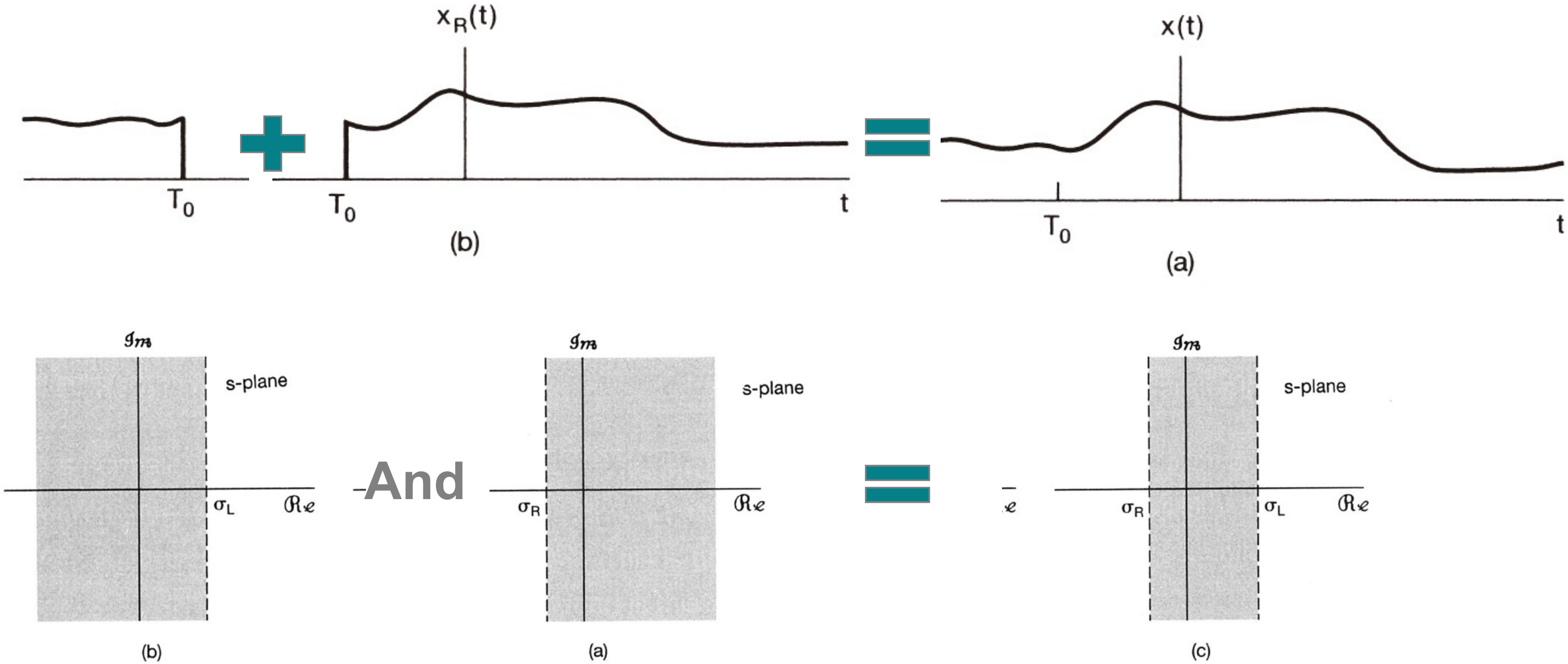


Figure 9.9 Two-sided signal divided into the sum of a right-sided and left-sided sig nal: (a) two-sided signal $x(t)$; (b) the right-sided signal equal to $x(t)$ for $t > T_0$ and equal to 0 for $t < T_0$; (c) the left-sided signal equal to $x(t)$ for $t < T_0$ and equal to 0 $t > T_0$.

9.2 The Region of Convergence For Laplace Transforms

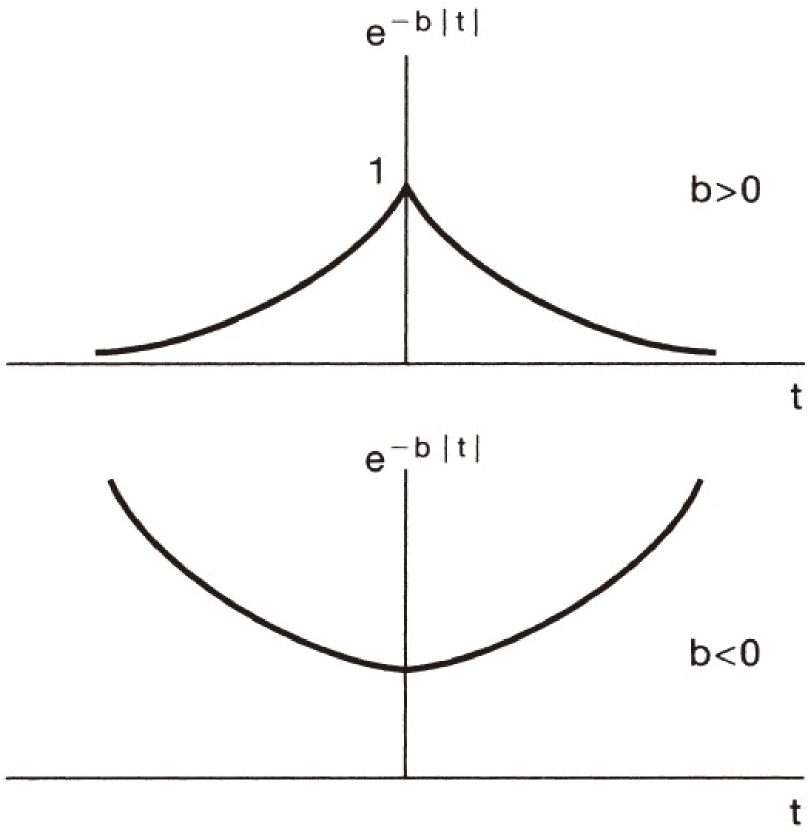


Example 9.7

Let

$$x(t) = e^{-b|t|}, \quad (9.47)$$

as illustrated in Figure 9.11 for both b > 0 and b < 0. Since this is a two-sided signal, let us divide it into the sum of a right-sided and left-sided signal; that is,



$$x(t) = e^{-bt}u(t) + e^{+bt}u(-t). \quad (9.48)$$

Example 9.7

From Example 9.1,

$$e^{-bt}u(t) \xleftrightarrow{L} \frac{1}{s+b}, \qquad \Re e\{s\} > -b, \tag{9.49}$$

and from Example 9.2,

$$e^{+bt}u(-t) \xleftrightarrow{L} \frac{-1}{s-b}, \qquad \Re e\{s\} < +b. \tag{9.50}$$

Example 9.7

Although the Laplace transforms of each of the individual terms in eq. (9.48) have a region of convergence, there is no common region of convergence if $b \leqq 0$, and thus, for those values of b, $x(t)$ has no Laplace transform. If $b > 0$, the Laplace transform of $x(t)$ is

$$e^{-b|t|} \xleftrightarrow{L} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2 - b^2}, \qquad -b < \Re e\{s\} < +b. \tag{9.51}$$

The corresponding pole-zero plot is shown in Figure 9.12, with the shading indicating the ROC.

Example 9.7

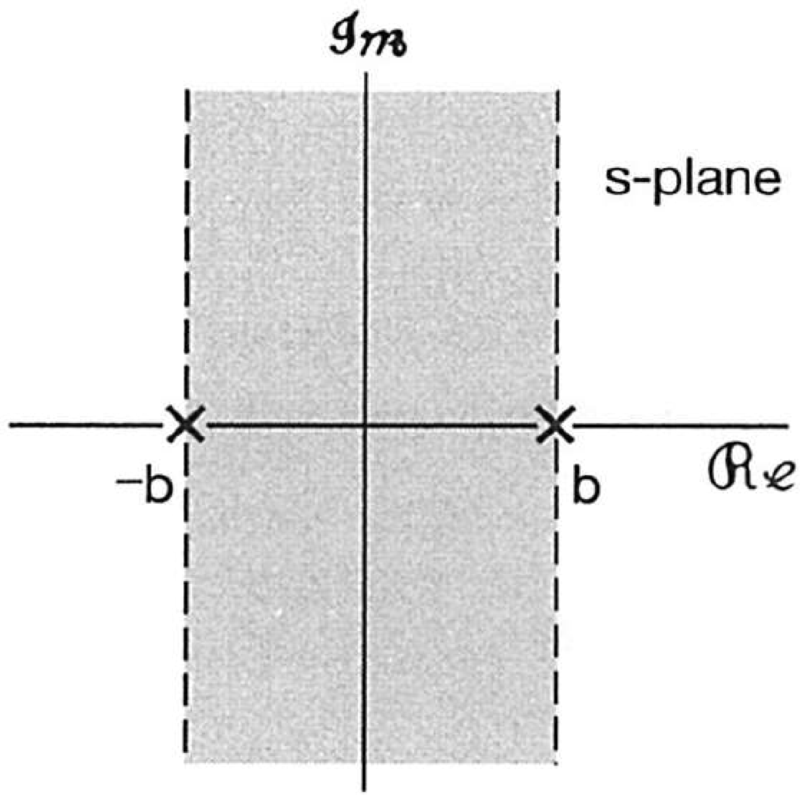


Figure 9.12 Pole-zero plot and ROC for Example 9.7.

9.2 The Region of Convergence For Laplace Transforms

- Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no piles of $X(s)$ are contained in the ROC.

  性質7：若$x(t)$的拉氏轉換$X(s)$為有理式，則其ROC必受極點限制範圍或延伸至無窮遠。此外，$X(s)$在ROC內沒有極點。

9.2 The Region of Convergence For Laplace Transforms

Due to Properties 4, 5, 7:

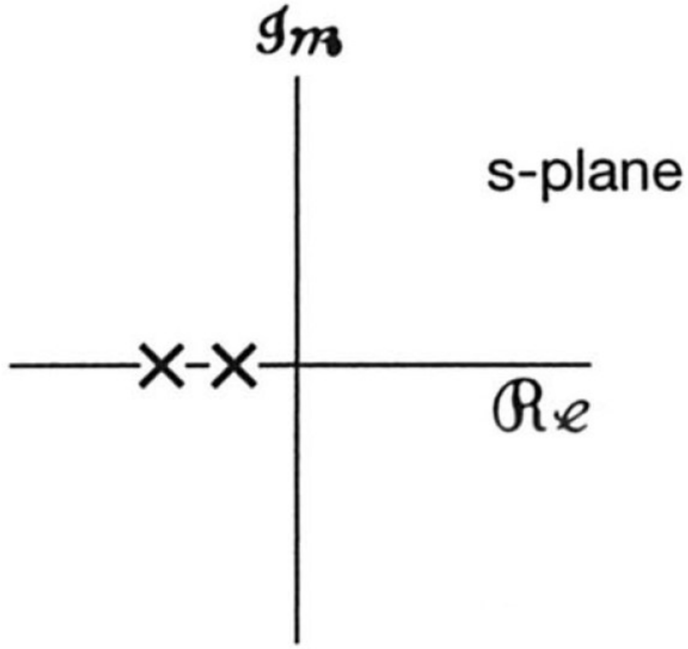
- Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

  性質8：若$x(t)$的拉氏轉換$X(s)$為有理式，則若$x(t)$為右邊訊號，其ROC為s平面上到最右的極點為止的區域。若$x(t)$為左邊訊號，其ROC為s平面上到最左的極點為止的區域。

Example 9.8

Let

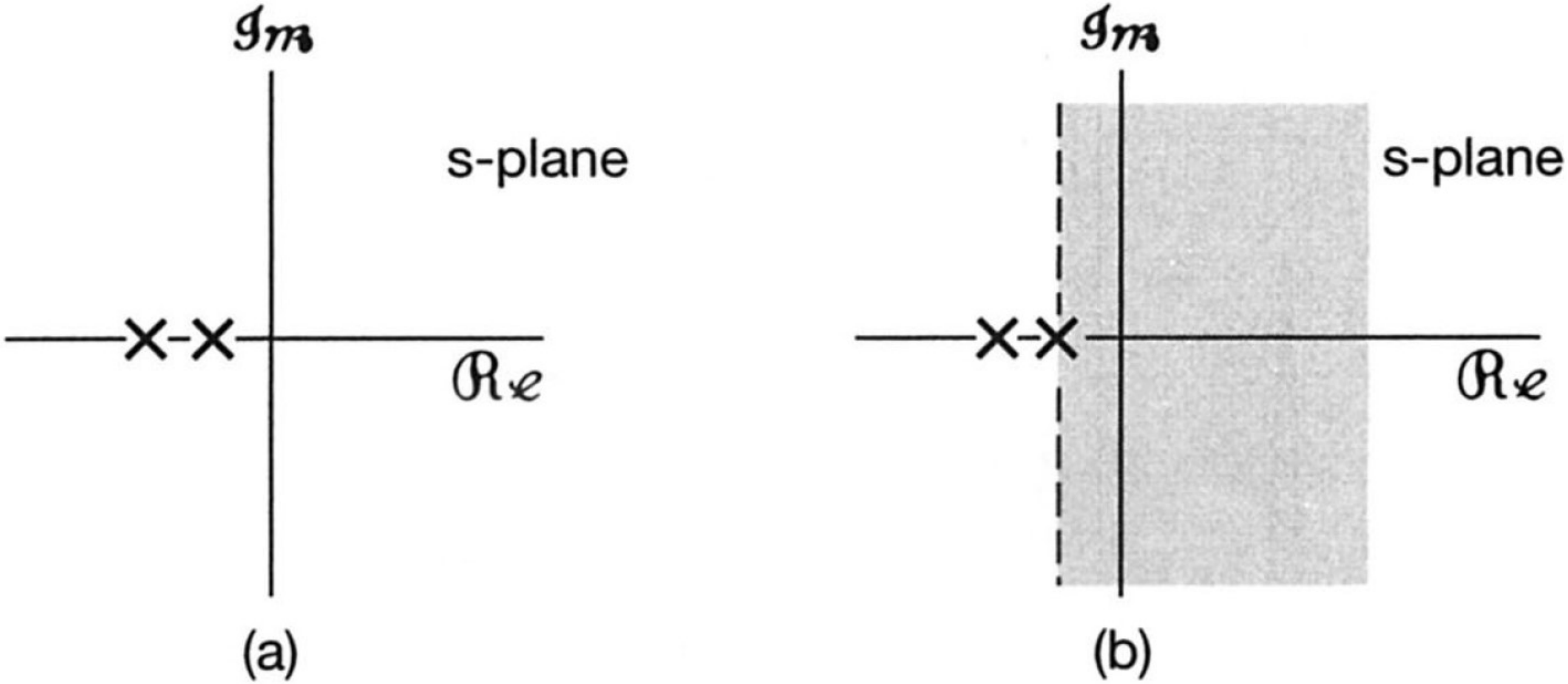
$$X(s) = \frac{1}{(s+1)(s+2)}, \quad (9.52)$$



(a)

with the associated pole-zero pattern in Figure 9.13(a). As indicated in Figures 9.13(b)-(d), there are three possible ROCs that can be associated with this algebraic expression, corresponding to three distinct signals. The signal associated with the pole-zero pattern in Figure 9.13(b) is right sided.

Example 9.8



Example 9.8

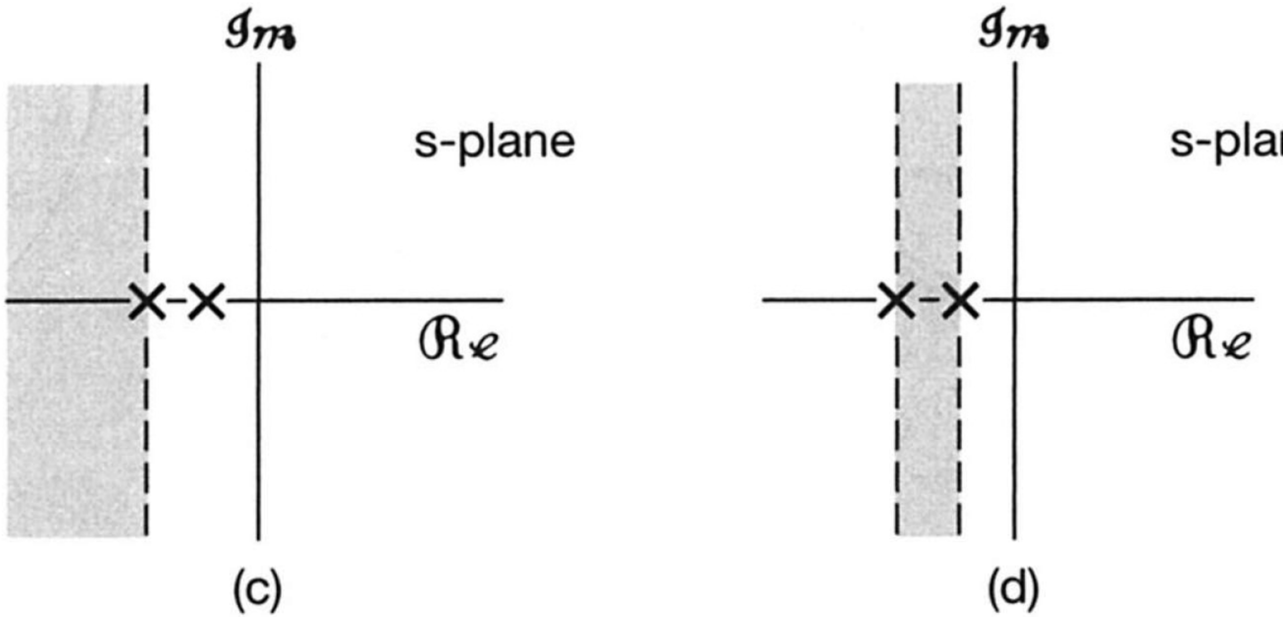
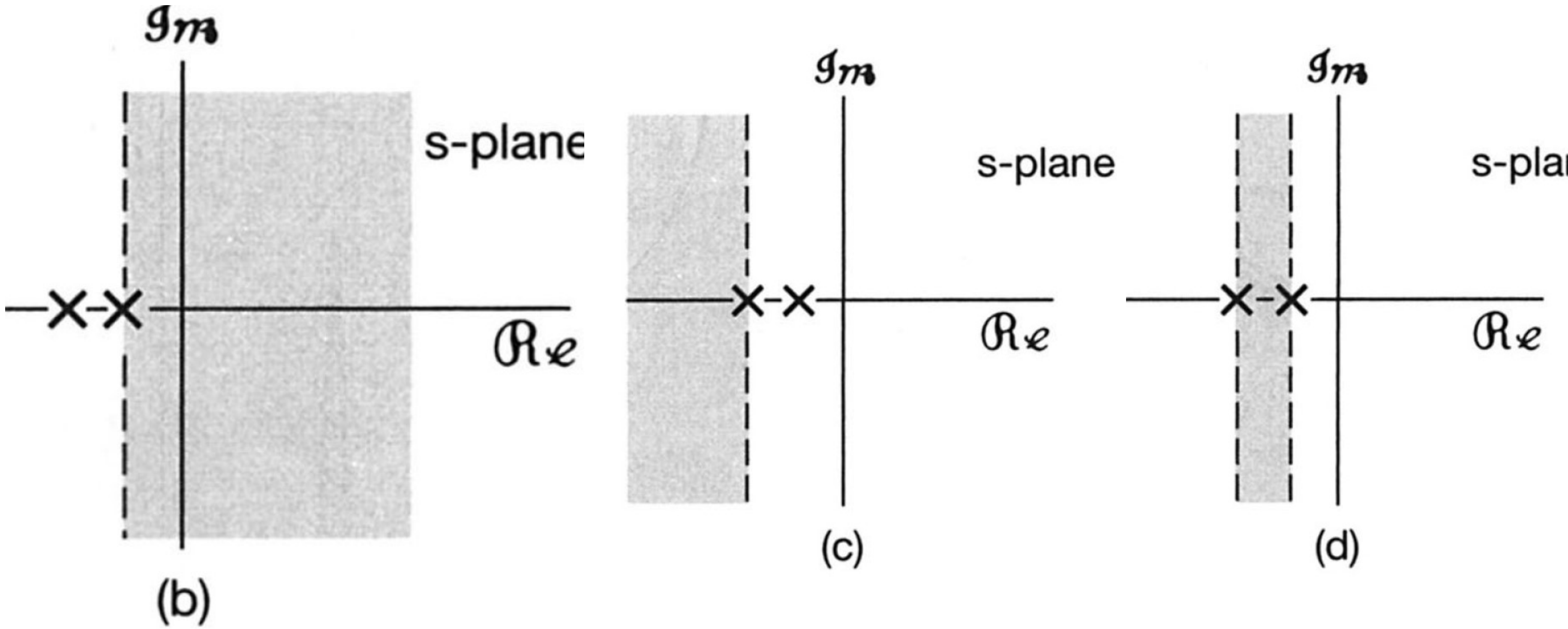


Figure 9.13 (a) Pole-zero pattern for Example 9.8; (b) ROC corresponding to a right-sided sequence; (c) ROC corresponding to a left-sided sequence; (d) ROC corresponding to a two-sided sequence.



For Figure 9.13(b), since the ROC includes the $j\omega$-axis, the Fourier transform of this signal converges. Figure 9.13(c) corresponds to a left-sided signal and Figure 9.13(d) to a two-sided signal. Neither of these two signals have Fourier transforms, since their ROCs do not include the $j\omega$-axis.

