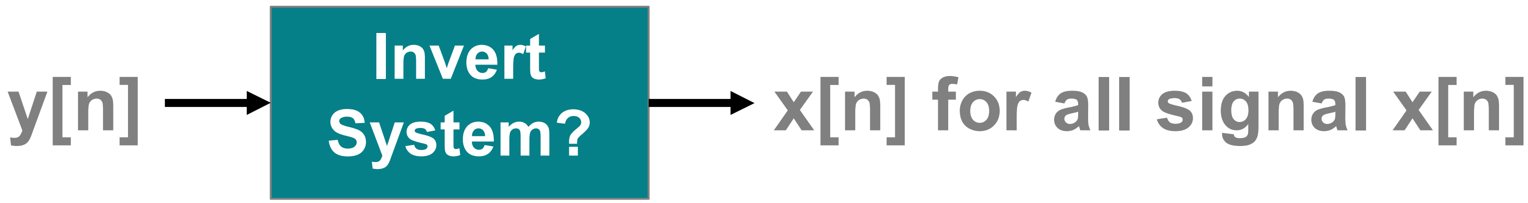
Chapter 7
Sampling

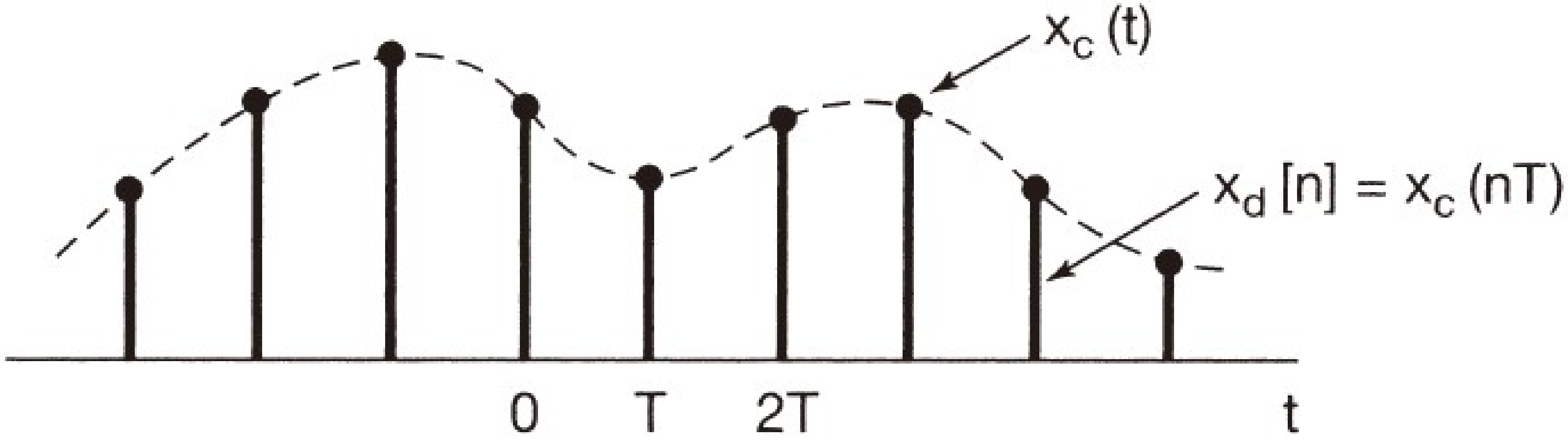
Min Sun

Is this system invertible?

- $y[n]=x[2n]$ (homework 1)



- $x_d[n]=x_c(Tn)$



7.0 Introduction

Under **certain conditions**, a continuous-time signal can be completely represented by and recoverable form knowledge of its values, or **samples, at points equally spaced in time**. This somewhat surprising property follows from a basic result that is referred to as the **sampling theorem**.

Much of the importance of the sampling theorem also lies in its role as a bridge between continuous-time signals and discrete-time signals.

In many contexts, processing **discrete-time signals is ore flexible and is often preferable**. This is due to the dramatic development of digital technology over the past few decades, resulting in inexpensive, lightweight, programmable, easily reproducible discrete-time system.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

In general, in the absence of any additional conditions or information, we would not expect that a signal could be uniquely specified by a sequence of equally spaced samples.

For example, $x_1(t)$, $x_2(t)$, and $x_3(t)$ all have the same samples

$$x_1(kT) = x_2(kT) = x_3(kT).$$

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

$$x_1(kT) = x_2(kT) = x_3(kT).$$

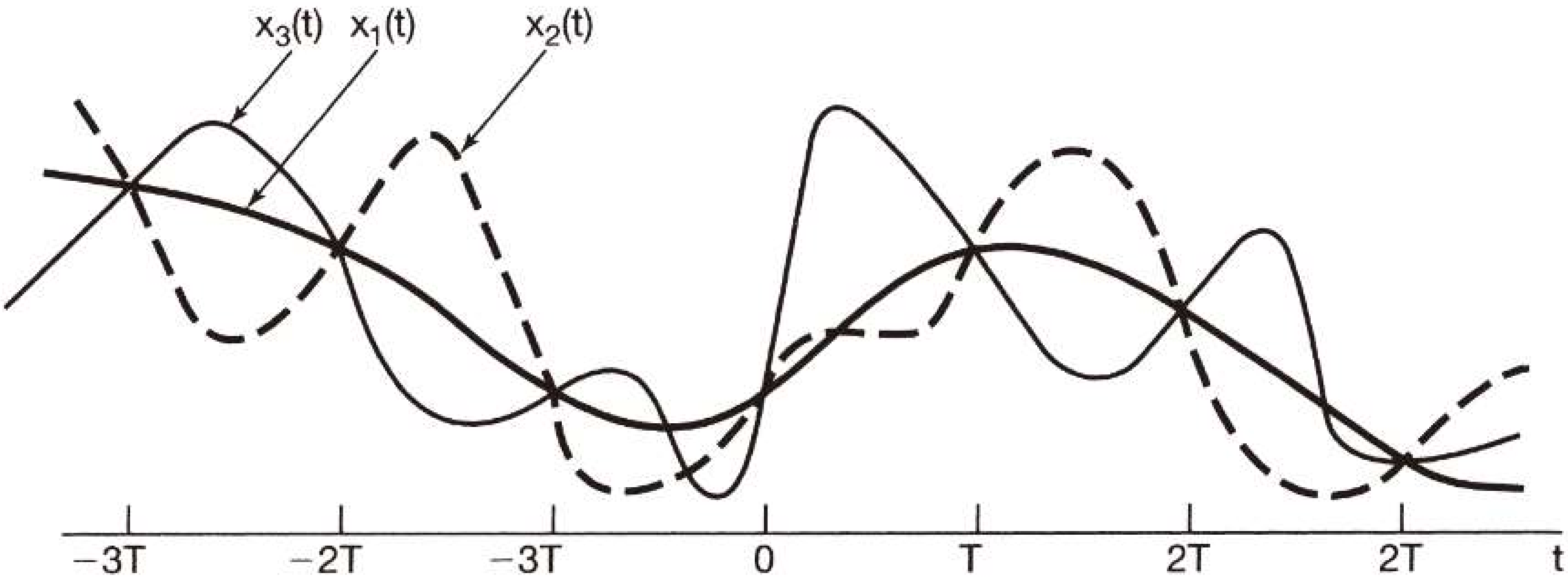


Figure 7.1 Three continuous-time signals with identical values at integer multiples of *T*.

7.1.1 Impulse-Train Sampling