9.3 The Inverse Laplace Transform

with s expressed as $s = \sigma + j\omega$, the Laplace transform of a signal *x(t)* is

$$X(\sigma + j\omega) = F\left\{x(t)e^{-\sigma t}\right\} = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt \qquad (9.53)$$

for values of $\boldsymbol{s = \sigma + j\omega}$ **in the ROC**. We can invert this relationship using the **inverse Fourier transform** as given in eq. (4.9). We have

$$x(t)e^{-\sigma t} = F^{-1}\left\{X(\sigma + j\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t}d\omega, \qquad (9.54)$$

$$x(t)e^{-\sigma t} = F^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t}d\omega,$$

or, multiplying both sides by $e^{\sigma t}$, we obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega. \tag{9.55}$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds. \tag{9.56}$$

反拉式轉換定義

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds \quad (9.56)$$

The contour of integration in eq. (9.56) is the **straight line** in the *s*-plane corresponding to all points s satisfying $\Re e\{s\}=\sigma$. This line is parallel to the $j\omega$-axis. Furthermore, we can choose **any** such line in the ROC

(9.56)式中是沿著*s*平面上的 $\Re e\{s\}=\sigma$ 直線積分，此一直線可選自ROC內的任意直線。

9.3 The Inverse Laplace Transform

assuming **no multiple-order poles**, and assuming that the order of the denominator polynomial is **greater** than the order of the numerator polynomial, we can expand *X(s)* in the form

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i} \tag{9.57}$$

Example 9.9

Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1. \tag{9.58}$$

To obtain the inverse Laplace transform, we first perform a **partial-fraction** expansion to obtain

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}. \tag{9.59}$$

Example 9.9

As discussed in the appendix, we can evaluate the coefficients A and B by multiplying both sides of eq. (9.59) by (s + 1) or (s + 2) and then equating coefficients of equal powers of s on both sides. Alternatively, we can use the relation

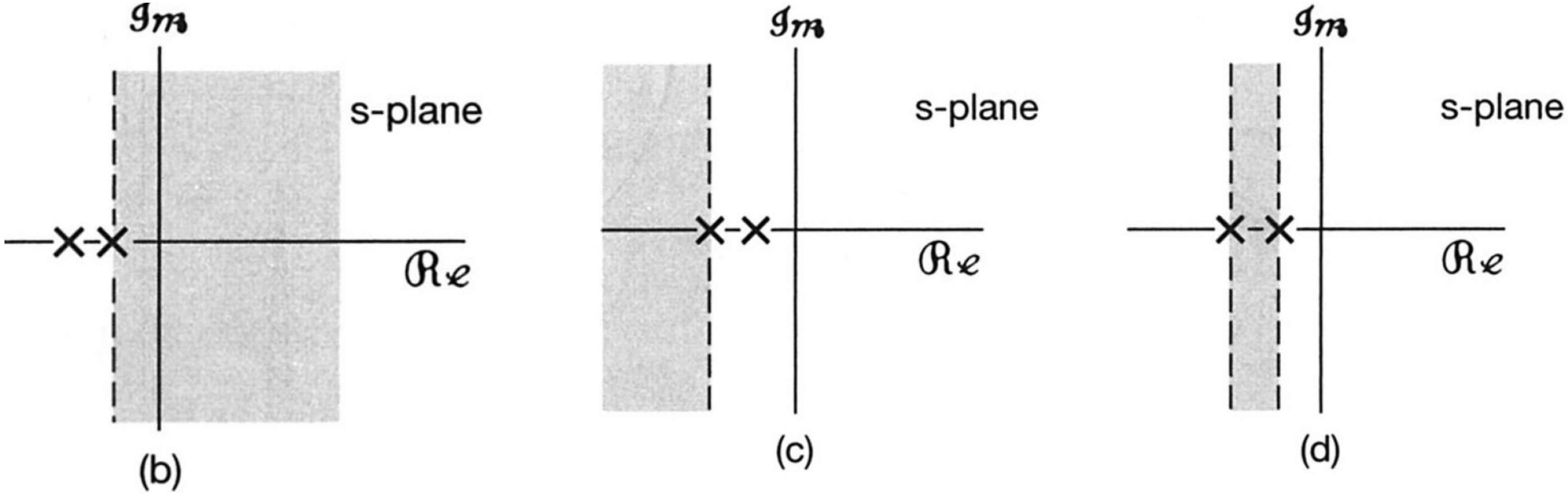
$$A = \left[(s+1)X(s)\right]\Big|_{s=-1} = 1, \qquad (9.60)$$

$$B = \left[(s+2)X(s)\right]\Big|_{s=-2} = -1. \qquad (9.61)$$

Thus, the partial-fraction expansion for *X(s)* is

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}. \qquad (9.62)$$

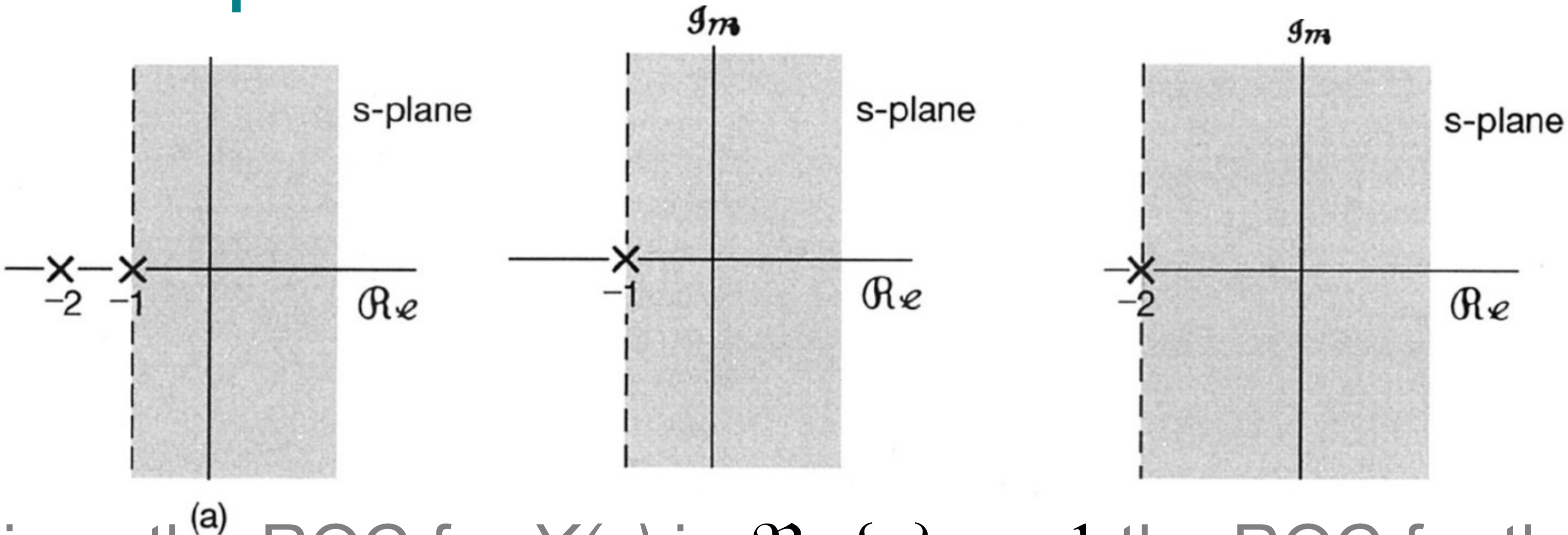
Example 9.9 $X(s)=\frac{1}{s+1}-\frac{1}{s+2}.$ $\Re e\{s\}>-1.$



From Examples 9.1 and 9.2, we know that there are two possible inverse transforms for a transform of the form 1/(s + a), **depending on whether the ROC** is to the **left** or the **right** of the pole. Consequently, we need to determine which ROC to associate with each of the individual first-order terms in eq. (9.62). This is done by reference to the properties of the ROC developed in Section 9.2.

Example 9.9

$$X(s)=\frac{1}{s+1}-\frac{1}{s+2}. \qquad \Re e\{s\}>-1.$$



Since the ROC for *X(s)* is $\Re e\{s\}>-1$, the ROC for the individual terms in the partial-fraction expansion of eq. (9.62) includes $\Re e\{s\}>-1$ The ROC for each term can then be extended to the left or right (or both) to be bounded by a pole or infinity. This is illustrated in Figure 9.14. Figure 9.14(a) shows the pole-zero plot and ROC for *X(s)*, as specified in eq. (9.58).

Example 9.9

$$e^{-t}u(t) \xleftrightarrow{L} \frac{1}{s+1}, \qquad \Re e\{s\} > -1, \tag{9.63}$$

$$e^{-2t}u(t) \xleftrightarrow{L} \frac{1}{s+2}, \qquad \Re e\{s\} > -2. \tag{9.64}$$

We thus obtain

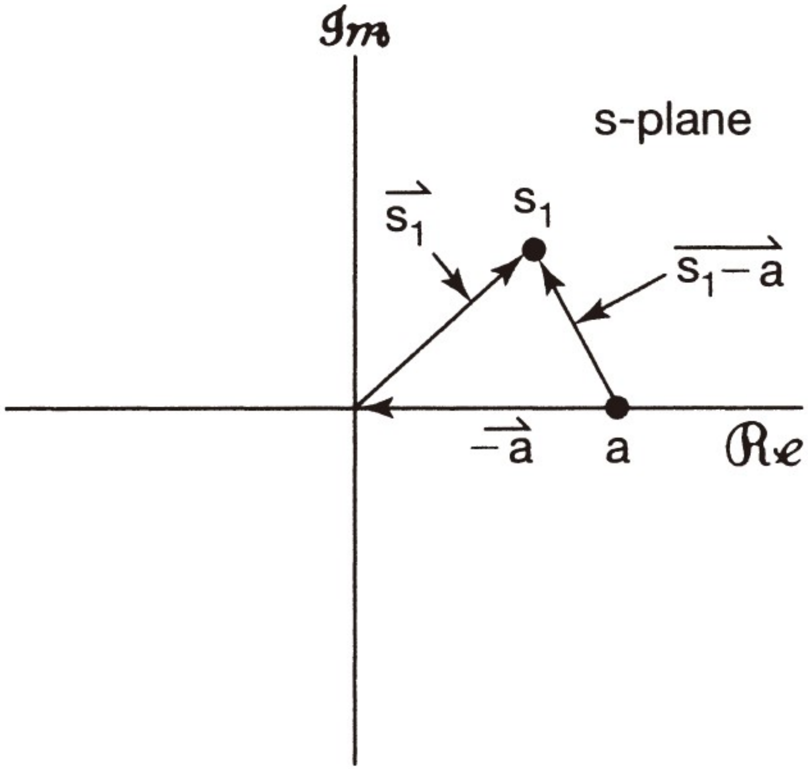
$$\left[e^{-t} - e^{-2t}\right]u(t) \xleftrightarrow{L} \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1. \tag{9.65}$$

9.3 The Inverse Laplace Transform

As discussed in the appendix, when *X(s)* has multiple-order poles, or when the denominator is not of higher degree than the numerator, the partial-fraction expansion of *X(s)* will include other terms in addition to the first-order terms considered in Examples 9.9-9.11.

若*X(s)*具有重極點，或分母次數不大於分子次數，則*X(s)*的部分分式展開將含有其他型式的項（見附錄）。

9.4 Geometric Evaluation of the Fourier Transform From the Pole-Zero plot

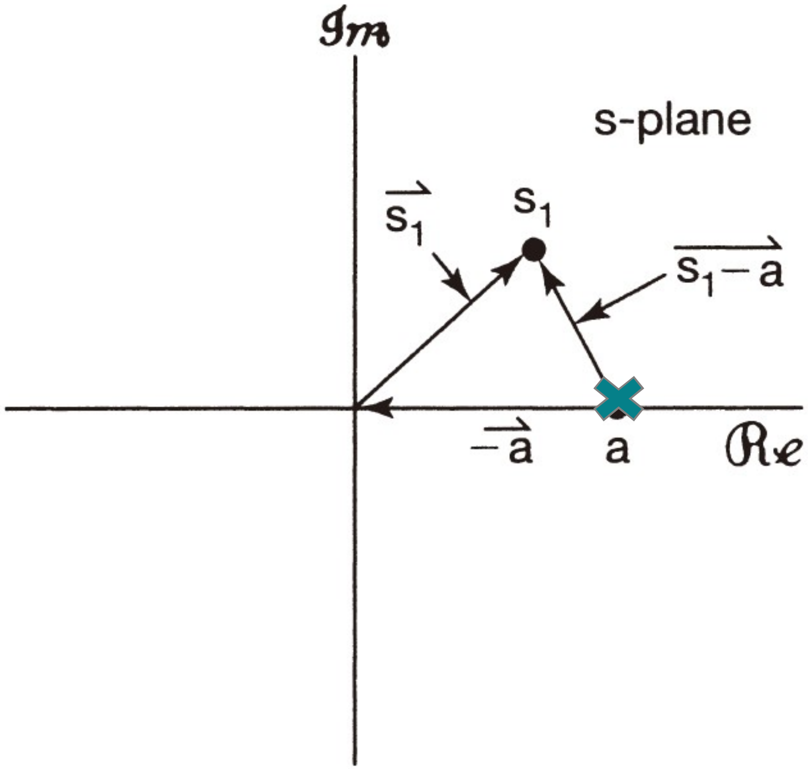


$$X(s) = s - a.$$

A zero when s=a

Figure 9.15 Complex plane representation of the vectors $\mathbf{s_1}$, $\mathbf{a}$, and $\mathbf{s_1} - \mathbf{a}$ representing the complex numbers s_1, a, and $s_1 - a$, respectively.

9.4 Geometric Evaluation of the Fourier Transform From the Pole-Zero plot



$$X(s) = s - a.$$

$$\hat{X}(s) = 1/(s-a).$$

A pole at s=a

$$|\hat{X}(s)| = |X(s)|$$

$$\angle \hat{X}(s) = -\angle X(s)$$

9.4 Geometric Evaluation of the Fourier Transform From the Pole-Zero plot

A more general rational Laplace transform consists of a product of pole and zero terms of the form discussed in the preceding paragraph; that is, it can be factored into the form

$$X(s) = M \frac{\prod_{i=l}^{R}(s-\beta_i)}{\prod_{j=l}^{P}(s-\alpha_j)}. \qquad (9.70)$$

9.4 Geometric Evaluation of the Fourier Transform From the Pole-Zero plot

$$X(s) = M \frac{\prod_{i=l}^{R}(s-\beta_i)}{\prod_{j=l}^{P}(s-\alpha_j)}.$$

To evaluate *X(s)* at $s = s_1$, each term in the product is represented by a vector from the zero or pole to the point s_1 . The magnitude of $X(s_1)$ is then the magnitude of the scale factor *M*, times the product of the lengths of the zero vectors (i.e., the vectors from the zeros to s_1), divided by the product of the lengths of the pole vectors (i.e., the vectors from the pole to s_1). The angle is the sum of the angles of zero vectors minus the sum of the angles of pole vectors.

Example 9.12

Let

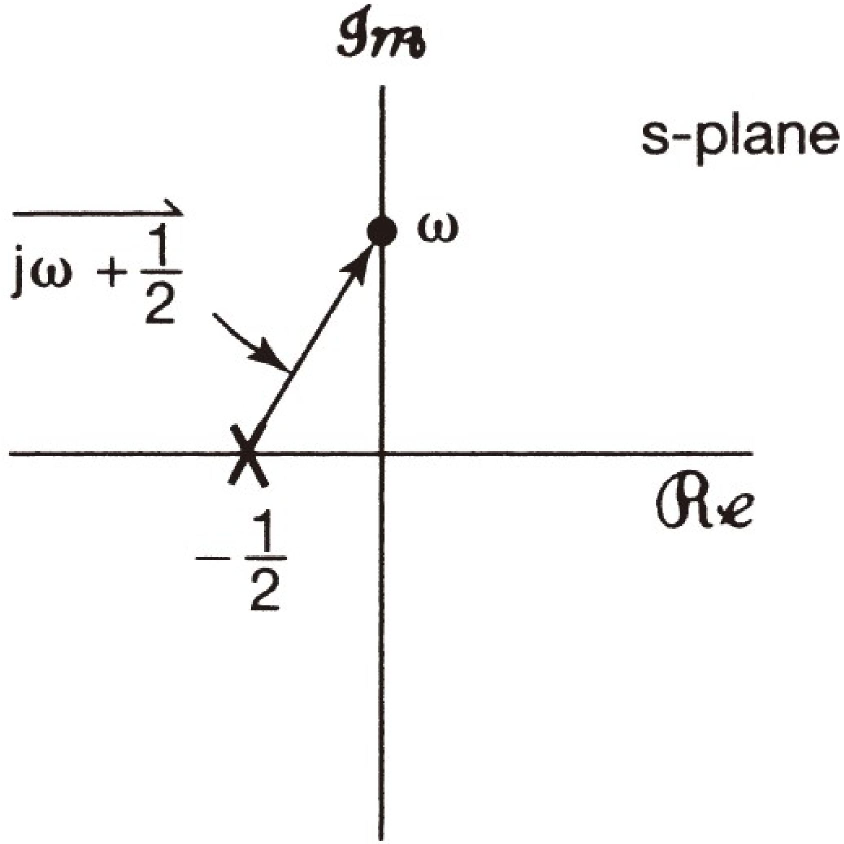
$$X(s) = \frac{1}{s + \frac{1}{2}}, \qquad \Re e\{s\} > -\frac{1}{2}. \tag{9.71}$$

The Fourier transform is $X(s)\big|_{s=j\omega}$. For this example, then, the Fourier transform is

$$X(j\omega) = \frac{1}{j\omega + 1/2}. \tag{9.72}$$

Example 9.12

The pole-zero plot for *X(s)* is shown in Figure 9.16. To determine the Fourier transform graphically, we construct the pole vector as indicated. The magnitude of the Fourier transform at frequency ω is the reciprocal of the length of the vector from the pole to the point $j\omega$ on the imaginary axis. The phase of the Fourier transform is the negative of the angle of the vector. Geometrically, from Figure 9.16, we can write



$$|X(j\omega)|^2 = \frac{1}{\omega^2 + (1/2)^2}$$

$$\sphericalangle X(j\omega) = -\tan^{-1} 2\omega.$$

9.4.1 First-Order Systems

$$\tau \frac{dy(t)}{dt} + y(t) = x(t), \quad H(j\omega) = \frac{1}{j\omega\tau + 1},$$

The impulse response for such a system is

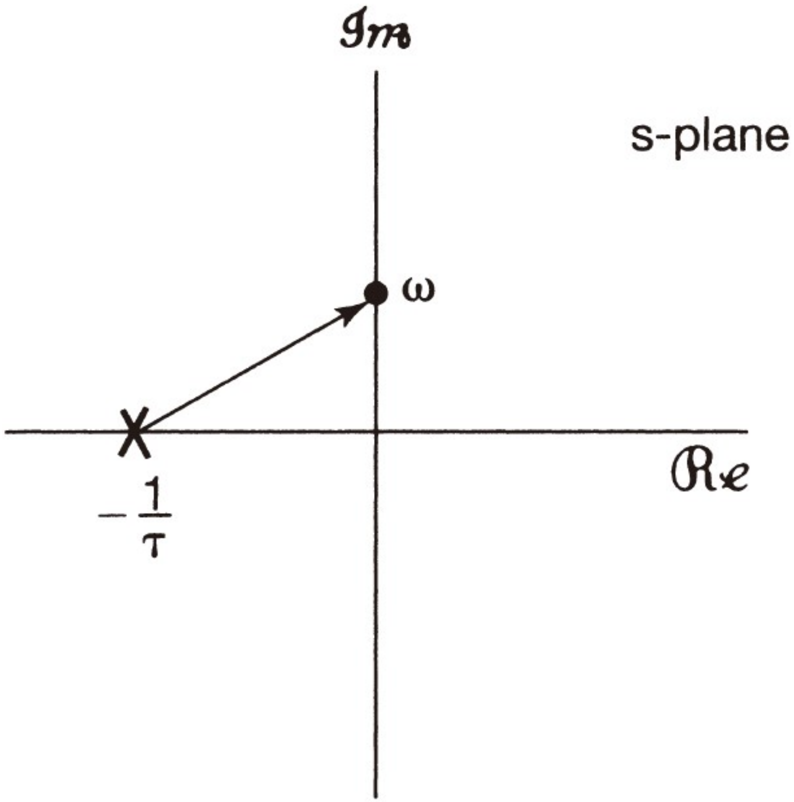
$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t), \tag{9.75}$$

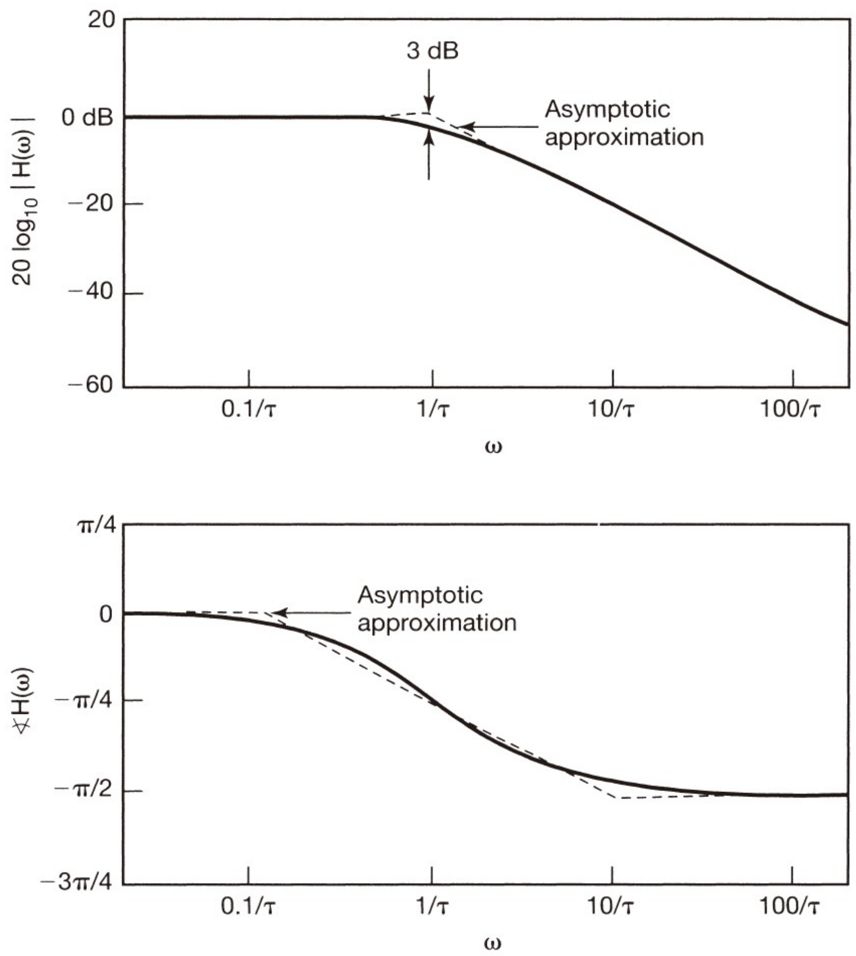
ant its Laplace transform is

$$H(s) = \frac{1}{s\tau + 1}, \qquad \Re e\{s\} > -\frac{1}{\tau}. \tag{9.76}$$

$$H(s) = \frac{1}{s\tau + 1}, \qquad \Re e\{s\} > -\frac{1}{\tau}.$$

9.4.1 First-Order Systems

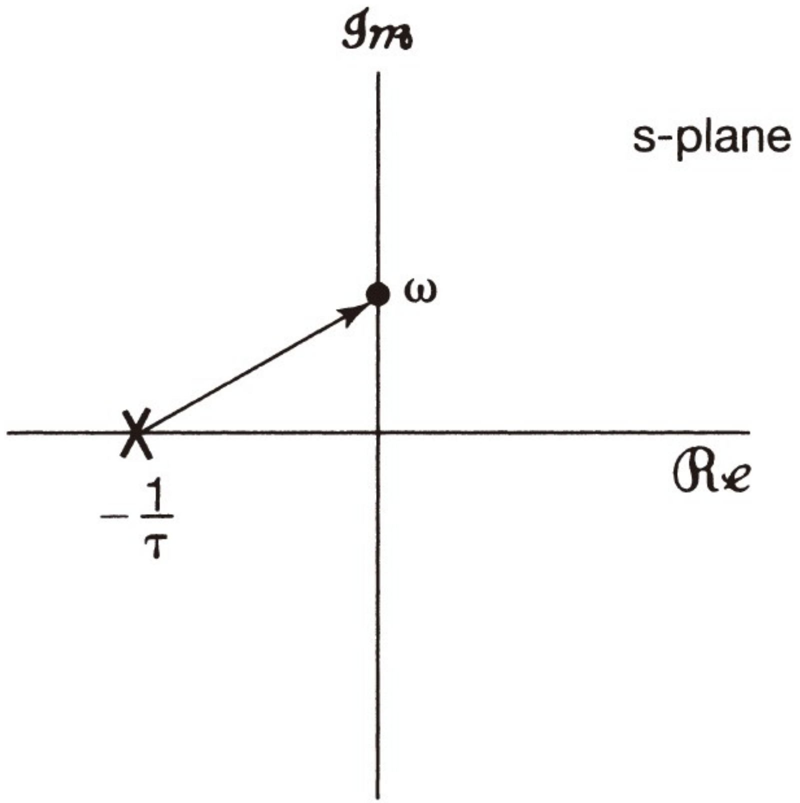


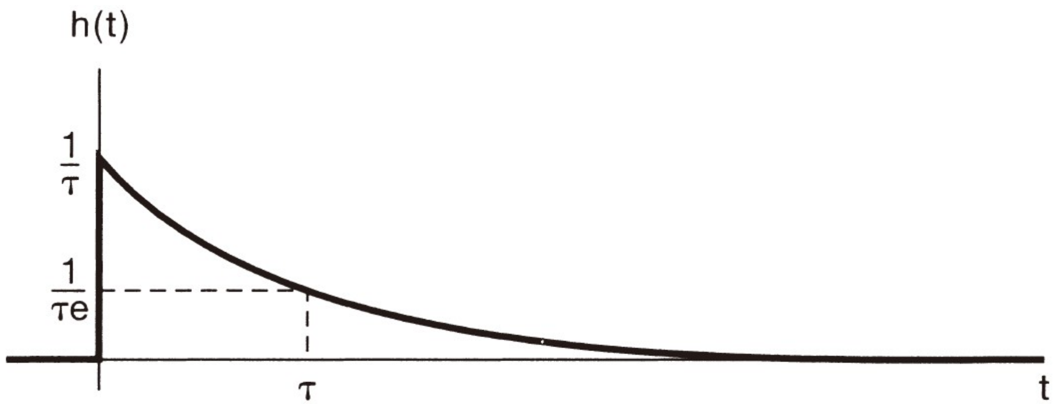


Figu
for a

$$20\log 2^{-1/2} = -10\log 2 \approx -10 * 0.3 = -3$$

9.4.1 First-Order Systems





(a)

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t),$$

Pole moves to the left corresponds to a decrease in time constant, resulting in a faster decay of impulse response, and a faster rise time in step response

9.4.2 Second-Order Systems

The impulse response and frequency response for the system, originally given in eqs. (6.37) and (6.33), respectively, are

$$h(t) = M\left[e^{c_1 t} - e^{c_2 t}\right]u(t), \qquad (9.77)$$

where

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1},$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}},$$

9.4.2 Second-Order Systems

and

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}. \qquad (9.78)$$

The Laplace transform of the impulse response is

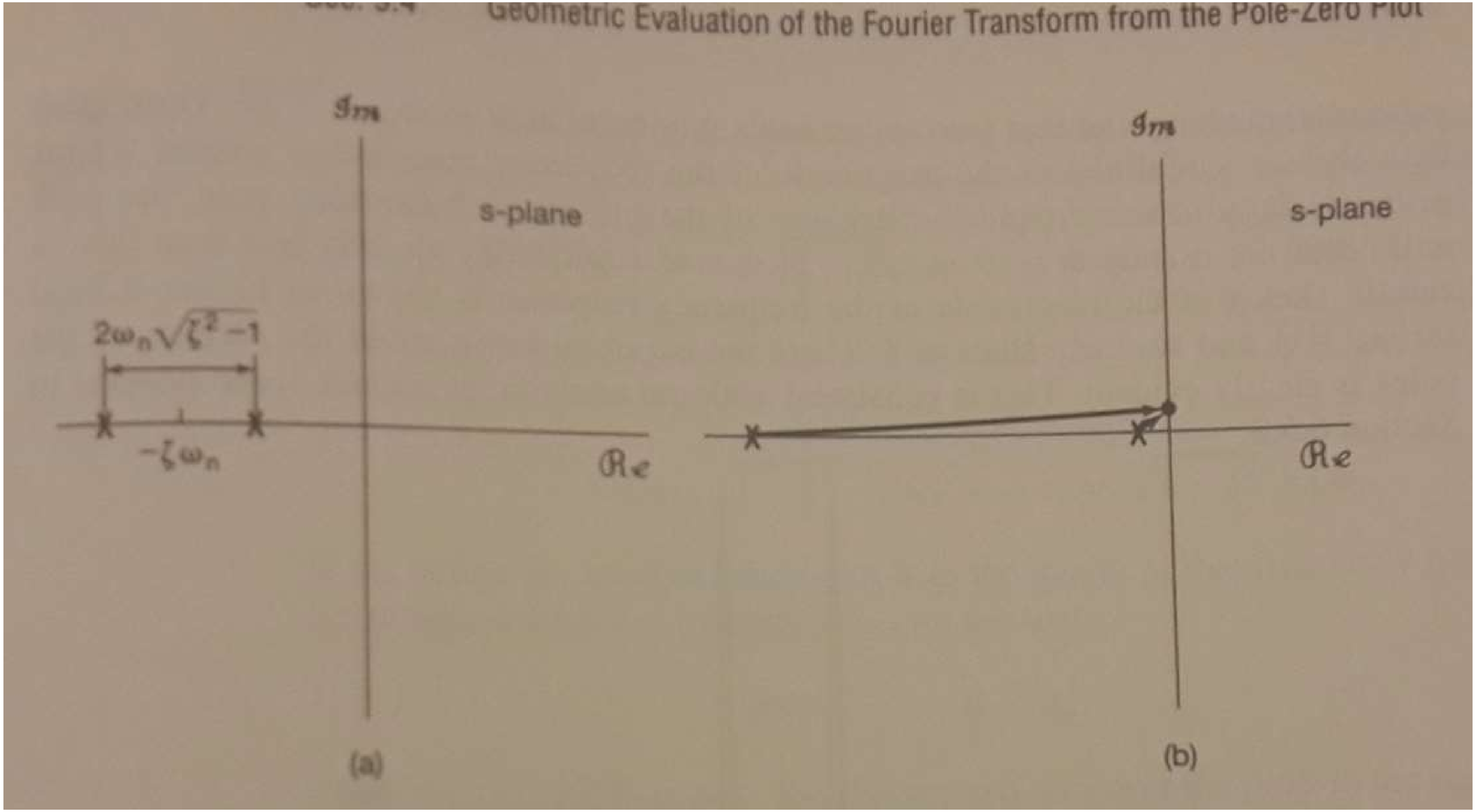
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}. \qquad (9.79)$$

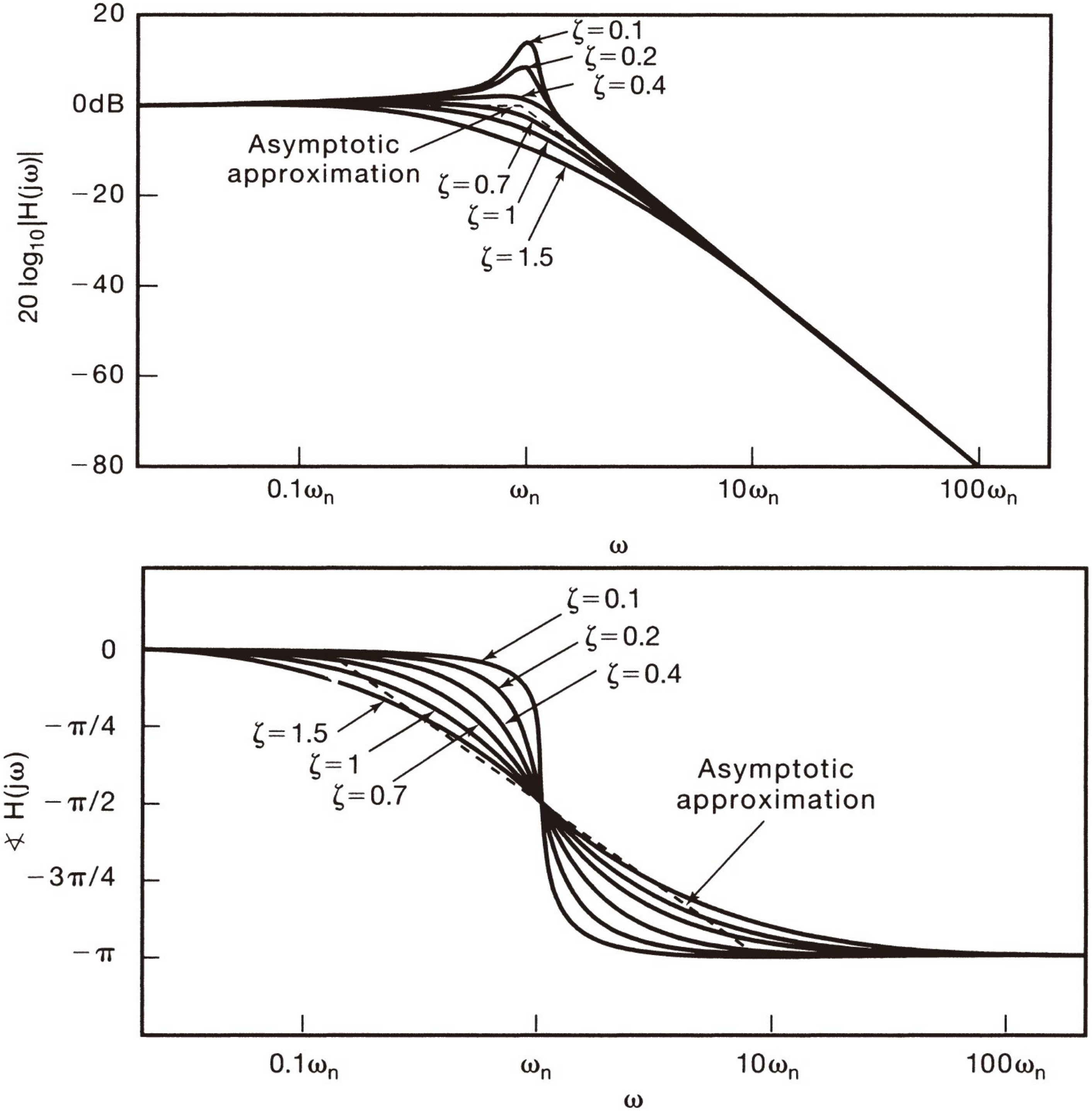
$$H(s) = \frac{\omega_n^2}{(s-c_1)(s-c_2)}.$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1},$$

ξ>1, c_1 and c_2 are real numbers





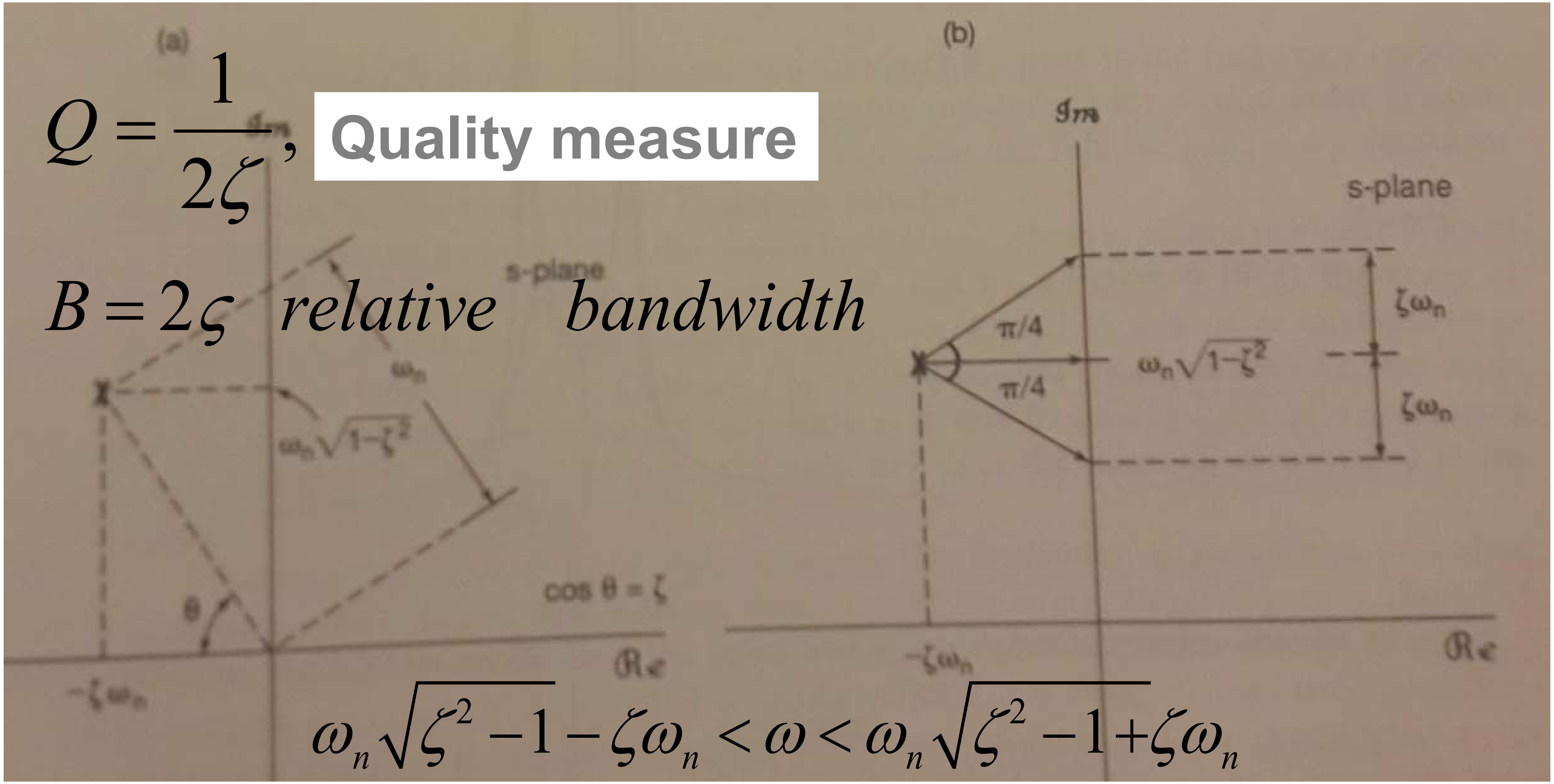
20
0dB
−20
−40
−60
−80
20 log10|H(jω)|
ζ = 0.1
ζ = 0.2
ζ = 0.4
Asymptotic approximation
ζ = 0.7
ζ = 1
ζ = 1.5
0.1ωn
ωn
10ωn
100ωn
ω
0
−π/4
−π/2
−3π/4
−π
∢ H(jω)
ζ = 0.1
ζ = 0.2
ζ = 0.4
ζ = 1.5
ζ = 1
ζ = 0.7
Asymptotic approximation
0.1ωn
ωn
10ωn
100ωn
ω

$$H(s)=\frac{\omega_n^2}{(s-c_1)(s-c_2)}.\qquad c_1=-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1},\quad c_2=-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1},$$

$1>\xi>0$, c_1 and c_2 are complex numbers

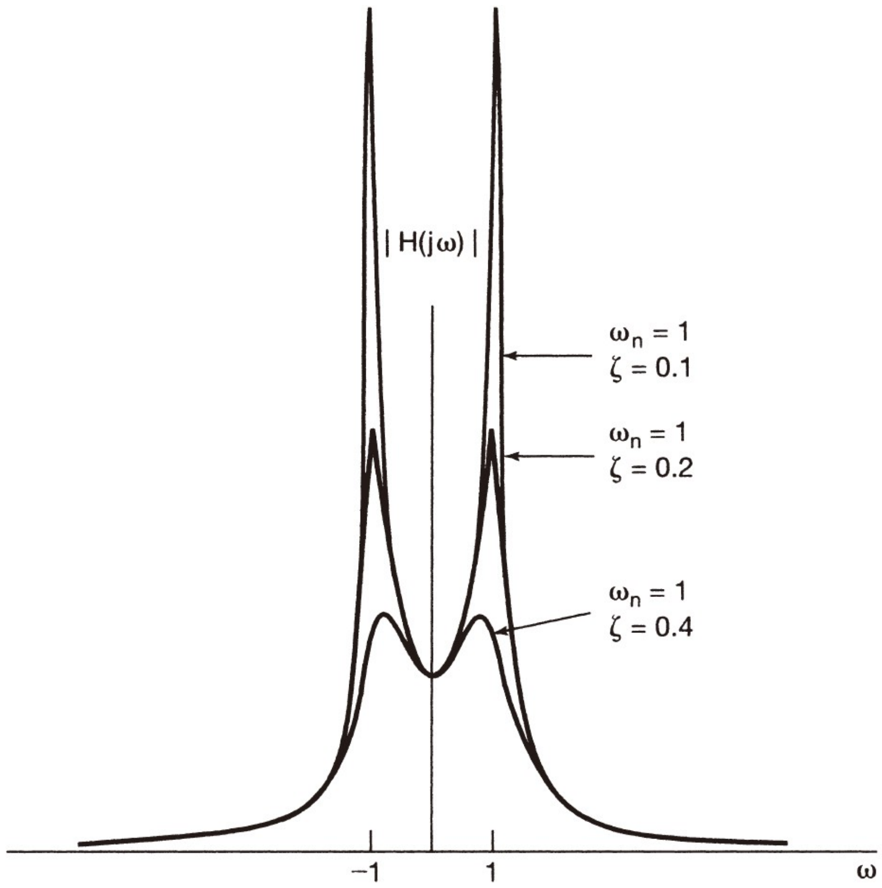
$$Q=\frac{1}{2\zeta},$$ **Quality measure**

$$B=2\varsigma \quad relative \quad bandwidth$$



$$\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n<\omega<\omega_n\sqrt{\zeta^2-1}+\zeta\omega_n$$

9.4.2 Second-Order Systems



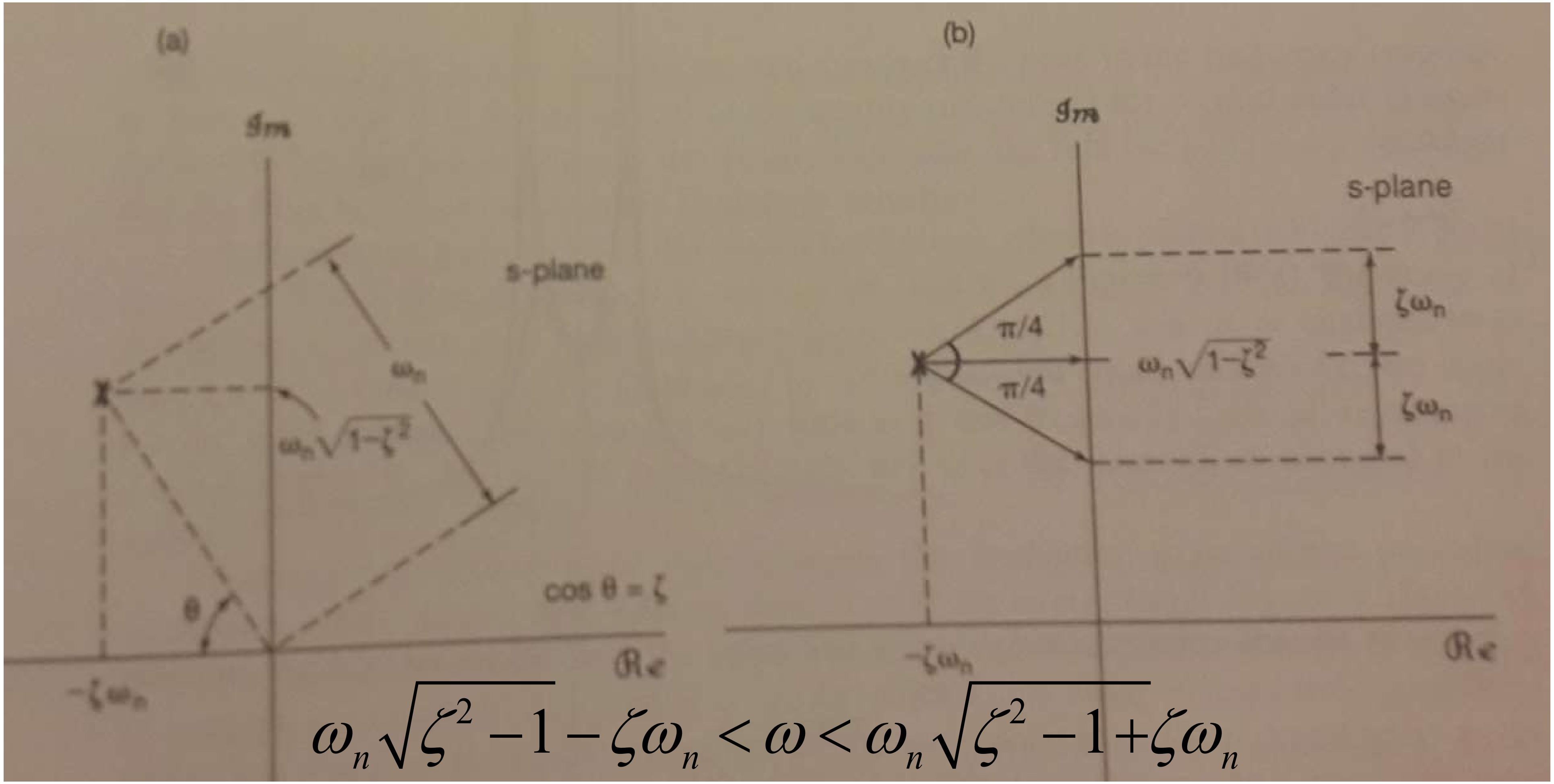
(a)

$$H(s) = \frac{\omega_n^2}{(s-c_1)(s-c_2)}.$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1},$$

$1>\xi>0$, c_1 and c_2 are complex numbers



$$\omega_n\sqrt{\zeta^2 - 1} - \zeta\omega_n < \omega < \omega_n\sqrt{\zeta^2 - 1} + \zeta\omega_n$$

9.4.2 Second-Order Systems

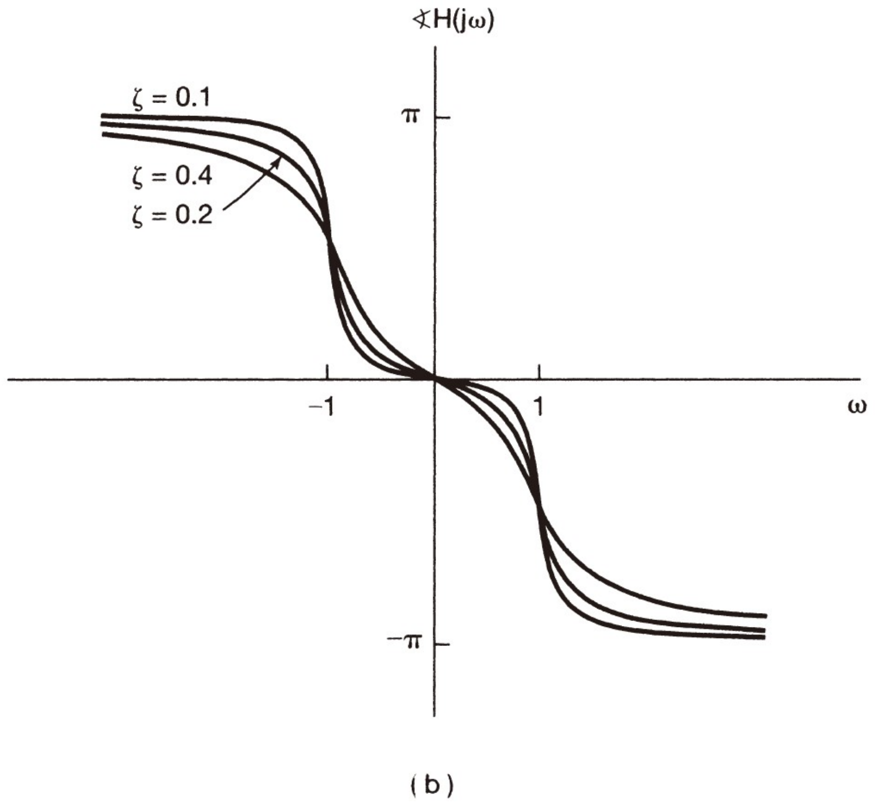
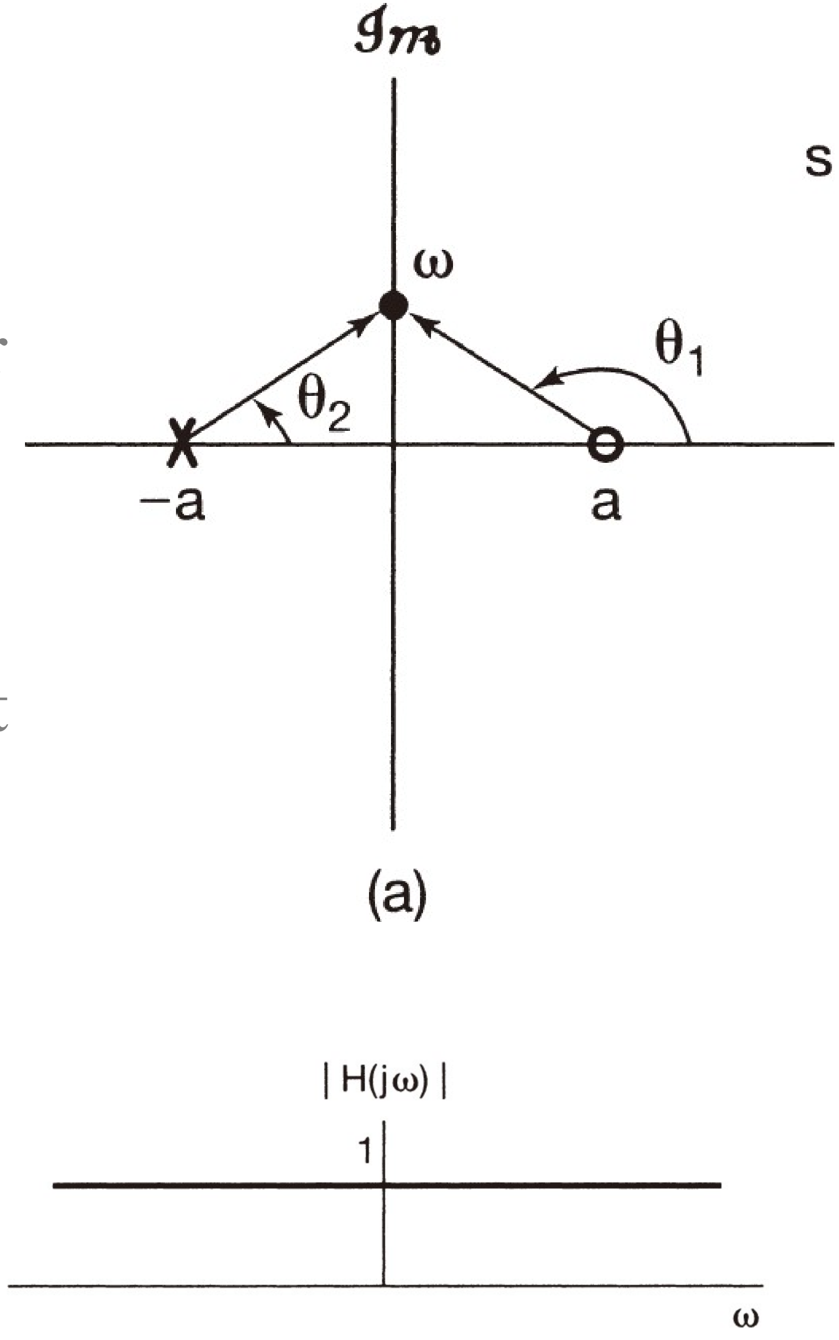
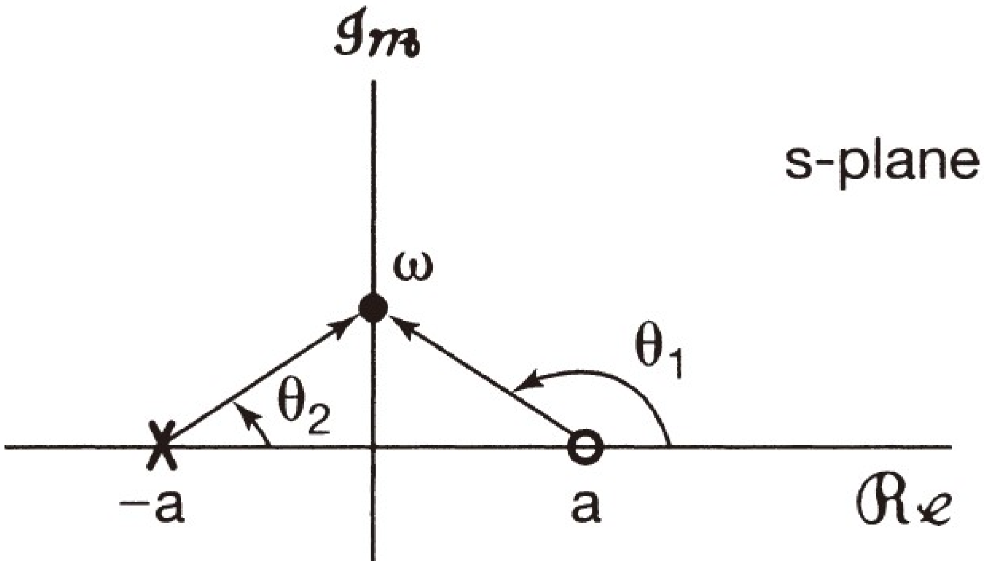


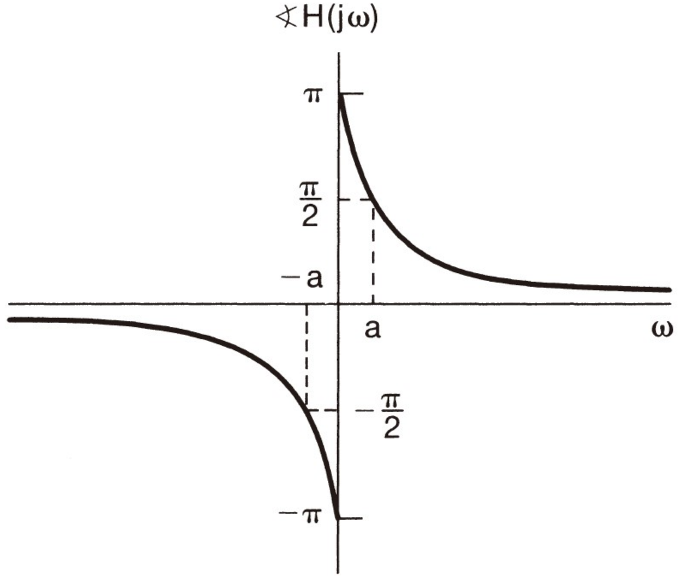
Figure 9.20 (a) Magnitude and (b) phase of the frequency response for a second-order system with $0 < \zeta < 1$.

9.4.3 All-Pass Systems

As a final illustration of the geometric evaluation of the frequency response, let us consider a system for which the Laplace transform of the impulse response has the pole-zero plot shown in Figure 9.21(a). Form this figure, it is evident that for any point along the $j\omega$-axis, the pole and zero vectors have equal length, and consequently, the magnitude of the frequency response is constant and independent of frequency.







The phase of the frequency response is $\theta_1 - \theta_2$, or, since $\theta_1 = \pi - \theta_2$

$$\sphericalangle H(j\omega) = \pi - 2\theta_2. \tag{9.80}$$

From Figure 9.21(a), $\theta_2 = \tan^{-1}(\omega / a)$, and thus,

$$\sphericalangle H(j\omega) = \pi - 2\tan^{-1}\left(\frac{\omega}{a}\right). \tag{9.81}$$

9.5.1 Linearity of the Laplace Transform

If

$x_1(t) \overset{L}{\longleftrightarrow} X_1(s)$ with a region of convergence that will be denoted as R_1

And

$x_2(t) \overset{L}{\longleftrightarrow} X_2(s)$ with a region of convergence that will be denoted as R_2

Then

$$ax_1(t) + bx_2(t) \overset{L}{\longleftrightarrow} aX_1(s) + bX_2(s), \quad \text{with ROC} \quad (9.82)$$

containing $R_1 \cap R_2.$

線性性質(重疊原理)

Example 9.13

In this example, we illustrate the fact that the ROC for the Laplace transform of a linear combination of signals can sometimes extend beyond the intersection of the ROCs for individual terms. Consider

$$x(t) = x_1(t) - x_2(t), \qquad (9.83)$$

Example 9.13

Where the Laplace transforms of $X_1(t)$ and $X_2(t)$ are, respectively,

$$X_1(s) = \frac{1}{s+1}, \qquad \Re e\{s\} > -1, \tag{9.84}$$

and

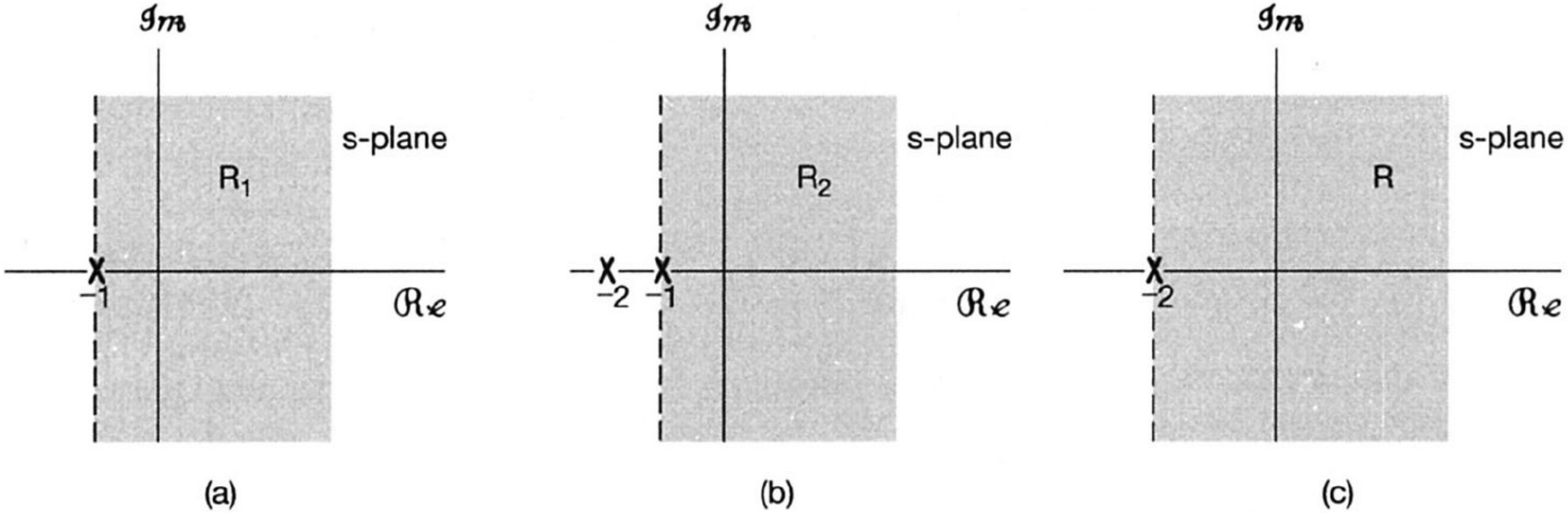
$$X_2(s) = \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1. \tag{9.85}$$

Example 9.13

The pole-zero plot, including the ROCs for $X_1(s)$ and $X_2(s)$, is shown in Figures 9.22(a) and (b). From eq. (9.82),

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}.$$

(9.86)



Thus, in the linear combination of $X_1(t)$ and $X_2(t)$, the pole at *s* = -1 is canceled by a zero at *s* = -1. The pole-zero plot for $X(s) = X_1(s) - X_2(s)$ is shown in Figure 9.22(c). The intersection of the ROCs for $X_1(s)$ and $X_2(s)$ is $\Re e\{s\} > -1$. However, since the **ROC is always bounded by a pole or infinity**, for this example the ROC for *X(s)* can be extended to the left to be bounded by the pole at *s* = -2, as a result of the pole-zero cancellation at *s* = -1.

9.5.2 Time Shifting

If

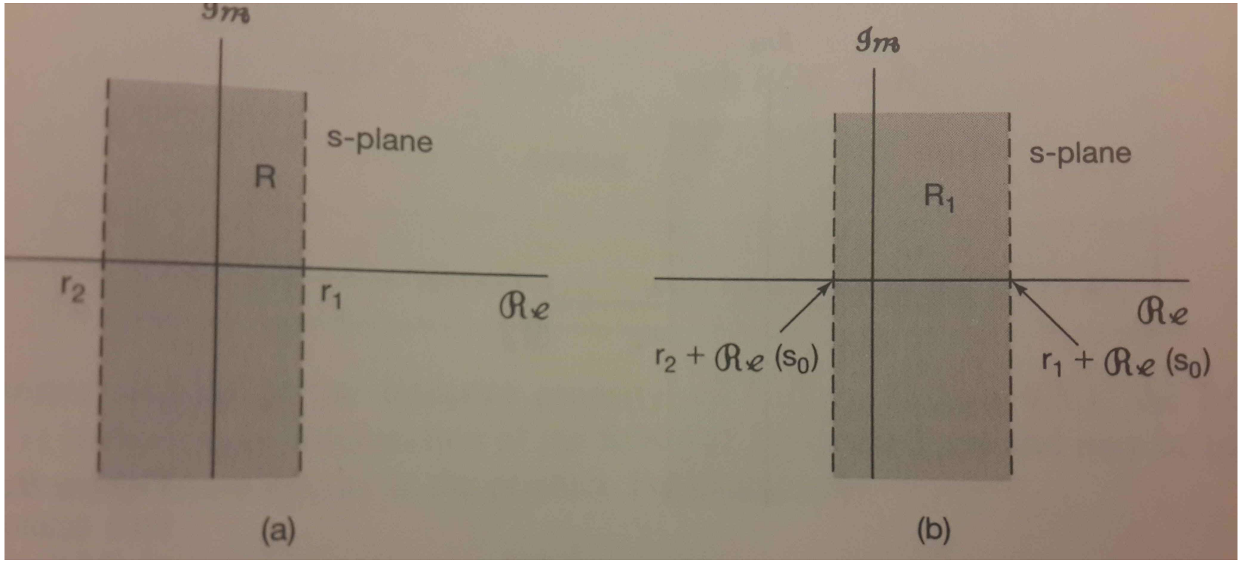
$$x(t) \overset{L}{\longleftrightarrow} X(s), \qquad with \qquad ROC = R,$$

then

$$x(t - t_0) \overset{L}{\longleftrightarrow} e^{-st_0} X(s), \qquad with \qquad ROC = R. \tag{9.87}$$

時間軸移位性質

9.5.3 Shifting in the s-Domain



If

$$x(t) \overset{L}{\longleftrightarrow} X(s), \quad with \quad ROC = R,$$

Then

$$e^{s_0 t}x(t) \overset{L}{\longleftrightarrow} X(s - s_0), \quad with \quad ROC = R + \Re e\{s_0\}. \quad (9.88)$$

$S_0 = j\omega_0$

In this case, eq. (9.88) becomes

$$e^{j\omega_0 t}x(t) \overset{L}{\longleftrightarrow} X(s - j\omega_0), \quad with \quad ROC = R. \quad (9.89)$$

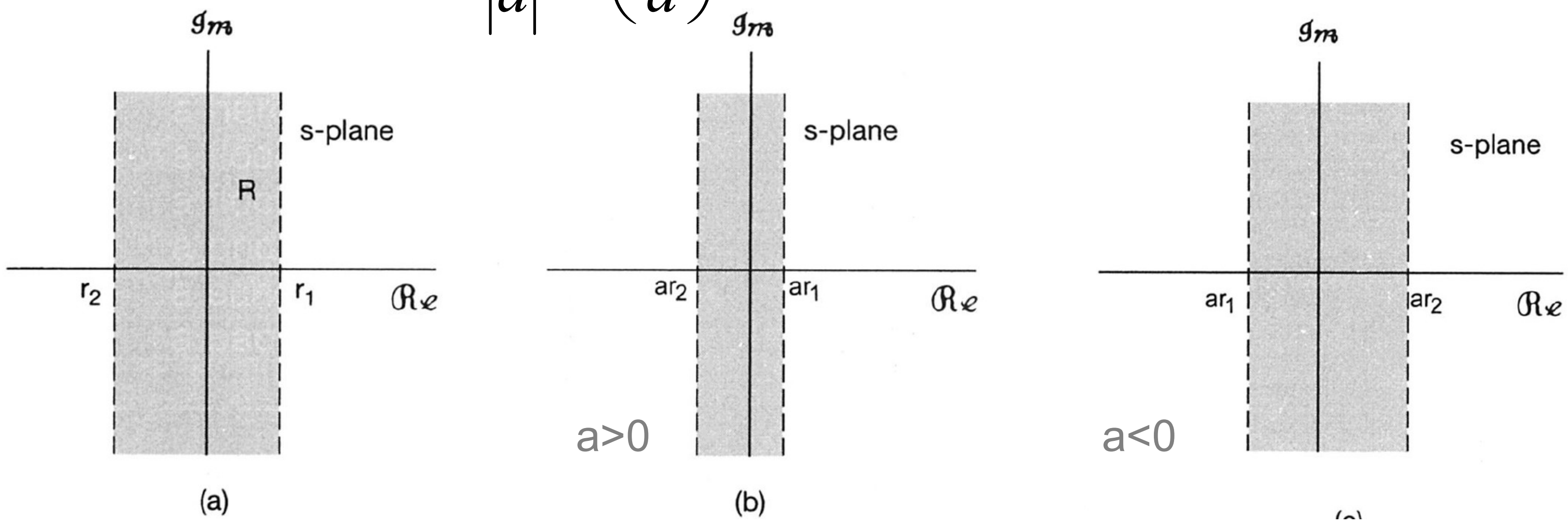
9.5.4 Time Scaling

If

$$x(t) \overset{L}{\longleftrightarrow} X(s), \qquad with \qquad ROC = R,$$

then

$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right), \qquad with \qquad ROC \qquad R_1 = aR. \tag{9.90}$$



9.5.4 Time Scaling

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad with \quad ROC \quad R_1 = aR.$$

Thus, time reversal of *x(t)* results in a reversal of the ROC. That is,

$$x(-t) \xleftrightarrow{\mathcal{L}} X(-s), \quad with \quad ROC = -R. \tag{9.91}$$

時間倒轉的拉式轉換

9.5.5 Conjugation

If

$$x(t) \stackrel{L}{\longleftrightarrow} X(s), \qquad with\ ROC = R, \tag{9.92}$$

Then

$$x^*(t) \stackrel{L}{\longleftrightarrow} X^*(s^*), \qquad with\ ROC = R. \tag{9.93}$$

共軛性質

Therefore,

$$X(s) = X^*(s^*) \text{ when x(t) is real.} \tag{9.94}$$

若x(t)為實值訊號，則 $X(s) = X^*(s^*)$

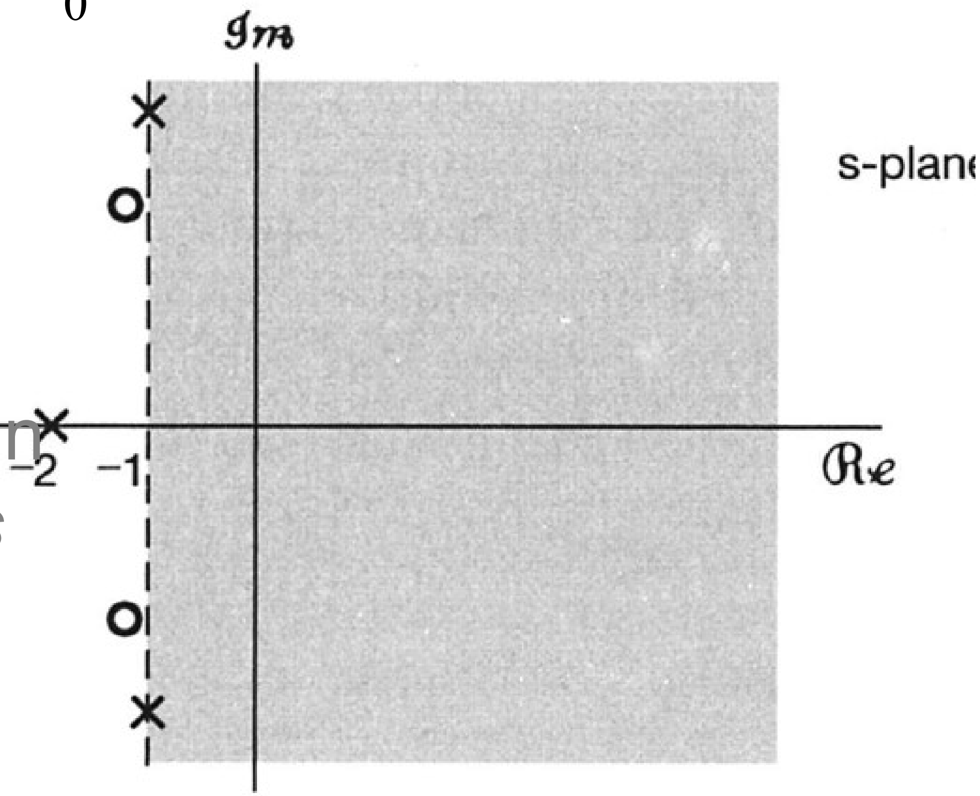
9.5.5 Conjugation

$$X(s) = X^*(s^*)$$

Consequently, if *x(t)* is real and if *X(s)* has a pole or zero at $s = s_0$ (i.e., if *X(s)* is unbounded or zero at $s = s_0$), then *X(s)* also has a pole or zero at the complex conjugate point $s = s_0^*$.
For example, the transform *X(s)* for the real signal *x(t)* in Example 9.4 has poles at *s* = 1 ± 3*j* and zeros at $s = (-5 \pm j\sqrt{71})/2$.

$$x(t) = e^{-2t}u(t)$$

$$+e^{-t}(\cos 3t)u(t).$$



9.5.6 Convolution Property

If

$$x_1(t) \stackrel{L}{\longleftrightarrow} X_1(s), \qquad with\ ROC = R_1,$$

and

$$x_2(t) \stackrel{L}{\longleftrightarrow} X_2(s), \qquad with\ ROC = R_2,$$

Then

$$x_1(t) * x_2(t) \stackrel{L}{\longleftrightarrow} X_1(s)X_2(s),$$ with ROC containing $R_1 \cap R_2$.

(9.95)

迴旋運算性質

9.5.6 Convolution Property

In a manner similar to the linearity property set forth in Section 9.5.1, the ROC of $X_1(s)X_2(s)$ includes the intersection of the ROCs of $X_1(s)$ and $X_2(s)$ and **may be larger if pole-zero cancellation** occurs in the product. For example, if

$$X_1(s) = \frac{s+1}{s+2}, \qquad \Re e\{s\} > -2, \tag{9.96}$$

and

$$X_2(s) = \frac{s+2}{s+1}, \qquad \Re e\{s\} > -1, \tag{9.97}$$

9.5.7 Differentiation in the Time Domain

If

$$x(t) \overset{L}{\longleftrightarrow} X(s), \qquad with\, ROC = R,$$

then

$$\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s), \qquad with\, ROC\, containing \quad R.$$

(9.98)

時域微分性質

9.5.7 Differentiation in the Time Domain

$$\frac{dx(t)}{dt} \stackrel{L}{\longleftrightarrow} sX(s), \quad \text{with ROC containing} \quad R.$$

This property follows by differentiating both sides of the inverse Laplace transform as expressed in equation (9.56). Specifically, let

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds.$$

Then

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds. \qquad (9.99)$$

9.5.8 Differentiation in the s-Domain

Differentiating both sides of the Laplace transform equation (9.3), i.e.,

$$X(s)=\int_{-\infty}^{+\infty} x(t)e^{-st}dt,$$

we obtain

$$\frac{dX(s)}{ds}=\int_{-\infty}^{+\infty}(-t)x(t)e^{-st}dt.$$

9.5.8 Differentiation in the s-Domain

Consequently, if

$$x(t) \overset{L}{\longleftrightarrow} X(s), \qquad with\, ROC = R,$$

then

$$-tx(t) \overset{L}{\longleftrightarrow} \frac{dX(s)}{ds}, \qquad with\, ROC = R. \tag{9.100}$$

s域微分性質（與時域微分性質互為對偶）

Example 9.14

$$-tx(t) \overset{L}{\longleftrightarrow} \frac{dX(s)}{ds}, \qquad with\ ROC = R.$$

Let us find the Laplace transform of

$$x(t) = te^{-at}u(t). \qquad (9.101)$$

Since

$$e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+a}, \qquad \Re e\{s\} > -a,$$

it follows from eq.(9.100) that

$$te^{-at}u(t) \overset{L}{\longleftrightarrow} -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \qquad \Re e\{s\} > -a.$$

(9.102)

Example 9.14

In fact, by repeated application of eq. (9.100), we obtain

$$\frac{t^2}{2}e^{-at}u(t)\overset{L}{\longleftrightarrow}\frac{1}{(s+a)^3}, \qquad \Re e\{s\}>-a, \tag{9.103}$$

and, more generally,

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)\overset{L}{\longleftrightarrow}\frac{1}{(s+a)^n}, \qquad \Re e\{s\}>-a. \tag{9.104}$$

9.5.9 Integration in the Time Domain

If

$$x(t) \stackrel{L}{\longleftrightarrow} X(s), \qquad with\ ROC = R,$$

Then

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{L}{\longleftrightarrow} \frac{1}{s}X(s),$$ with ROC **containing**

$$R \cap \{\Re e\{s\} > 0\}. \tag{9.106}$$

時域積分性質

9.5.9 Integration in the Time Domain

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{L} \frac{1}{s}X(s),$$

This property is the inverse of the differentiation property set forth in Section 9.5.7. It can be derived using the **convolution** property presented in Section 9.5.6. Specifically,

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t). \qquad (9.107)$$

此性質為時域微分性質的逆向性質。

9.5.9 Integration in the Time Domain

$$e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+a}, \qquad \Re e\{s\} > -a,$$

From Example 9.1, with a = 0,

$$u(t) \overset{L}{\longleftrightarrow} \frac{1}{s}, \qquad \Re e\{s\} > 0, \tag{9.108}$$

and thus, from the convolution property,

$$u(t) * x(t) \overset{L}{\longleftrightarrow} \frac{1}{s} X(s), \tag{9.109}$$

ROC including

$$R \cap \{\Re e\{s\} > 0\}.$$

9.5.10 The Initial- and Final-Value Theorems

If x(t)=0 for t<0 and x(t) contains no impulses or higher order singularities at the origin,

the initial value $x(0^+)$—i.e., *x(t)* as *t* approaches zero from positive values of *t*. Specifically the *initial-value theorem* states that

$$x(0^+) = \lim_{s\to\infty} sX(s), \qquad (9.110)$$

初值定理（利用s域函數求時域的初值）

9.5.10 The Initial- and Final-Value Theorems

Also, it $x(t) = 0$ for $t < 0$ and, in addition, $x(t)$ has a finite limit as $t \to \infty$, then the *final-value theorem* says that

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s). \qquad (9.111)$$

See Problem 9.53

終值定理（利用s域函數求時域的終值）

Example 9.16

$$\frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}$$

$$x(t)=e^{-2t}u(t) + e^{-t}(\cos 3t)u(t).$$

The initial- and final-value theorems can be in checking the correctness of the Laplace transform calculations for a signal. For example, consider the signal x(t) in Example 9.4. From eq. (9.24), we see that $x(0^+)=2$. Also, using eq. (9.29), we find that

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\frac{2s^3+5s^2+12s}{s^3+4s^2+14s+20}=2,$$

which is consistent with the initial-value theorem in eq. (9.110).

9.5.11 Table of Properties

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM 表 9.1 常用的拉氏轉換性質

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{Re}\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

9.6 Some Laplace Transform Pairs

表 9.2　基本函數的拉氏轉換

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$

9.7 Analysis And Characterization of LTI Systems Using The Laplace Transform

Specifically, the Laplace transforms of the input and output of an LTI system are related through multiplication by the Laplace transform of the impulse response of the system. Thus,

LTI系統的輸入、輸出、脈衝響應的拉氏轉換關係為：

$$Y(s) = H(s)X(s). \tag{9.112}$$

1.4.2 The Continuous-Time Unit Step and Unit Impulse Functions

If the input to the LTI system is $x(t)=e^{st}$, with s in the ROC of H(s), then the output will be $H(s)e^{st}$.

e^{st} is the eigenfunction of the system with eigenvalue equal to the Laplace transform of the impulse response h(t).

H(s) is also referred to as "**system function**" or "**transfer function**".

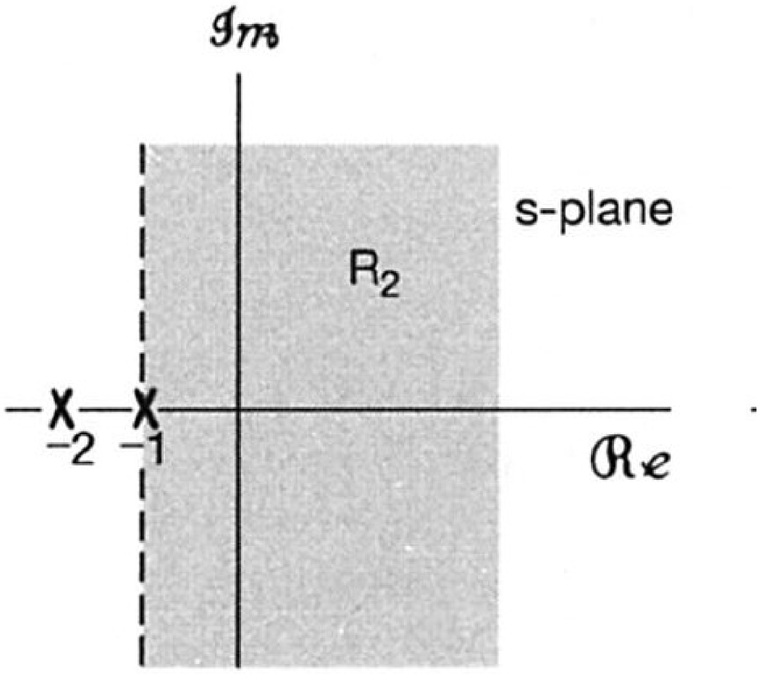
If $s=j\omega$ is included in the ROC, $H(j\omega)$ is the **frequency response** of the LTI system.

9.7.1 Causality

For a causal LTI system, the impulse response h(t) is zero for t<0, and thus is **right-sided.**

From Sec. 9.2, the ROC associated with the system function for a **causal** system is a **right-half plane** to the right of the rightmost pole.

The **converse** statement is not always true. It is only true for a system with **rational** H(s) “system function”.



Example 9.17

Consider a system with impulse response

$$h(t) = e^{-t}u(t). \tag{9.113}$$

Since $h(t) \neq 0$ for $t < 0$, this system is causal. Also, the system function can be obtained from Example 9.1:

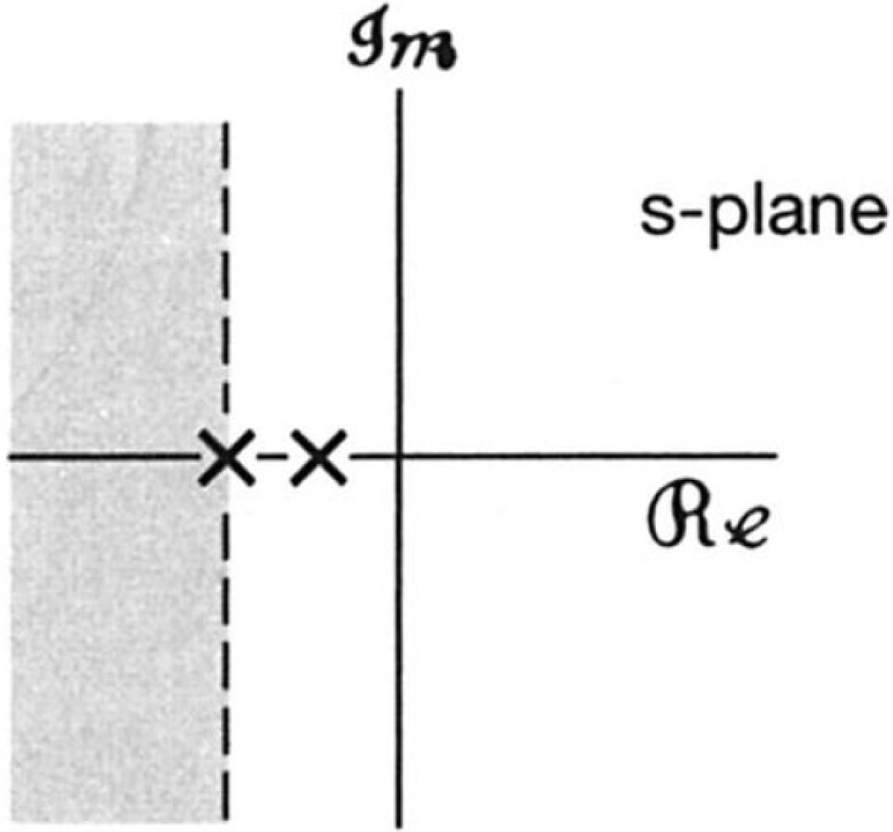
$$H(s) = \frac{1}{s+1}, \qquad \Re e\{s\} > -1. \tag{9.114}$$

Example 9.17

In this case, the system function is **rational** and the ROC in eq. (9.114) is to the **right of the rightmost pole**, consistent with our statement that causality for systems with rational system functions is equivalent to the ROC being to the right of the rightmost pole.

9.7.1 Causality

In an exactly analogous manner, we can deal with the concept of anticausality. A system is anticausal if its impulse response $h(t) = 0$ for $t > 0$. Since in that case $h(t)$ would be **left sided**, we know from Section 9.2 that the ROC of the system function $H(s)$ would have to be a left-half plane to the left most pole



(c)

9.7.2 Stability

The ROC of *H(s)* can also be related to the **stability** of a system. As mentioned in Section 2.3.7, the stability of an LTI system is equivalent to **its impulse response being absolutely integrable**, in which case (Section 4.4) the **Fourier transform of the impulse response converges**.

An LTI system is stable if and only if the ROC of its system function *H(s)* includes the entire *jω*-axis [i.e., $\Re e\{s\}=0$].

Example 9.20

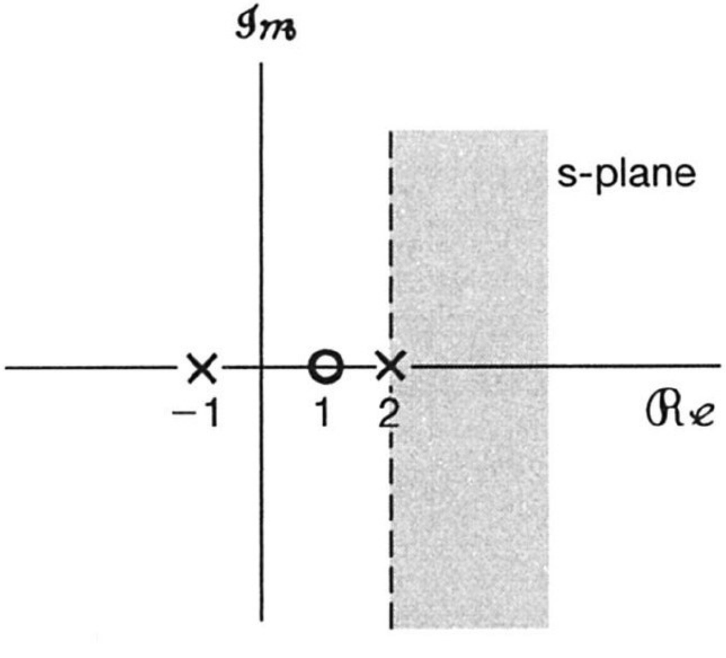
Let us consider an LTI system with system function

$$H(s) = \frac{s-1}{(s+1)(s-2)}. \qquad (9.119)$$

Since the ROC has not been specified, we know from our discussion in Section 9.2 that there are several different ROCs and, consequently, several different system impulse responses that can be associated with the algebraic expression for *H(s)* given in eq. (9.119).

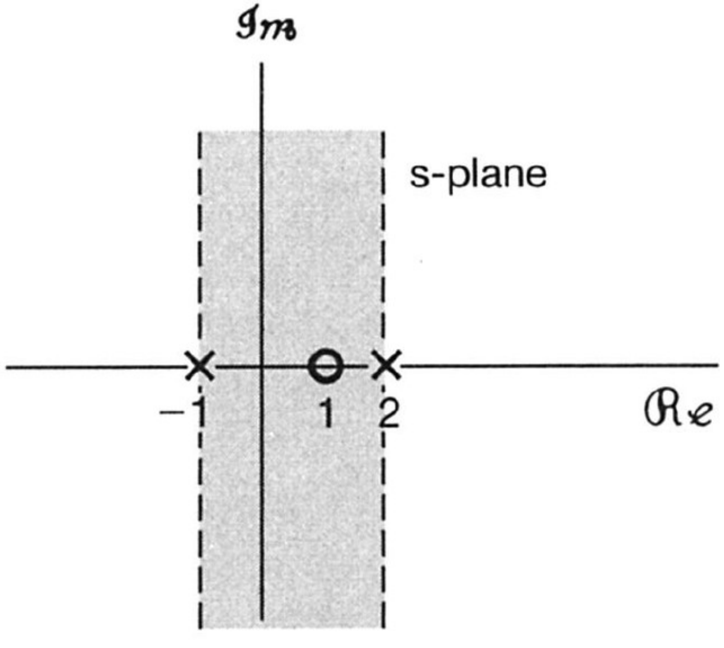
If, however, we have information about the **causality** or **stability** of the system, the appropriate ROC can be identified. For example, if the system is known to be **causal**, the ROC will be that indicated in Figure 9.25(a), with impulse response

$$h(t) = \left(\frac{2}{3} e^{-1} + \frac{1}{3} e^{2t} \right) u(t). \qquad (9.120)$$



$$H(s) = \frac{s-1}{(s+1)(s-2)}.$$

Note that this particular choice of ROC does **not include the *jω*-axis**, and consequently, the corresponding system is **unstable** (as can be checked by observing that *h(t)* is not absolutely integrable). On the other hand, if the system is known to be **stable**, the ROC is that given in Figure 9.25(b), and the corresponding impulse response is



$$H(s)=\frac{s-1}{(s+1)(s-2)}.$$

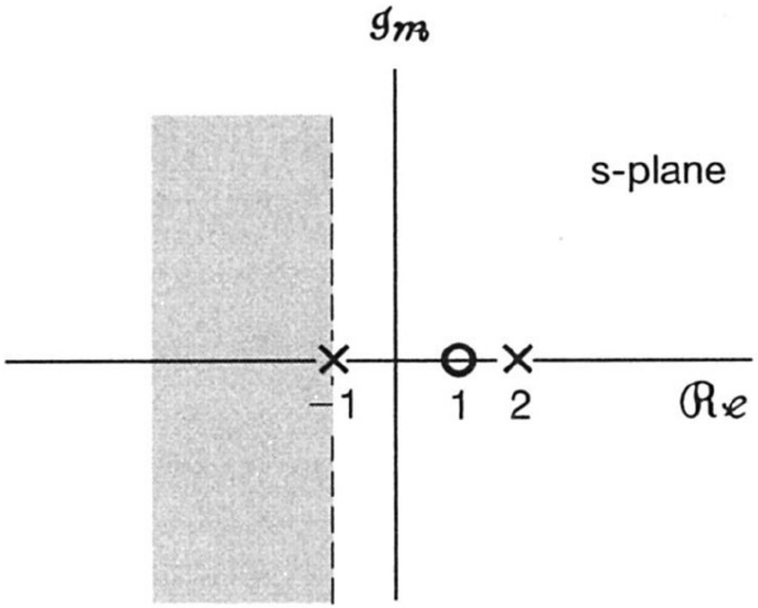
$$h(t)=\frac{2}{3}e^{-t}u(t)-\frac{1}{3}e^{2t}u(-t),$$

which is absolutely integrable.

Example 9.20

Finally, for the ROC in Figure 9.25(c), the system is anticausal and unstable, with

$$h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t).$$



9.7.2 Stability

A **causal** system with **rational** system function *H(s)* is **stable** if and only if **all of the poles** of *H(s)* lie in the **left-half of the s-plane**—i.e., all of the poles have **negative** real parts.

—具有有理式系統函數*H(s)*的因果系統，若且唯若*H(s)*的所有極點均位於*s*平面的左半平面。

Example 9.21

$$e^{-t}u(t)(9.113) \qquad H(s)=\frac{1}{s+1}, \qquad \Re e\{s\} > -1(9.114)$$

Consider again the causal system in Example 9.17. The impulse response in eq. (9.113) is absolutely integrable, ad thus the system is **stable**. Consistent with this, we see that the pole of *H(s)* in eq. (9.114) is at *s* = -1, which is in the **left**-half of the *s*-plane. In contrast, the causal system with impulse response

$$h(t)=e^{2t}u(t)$$

is **unstable**, since *h(t)* is not absolutely integrable. Also, in this case

$$H(s)=\frac{1}{s-2}, \qquad \Re e\{s\} > 2,$$

so the system has a pole at *s* = 2 in the **right** half of the *s*-plane.

9.7.3 LTI Systems Characterized by Linear Constant-Coefficient

- FT can be applied to obtain the frequency response of an LTI system characterized by a linear constant-coefficient differential equation.
- LT can be applied in a similar way

Example 9.23

Consider an LTI system for which the input *x(t)* and output *y(t)* satisfy the linear constant-coefficient differential equation

$$\frac{dy(t)}{dt} + 3y(t) = x(t). \tag{9.126}$$

Example 9.23

$$\frac{dy(t)}{dt} + 3y(t) = x(t).$$

$$\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s),$$

$$with \quad ROC \quad containing \quad R.$$

Applying the Laplace transform to both sides of eq. (9.126), and using the **linearity** and **differentiation** properties set forth in Sections 9.5.1 and 9.5.7, respectively [(eqs. (9.82) and (9.98)], we obtain the algebraic equation

$$sY(s) + 3Y(s) = X(s). \qquad (9.127)$$

Example 9.23 $Y(s) = H(s)X(s).$

Since, from eq. (9.112), the system function is

$$H(s) = \frac{Y(s)}{X(s)},$$

we obtain, for this system,

$$H(s) = \frac{1}{s+3}. \tag{9.128}$$

$$sY(s) + 3Y(s) = X(s).$$

Example 9.23

This, then, provides the algebraic expression for the system function, but **not the region of convergence**. In fact, as we discussed in Section 2.4, the differential equation itself is not a complete specification of the LTI system, and there are, in general, different impulse responses, all consistent with the differential equation. If, in addition to the differential equation, we know that the system is **causal**, then the ROC can be inferred to be to the right of the rightmost pole, which in this case corresponds to $\Re e\{s\} > -3$.

Example 9.23

$$H(s) = \frac{1}{s+3}.$$

If the system were known to be anticausal, then the ROC associated with H(s) would be $\Re e\{s\} < -3$. The corresponding impulse response in the causal case is

$$h(t) = e^{-3t}u(t), \quad (9.129)$$

whereas in the anticausal case it is

$$h(t) = -e^{-3t}u(-t). \quad (9.130)$$

9.7.3 LTI Systems Characterized by Linear Constant-Coefficient

Consider a general linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \qquad (9.131)$$

N階線性常係數微分方程

9.7.3 LTI Systems Characterized by Linear Constant-Coefficient

Applying the Laplace transform to both sides and using the linearity and differentiation properties repeatedly, we obtain

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s), \qquad (9.132)$$

or

$$H(s) = \frac{\left\{\sum_{k=0}^{M} b_k s^k\right\}}{\left\{\sum_{k=0}^{N} a_k s^k\right\}}. \qquad (9.133)$$

N階LTI系統的轉移函數與微分方程係數的關係

9.7.3 LTI Systems Characterized by Linear Constant-Coefficient

Thus, the system function for a system specified by a differential equation is always **rational**, with zeros at the solutions of

$$\sum_{k=0}^{M} b_k s^k = 0 \tag{9.134}$$

系統的零點滿足(9.134)式(即*H(s)*的分子令為0)。

and poles at the solutions of

$$\sum_{k=0}^{N} a_k s^k = 0. \tag{9.135}$$

系統的極點滿足(9.135)式(即*H(s)*的分母令為0)。

Example 9.24

An RLC circuit whose capacitor voltage and inductor current are **initially zero** (causual) constitutes an LTI system describable by a linear constant-coefficient differential equation. Consider the series RLC circuit in Figure 9.27. Let the voltage across the voltage source be the input signal x(t), and let the voltage measured across the capacitor be the output signal y(t).

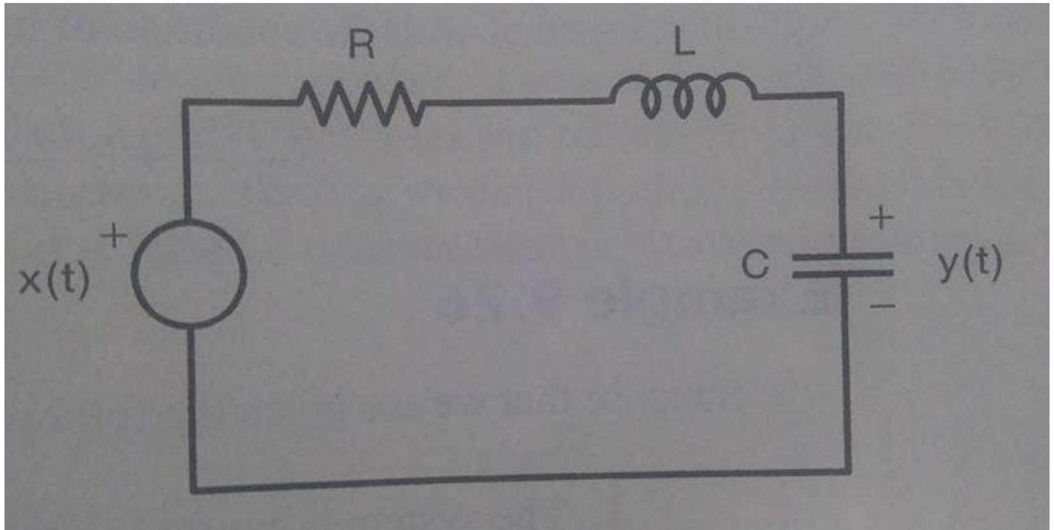


Figure 9.27 A series *RLC* circuit.

Example 9.24

$$H(s) = \frac{\left\{\sum_{k=0}^{M} b_k s^k\right\}}{\left\{\sum_{k=0}^{N} a_k s^k\right\}}.$$

Equating the sum of the voltages across the resistor, inductor, and capacitor with the source voltage, we obtain

$$RC\frac{dy(t)}{dt} + LC\frac{d^2 y(t)}{dt^2} + y(t) = x(t). \qquad (9.136)$$

Applying eq. (9.133), we obtain

$$H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)} \qquad (9.137)$$

$$= \frac{1/LC}{(s-c_1)(s-c_2)}; c_{1,2} = -(R/L) \pm \sqrt{(R/L)^2 - 4/LC}$$

$$H(s)=\frac{1/LC}{s^2+(R/L)s+(1/LC)}$$

$$=\frac{1/LC}{(s-c_1)(s-c_2)};c_{1,2}=-(R/L)\pm\sqrt{(R/L)^2-4/LC}$$

As shown in Problem 9.64, if the values of R, L, and C are all **positive**, the poles of this system function will have **negative real parts**, and consequently, the system will be stable.

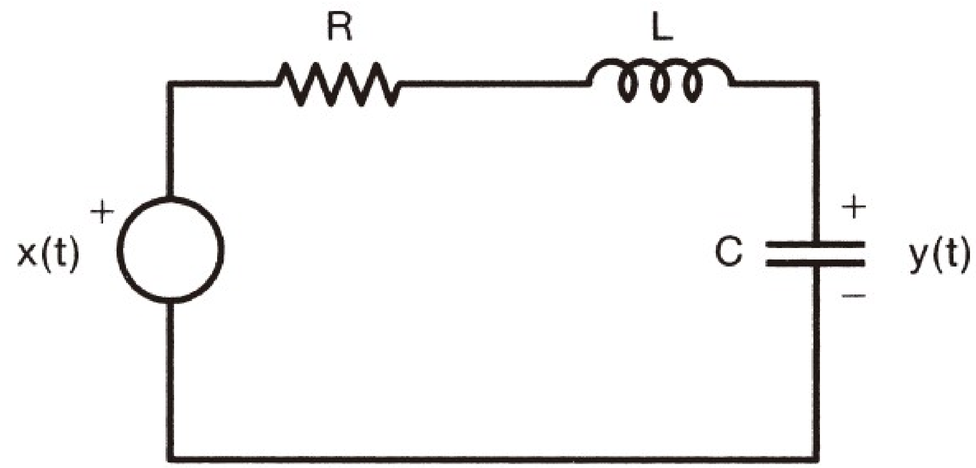


Figure 9.27 A series *RLC* circuit.

Example 9.25

Suppose we know that if the input to an LTI system is

$$x(t) = e^{-3t}u(t),$$

then the output is

$$y(t) = \left[e^{-t} - e^{-2t}\right]u(t).$$

由已知輸入及輸出，推導LTI系統的系統函數及微分方程的範例。

已知*x(t)*及*y(t)*

Example 9.25

As we now show, from this knowledge we can determine the system function for this system and from this can immediately deduce a number of other properties of the system.

Taking Laplace transforms of *x(t)* and *y(t),* we get

$$X(s) = \frac{1}{s+3}, \qquad \Re e\{s\} > -3,$$

and

$$Y(s) = \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1.$$

求得拉氏轉換*X(s)*及*Y(s)*。

Example 9.25 $Y(s) = H(s)X(s).$

From eq. (9.112), we can then conclude that

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}.$$

代入定義得系統函數*H(s)*。

Example 9.25

Furthermore, we can also determine the ROC for this system. In particular, we know from the convolution property set forth in Section 9.5.6 that the ROC of *Y(s)* must include **at least the intersections for the ROCs** of *X(s)* and *H(s)*. Examining the three possible choices for the ROC of *H(s)* (i.e., to the left of the pole at *s* = -2, between the poles at -2 and -1, and to the right of the pole at *s* = -1), we see that the only choice that is consistent with the ROCs of *X(s)* and *Y(s)* is $\Re e\{s\} > -1$.

Example 9.25

$$H(s) = \frac{s+3}{s^2+3s+2}.$$

Since this is to the right of the rightmost pole of *H(s)*, we conclude that *H(s)* is **causal**, and since both poles of *H(s)* have negative real parts, it follows that the system is **stable**. Moreover, from the relationship between eqs. (9.131) and (9.133), we can specify the differential equation that, together with the condition of initial rest, characterizes the system:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t).$$

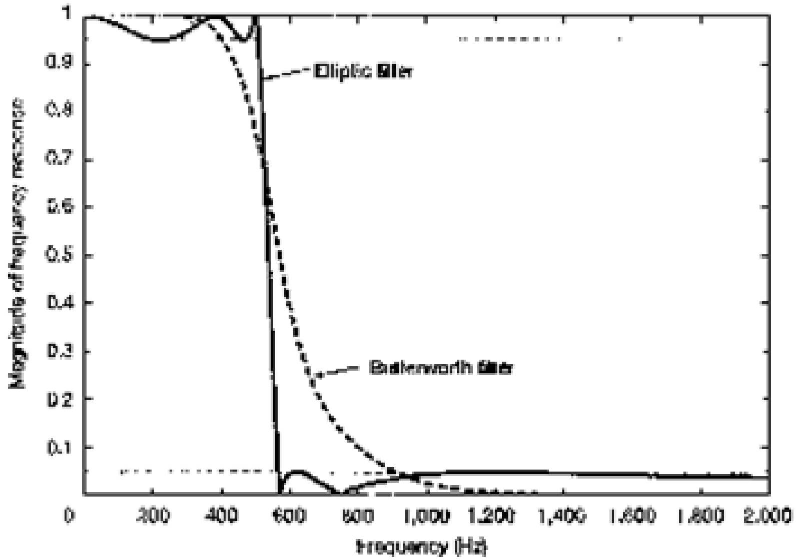
由*H(s)*分子與分母係數及*s*的次數可寫出微分方程。

9.7.5 Butterworth Filters

An Nth-order lowpass Butterworth filter has a frequency response the square of whose magnitude is given by

$$|B(j\omega)|^2 = \frac{1}{1+(j\omega/j\omega_c)^{2N}},$$

(9.140)



*N*階低通巴特沃斯濾波器的頻率響應*B(jω)*滿足（9.140）式。

9.7.5 Butterworth Filters

where N is the order of the filter. From eq. (9.140), we would like to determine the system function $B(s)$ that gives rise to $|B(j\omega)|^2$. We first note that, by definition,

$$|B(j\omega)|^2 = B(j\omega)B^*(j\omega). \qquad (9.141)$$

9.7.5 Butterworth Filters

If we restrict the **impulse response** of the Butterworth filter to be **real**, then from the property of conjugate symmetry for Fourier transforms,

$$B^{*}(j\omega) = B(-j\omega), \tag{9.142}$$

so that

$$B(j\omega)B(-j\omega) = \frac{1}{1+(j\omega / j\omega_c)^{2N}}. \tag{9.143}$$

9.7.5 Butterworth Filters

Next, we note that $B(s)\big|_{s=j\omega} = B(j\omega)$, and consequently, from eq. (9.143),

$$B(s)B(-s) = \frac{1}{1+(s/j\omega_c)^{2N}}. \quad (9.144)$$

The roots of the denominator polynomial corresponding to the combined **poles** of $B(s)B(-s)$ are at

$$s = (-1)^{1/2N}(j\omega_c). \quad (9.145)$$

9.7.5 Butterworth Filters

$$B(s)B(-s) = \frac{1}{1+(s/j\omega_c)^{2N}}.$$

Equation (9.145) is satisfied for any value $s = s_p$ for which

$$|s_p| = \omega_c \qquad (9.146)$$

and

$$\measuredangle s_p = \frac{\pi(2k+1)}{2N} + \frac{\pi}{2}, \quad k \text{ an integer}; \qquad (9.147)$$

that is,

$$s_p = \omega_c \exp\left(j\left[\frac{\pi(2k+1)}{2N} + \pi/2\right]\right). \qquad (9.148)$$

巴特沃斯濾波器的極點

9.7.5 Butterworth Filters

$$s_p = \omega_c \exp\left(j\left[\frac{\pi(2k+1)}{2N} + \pi/2 \right] \right).$$

In general, the following observation can be made about these poles:

1. There are $2N$ poles equally spaced in angle on a circle of radius ω_c in the s-plane.
2. A pole never lies on the $j\omega$-axis and occurs on the σ-axis for N odd, but not for N even.
3. The angular spacing between the poles of $B(s)B(-s)$ is π/N radians.

9.7.5 Butterworth Filters

$$B(s)B(-s) = \frac{1}{1+(s/j\omega_c)^{2N}}.$$

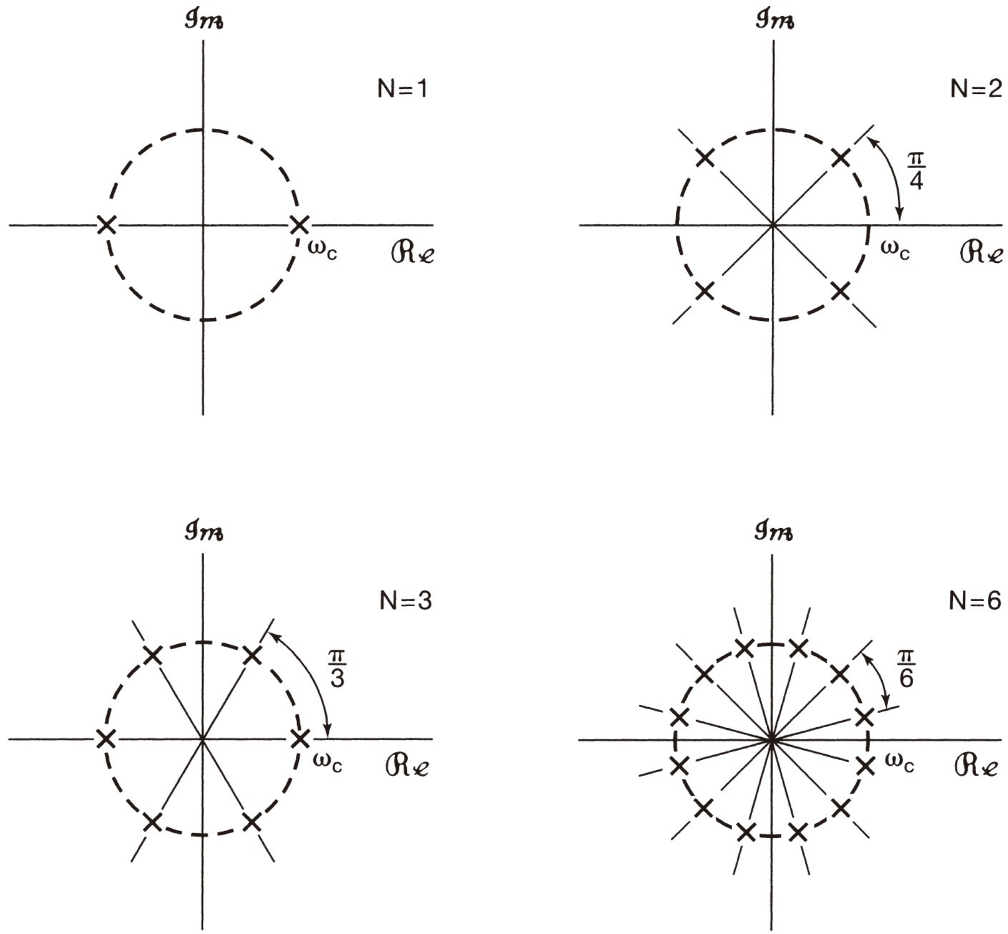


Figure 9.28 Position of the poles of $B(s)B(-s)$ for $N = 1, 2, 3$, and 6.

9.7.5 Butterworth Filters

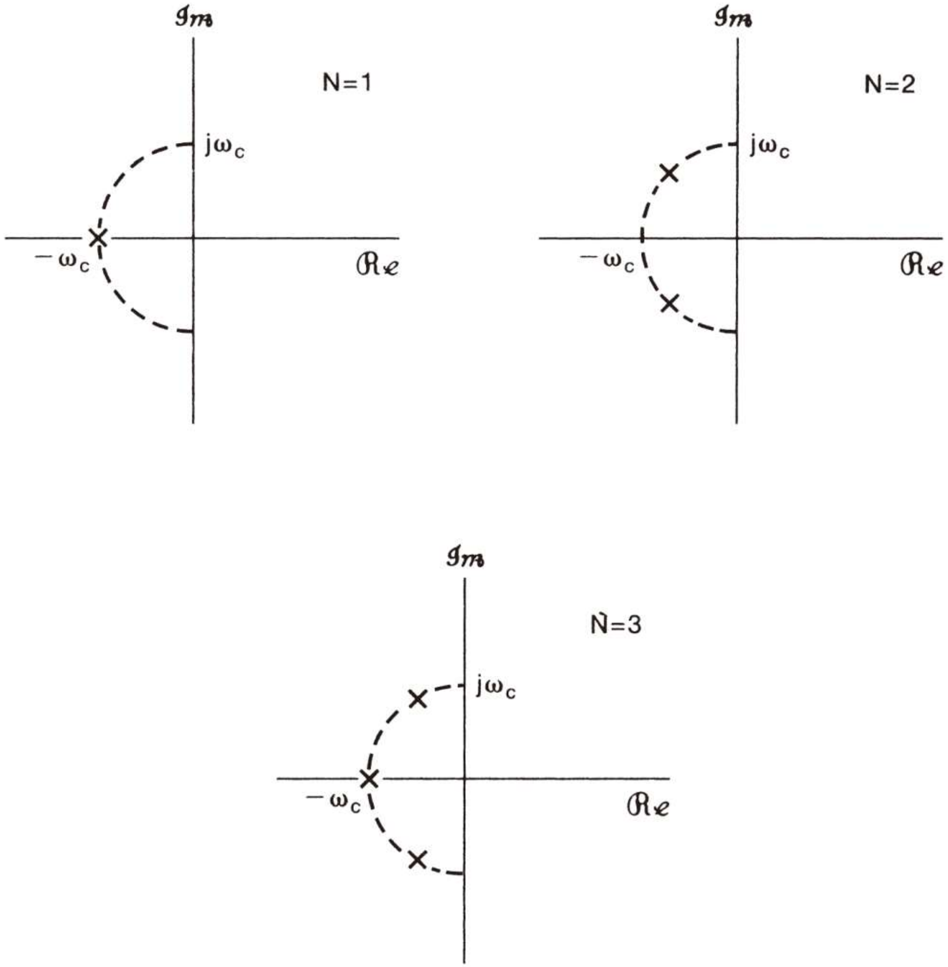


Figure 9.29 Position of the poles of $B(s)$ for $N = 1$, 2, and 3.

9.7.5 Butterworth Filters

In Figure 9.29 we show the poles associated with *B(s)* for each of these values of *N*. The corresponding transfer functions are:

N = 1: $$B(s) = \frac{\omega_c}{s + \omega_c}; \qquad (9.149)$$

N = 2: $$B(s) = \frac{\omega_c^2}{(s + \omega_c e^{j(\pi/4)})(s + \omega_c e^{-j(\pi/4)})} \qquad (9.150)$$

$$= \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2};$$

9.7.5 Butterworth Filters

$$\text{N} = 3: \; B(s) = \frac{\omega_c^3}{(s+\omega_c)(s+\omega_c e^{j(\pi/3)})(s+\omega_c e^{-j(\pi/3)})}$$

$$= \frac{\omega_c^3}{(s+\omega_c)(s^2+\omega_c s+\omega_c^2)} \qquad (9.151)$$

$$= \frac{\omega_c^3}{s^3+2\omega_c s^2+2\omega_c^2 s+\omega_c^3}.$$

9.7.5 Butterworth Filters

Specifically, for the foregoing three values of N, the corresponding differential equations are:

$$\text{N} = 1: \quad \frac{dy(t)}{dt} + \omega_c y(t) = \omega_c x(t); \qquad (9.152)$$

$$\text{N} = 2: \quad \frac{d^2 y(t)}{dt^2} + \sqrt{2}\omega_c \frac{dy(t)}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t); \qquad (9.153)$$

$$\text{N} = 3: \quad \frac{d^3 y(t)}{dt^3} + 2\omega_c \frac{d^2 y(t)}{dt^2} + 2\omega_c^2 \frac{dy(t)}{dt} + \omega_c^3 y(t) = \omega_c^3 x(t). \qquad (9.154)$$

9.8 System Function Algebra and Block Diagram Representations

The use of the Laplace transform allows us to replace time-domain operations such as differentiation, convolution, time shifting, and so on, with **algebraic operations**.

時域的計算如微分、迴旋運算、時間移位……等較為複雜，可改用拉氏轉換，只需以簡單的代數運算即可。

9.8.1 System Functions for Interconnections of LTI Systems

The impulse response of the overall system is

$$h(t) = h_1(t) + h_2(t), \tag{9.155}$$

and from the linearity of the Laplace transform,

$$H(s) = H_1(s) + H_2(s). \tag{9.156}$$

系統並接，則系統函數為各系統函數之和。

9.8.1 System Functions for Interconnections of LTI Systems

Similarly, the impulse response of the series interconnection in Figure 9.30(b) is

$$h(t) = h_1(t) * h_2(t), \tag{9.157}$$

and the associated system function is

$$H(s) = H_1(s)H_2(s). \tag{9.158}$$

系統串接，則系統函數為各系統函數之積。

9.8.1 System Functions for Interconnections of LTI Systems

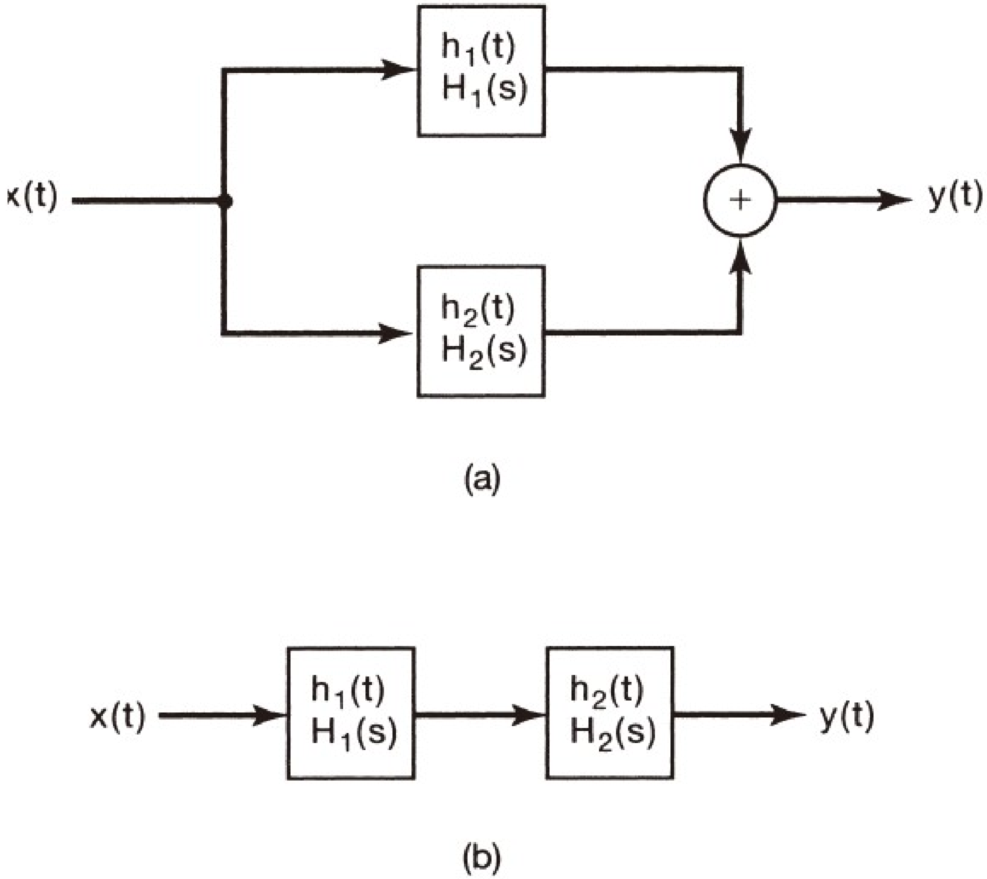
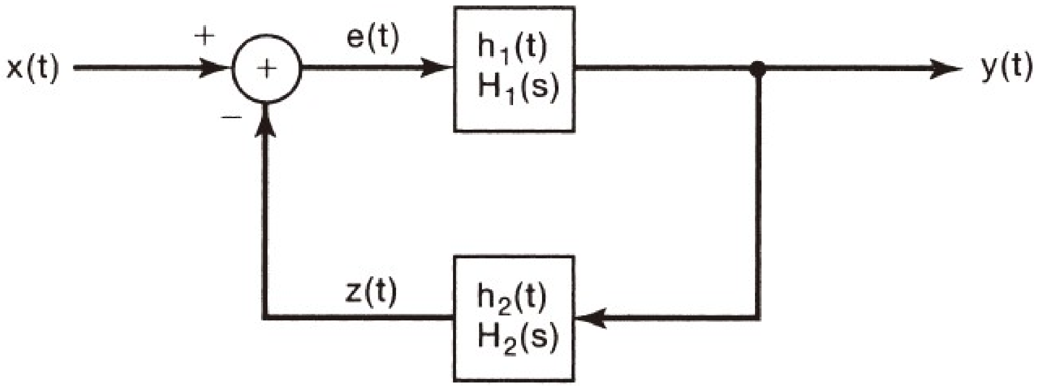


Figure 9.30 (a) Parallel interconnection of two LTI systems; (b) series combination of two LTI systems.



While analysis of the system in the time domain is not particularly simple, determining the overall system function from input x(t) to output y(t) is a straightforward algebraic manipulation. Specifically, from Figure 9.31,

$$Y(s) = H_1(s)E(s), \qquad (9.159)$$

$$E(s) = X(s) - Z(s), \qquad (9.160)$$

and

$$Z(s) = H_2(s)Y(s), \qquad (9.161)$$

9.8.1 System Functions for Interconnections of LTI Systems

from which we obtain the relation

$$Y(s) = H_1(s)\left[X(s) - H_2(s)Y(s)\right], \qquad (9.162)$$

or

$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}. \qquad (9.163)$$

單迴路系統的整體系統函數

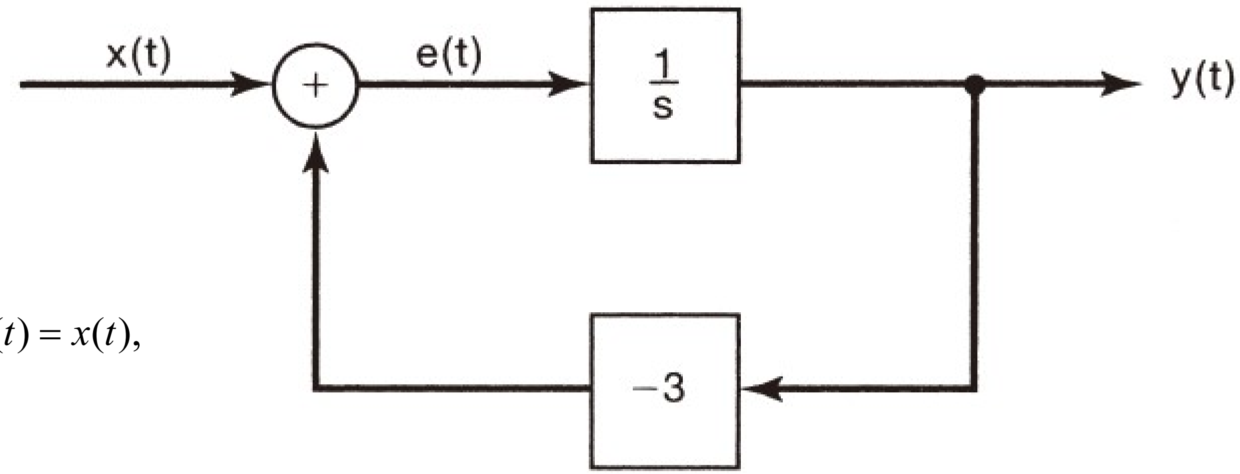
Example 9.28

Consider the causal LTI system with system function

$$H(s) = \frac{1}{s+3}.$$

From Section 9.7.3, we know that this system can also be described by the differential equation

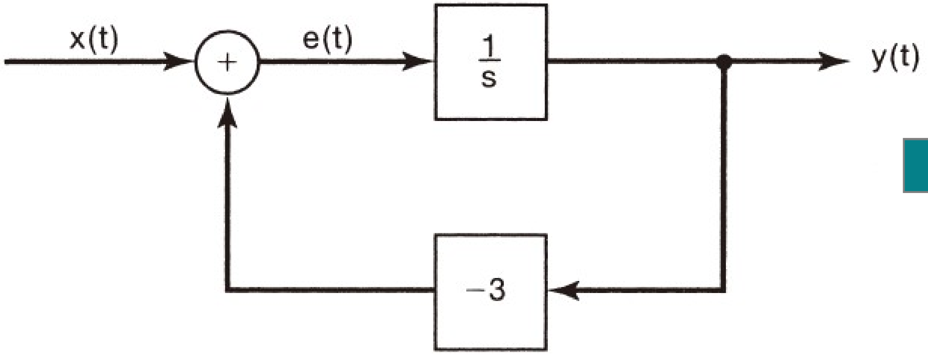
$$\frac{dy(t)}{dt} + 3y(t) = x(t),$$



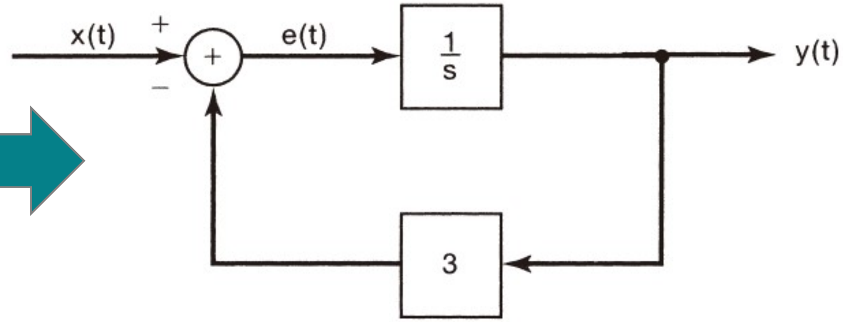
(a)

$$\frac{dy(t)}{dt} + 3y(t) = x(t),$$

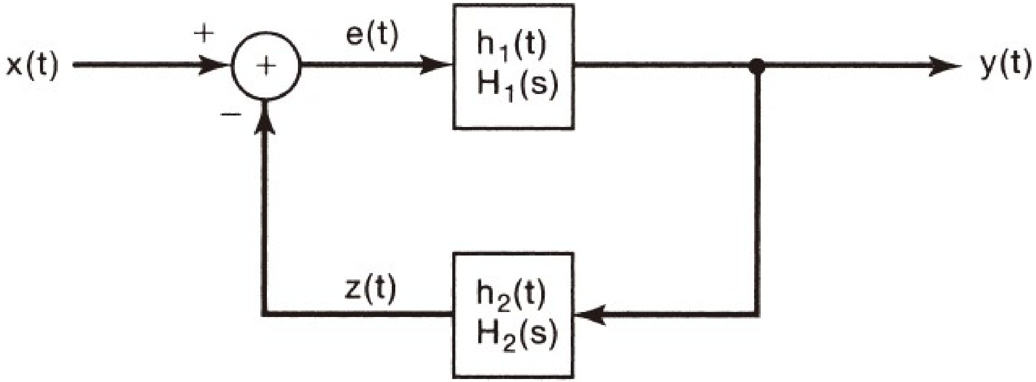
together with the condition of **initial rest**. In Section 2.4.3 we constructed a block diagram representation, shown in Figure 2.32, for a first-order system such as this. An equivalent block diagram (corresponding to Figure 2.32 with a = 3 and b = 1) is shown in Figure 9.32(a). Here, 1/s is the system function of a system with impulse response $u(t)$, i.e., it is the system function of an **integrator**.



(a)



(b)



$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1+H_1(s)H_2(s)} = \frac{1/s}{1+3/s} = \frac{1}{s+3}$$

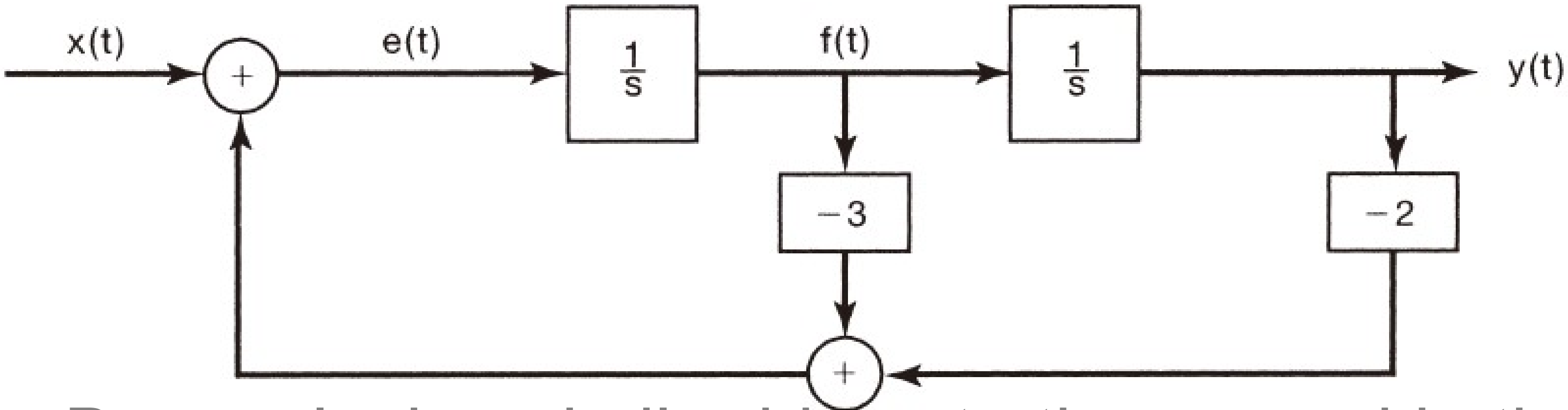
Example 9.30

Consider next a causal second-order system with system function

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}. \qquad (9.165)$$

The input *x(t)* and output *y(t)* for this system satisfy the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t). \qquad (9.166)$$



By employing similar ideas to those used in the preceding examples, we obtain the block diagram representation for this system shown in Figure 9.34(a). Specifically, since the input to an integrator is the derivative of the output of the integrator, the signals in the block diagram are related by

$$f(t) = \frac{dy(t)}{dt},$$

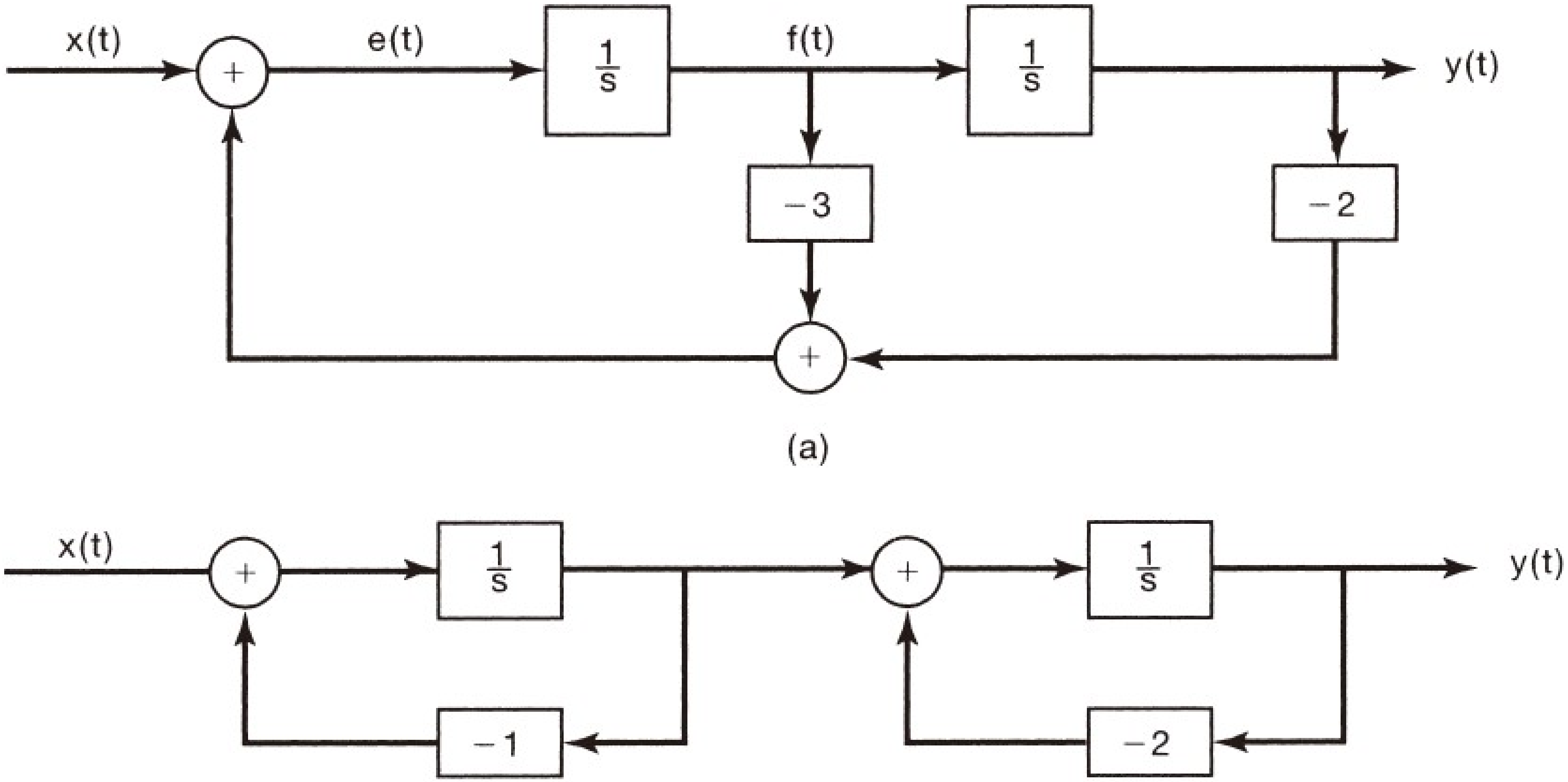
$$e(t) = \frac{df(t)}{dt} = \frac{d^2y(t)}{dt}.$$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t).$$

$$= e(t) + 3f(t) + 2y(t)$$

$$e(t) = -3f(t) - 2y(t) + x(t)$$

Example 9.30



(a)

(b)

Example 9.30

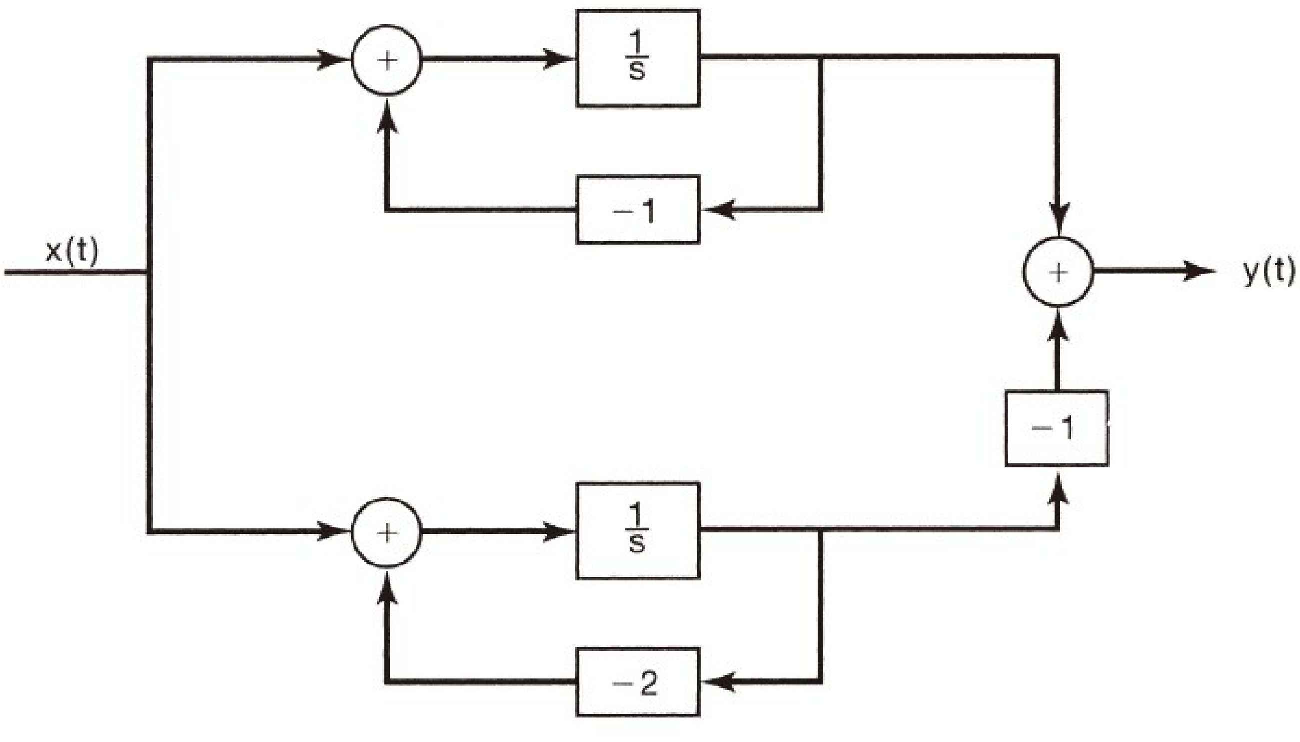
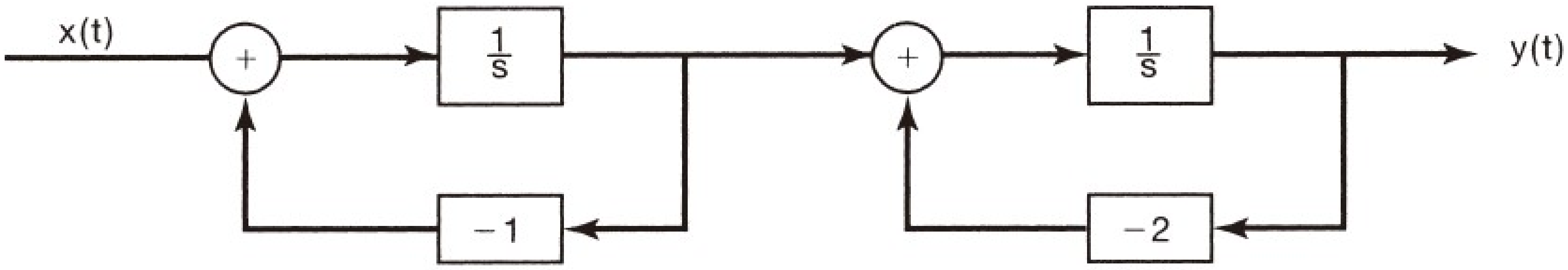


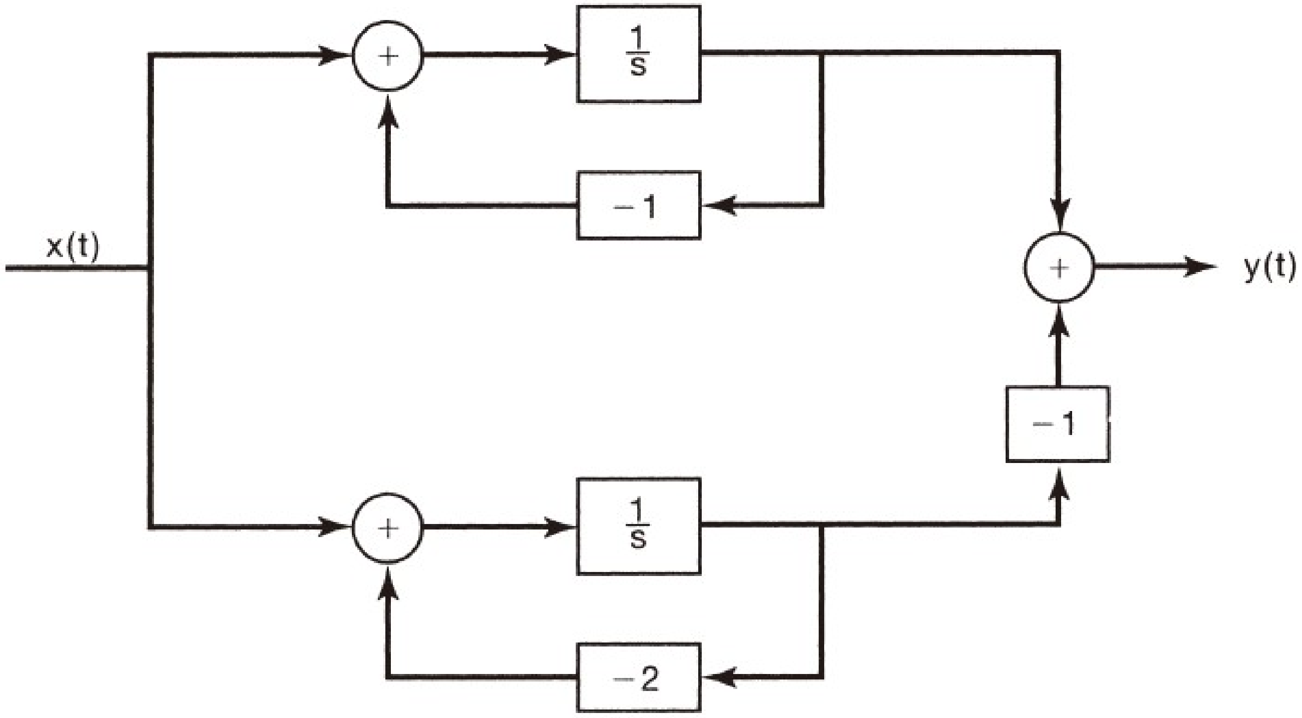
Figure 9.34 Block diagram representations for the system in Example 9.30: (a) direct form; (b) cascade form; (c) parallel form.



(b)

The block diagram in this figure is sometimes referred to as a **direct-form** representation, since the coefficients appearing in the diagram can be directly identified with the coefficients appearing in the system function or, equivalently, the differential equation. Other block diagram representations of practical importance also can be obtained after a modest amount of system function algebra. Specifically, *H(s)* in eq. (9.165) can be rewritten as

$$H(s)=\left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right), \quad \frac{dy(t)}{dt}+1y(t)=x(t),$$



(c)

Alternatively, by performing a partial-fraction expansion of H(s), we obtain

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2},$$

which leads to the parallel-form representation depicted in Figure 9.34(c).

9.9 The unilateral Laplace Transform

Analyzing causal system by LCCDE with nonzero initial conditions (not initial rest).

The unilateral Laplace transform of a continuous-time signal *x(t)* is defined as

$$\mathrm{X}(s) \overset{\Delta}{=} \int_{0^-}^{\infty} x(t)e^{-st}dt, \qquad (9.170)$$

單邊拉氏轉換的定義

9.9 The unilateral Laplace Transform

where the lower limit of integration, 0^- ,

積分下限為 0^- 。

Laplace transform:

$$x(t) \xleftrightarrow{UL} X(s) = UL\{x(t)\}. \qquad (9.171)$$

時域函數與其拉氏轉換的對應符號

Example 9.33

Consider next

$$x(t) = e^{-a(t+1)}u(t+1). \qquad (9.174)$$

The **bilateral** transform *X(s)* for this example can be obtained from Example 9.1 and the time-shifting property (Section 9.5.2):

$$X(s) = \frac{e^{s}}{s+a}, \qquad \Re e\{s\} > -a. \qquad (9.175)$$

Example 9.33

By contrast, the unilateral transform is

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} e^{-a(t+1)} u(t+1) e^{-st}\, dt \\ &= \int_{0^-}^{\infty} e^{-a} e^{-t(s+a)}\, dt \\ &= e^{-a} \frac{1}{s+a}, \qquad \Re e\{s\} > -a. \end{aligned} \tag{9.176}$$

$$X(s) = \frac{e^{s}}{s+a}, \qquad \Re e\{s\} > -a.$$

Example 9.33

Thus, in this example, the unilateral and bilateral Laplace transforms are clearly different. In fact, we should recognize *X(s)* as the bilateral transform not of *x(t)*, but of ***x(t)u(t)***, consistent with our earlier comment that the unilateral transform is the bilateral transform of a signal whose values for $t < 0^-$ have been set to zero.

Example 9.36

Consider the unilateral transform

$$X(s) = \frac{s^2 - 3}{s + 2}. \quad (9.181)$$

Since the degree of the numerator of X(s) is not strictly less than the degree of the denominator, we expand X(s) as

$$X(s) = A + Bs + \frac{C}{s + 2}. \quad (9.182)$$

Example 9.36

$$X(s) = A + Bs + \frac{C}{s+2}. \quad X(s) = \frac{s^2-3}{s+2}.$$

Equating eqs. (9.181) and (9.182), and clearing denominators, we obtain

$$s^2 - 3 = (A + Bs)(s+2) + C, \qquad (9.183)$$

and equating coefficients for each power of *s* yields

$$X(s) = -2 + s + \frac{1}{s+2}, \qquad (9.184)$$

Example 9.36

with an ROC of $\Re e\{s\} > -2$. Taking inverse transforms of each term results in

$$x(t) = -2\delta(t) + u_1(t) + e^{-2t}u(t) \qquad for \qquad t > 0^- .$$

(9.185)

9.9.2 Properties of the Unilateral Laplace Transform

Table 9.3 summarizes these properties. Note that we have not included a column explicitly identifying the ROC for the unilateral Laplace transform for each signal, since the ROC of any unilateral Laplace transform is always a **right-half plane**.

表9.3為單邊拉氏轉換的性質，因單邊拉式轉換的ROC必為右半平面，故ROC欄省略。

9.9.2 Properties of the Unilateral Laplace Transform

注意迴旋運算定理的大前提為對於t < 0，$x_1(t)$及$x_2(t)=0$

TABLE 9.3 PROPERTIES OF THE UNILATERAL LAPLACE TRANSFORM

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathfrak{X}(s)$ $\mathfrak{X}_1(s)$ $\mathfrak{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathfrak{X}_1(s) + b\mathfrak{X}_2(s)$
Shifting in the s-domain	$e^{s_0 t}x(t)$	$\mathfrak{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a}\mathfrak{X}\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$x^*(s)$
Convolution (assuming that $x_1(t)$ and $x_2(t)$ are identically zero for $t < 0$)	$x_1(t) * x_2(t)$	$\mathfrak{X}_1(s)\mathfrak{X}_2(s)$
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$s\mathfrak{X}(s) - x(0^-)$
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}\mathfrak{X}(s)$
Integration in the time domain	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\mathfrak{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} s\mathfrak{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathfrak{X}(s)$$

9.9.2 Properties of the Unilateral Laplace Transform

The convolution property for unilateral transforms also is quite similar to the corresponding property for bilateral transforms. This property states that if

$$x_1(t) = x_2(t) = 0 \quad \text{for all} \quad t < 0, \qquad (9.186)$$

then

$$x_1(t) * x_2(t) \xleftrightarrow{UL} X_1(s)X_2(s). \qquad (9.187)$$

9.9.2 Properties of the Unilateral Laplace Transform

An example of this is the integration property in Table 9.3: If *x(t)* = 0 for *t* < 0, then

$$\int_{0^-}^{t} x(\tau)d\tau = x(t) * u(t) \xleftrightarrow{UL} X(s)U(s) = \frac{1}{s}X(s)$$

(9.188)

Example 9.37

$$H(s) = \frac{\left\{\sum_{k=0}^{M} b_k s^k\right\}}{\left\{\sum_{k=0}^{N} a_k s^k\right\}}.$$

Suppose a causal LTI system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t), \quad (9.189)$$

together with the condition of initial rest. Using eq. (9.133), we find that the system function for this system is

$$\mathrm{H}(s) = \frac{1}{s^2 + 3s + 2}. \quad (9.190)$$

Example 9.37

Let the input to this system be $x(t) = \alpha u(t)$. In this case, the unilateral (and bilateral) Laplace transform of the output $y(t)$ is

$$\begin{aligned} \mathrm{Y}(s) = \mathrm{H}(s)\mathrm{X}(s) &= \frac{\alpha}{s(s+1)(s+2)} \\ &= \frac{\alpha/2}{s} - \frac{\alpha}{s+1} + \frac{\alpha/2}{s+2}. \end{aligned} \tag{9.191}$$

Applying Example 9.32 to each term of eq. (9.191) yields

$$y(t) = \alpha\left[\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right]u(t). \tag{9.192}$$

9.9.2 Properties of the Unilateral Laplace Transform

$$\int uv'dx = \int (uv)'dx - \int u'vdx$$

Consider a signal *x(t)* with unilateral Laplace transform *X(s)*. Then, **integrating by parts**, we find that the unilateral transform of *dx(t)/dt* is given by

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st} dt \qquad (9.193)$$

$$= s\mathrm{X}(s) - x(0^-).$$

(9.193)及(9.194)式微分性質中，單邊拉氏轉換比雙邊拉氏轉換多了有關初始值的項。

9.9.2 Properties of the Unilateral Laplace Transform

Similarly, a second application of this would yield the unilateral Laplace transform of $d^2x(t)/dt^2$, i.e.,

$$s^2\mathrm{X}(s) - sx(0^-) - x'(0^-), \qquad (9.194)$$

$$L(x'(t)) = \mathrm{X}'(s) = s\mathrm{X}(s) - x(0^-)$$

$$L(x''(t)) = s\mathrm{X}'(s) - x'(0^-) = s^2\mathrm{X}(s) - sx(0^-) - x'(0^-)$$

Example 9.38

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t) \quad (9.189)$$

單邊拉氏轉換主要用途之一為，求解已知初始條件下的線性常係數分方程式。

Consider the system characterized by the differential equation (9.189) with initial conditions

$$y(0^-)=\beta, \qquad y'(0^-)=\gamma. \qquad (9.195)$$

Example 9.38

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t)(9.189)$$

$$L(x'(t))=X'(s)=sX(s)-x(0^-)$$

$$L(x''(t))=s^2X(s)-sx(0^-)-x'(0^-)$$

Let *x(t) = αu(t).* Then, applying the unilateral transform to both sides of eq. (9.189), we obtain

$$s^2Y(s)-\beta s-\gamma+3sY(s)-3\beta+2Y(s)=\frac{\alpha}{s}, \qquad (9.196)$$

or

$$Y(s)=\frac{\beta(s+3)}{(s+1)(s+2)}+\frac{\gamma}{(s+1)(s+2)}+\frac{\alpha}{s(s+1)(s+2)}, \qquad (9.197)$$

where *Y(s)* is the unilateral Laplace transform of *y(t).*

$$Y(s)=\frac{\beta(s+3)}{(s+1)(s+2)}+\frac{\gamma}{(s+1)(s+2)}+\frac{\alpha}{s(s+1)(s+2)},$$

$$\mathrm{Y}(s)=\mathrm{H}(s)\mathrm{X}(s)=\frac{\alpha}{s(s+1)(s+2)}$$

$$=\frac{\alpha/2}{s}-\frac{\alpha}{s+1}+\frac{\alpha/2}{s+2}.$$

Referring to Example9.37 and, in particular, to eq. (9.191), we see that the last term on the right-hand side of eq. (9.197) is precisely the unilateral Laplace transform of the response of the system when the initial conditions in eq. (9.195) are both zero ($\beta = \gamma = 0$). That is, the last term represents the response of the causal LTI system described by eq. (9.189) and the condition of initial rest. This response is often referred to as the ***zero-state response***—i.e., the response when the initial state (the set of initial conditions in eq. (9.195)) is zero.

$$Y(s) = \frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)} + \frac{\alpha}{s(s+1)(s+2)},$$

$$\mathrm{Y}(s) = \mathrm{H}(s)\mathrm{X}(s) = \frac{\alpha}{s(s+1)(s+2)}$$

$$= \frac{\alpha/2}{s} - \frac{\alpha}{s+1} + \frac{\alpha/2}{s+2}.$$

An analogous interpretation applies to the first two terms on the right-hand side of eq. (9.197). These terms represent the unilateral transform of the response of the system when the input is zero (α =0). This response is commonly referred to as the ***zero-input*** *response*. Note that the zero-input response is a linear function of the values of the initial conditions (e.g., doubling the values of both β and γ doubles the zero-input response).

Example 9.38

For example, if $\alpha = 2$, $\beta = 3$, and $\gamma = -5$, then performing a partial-fraction expansion for eq. (9.197) we find that

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}. \qquad (9.198)$$

Applying Example 9.32 to each term then yields

$$y(t) = \left[1 - e^{-t} + 3e^{-2t}\right]u(t) \quad \text{for } t > 0. \qquad (9.199)$$

9.10 Summary

- Laplace Transform and its relation with FT
- ROC and properties
- Geometric Evaluation of Frequency Response
- Properties of LT
- Causality and Stability from ROC
- Solving LCCDE using LT
- Unilateral LT handling non initial rest condition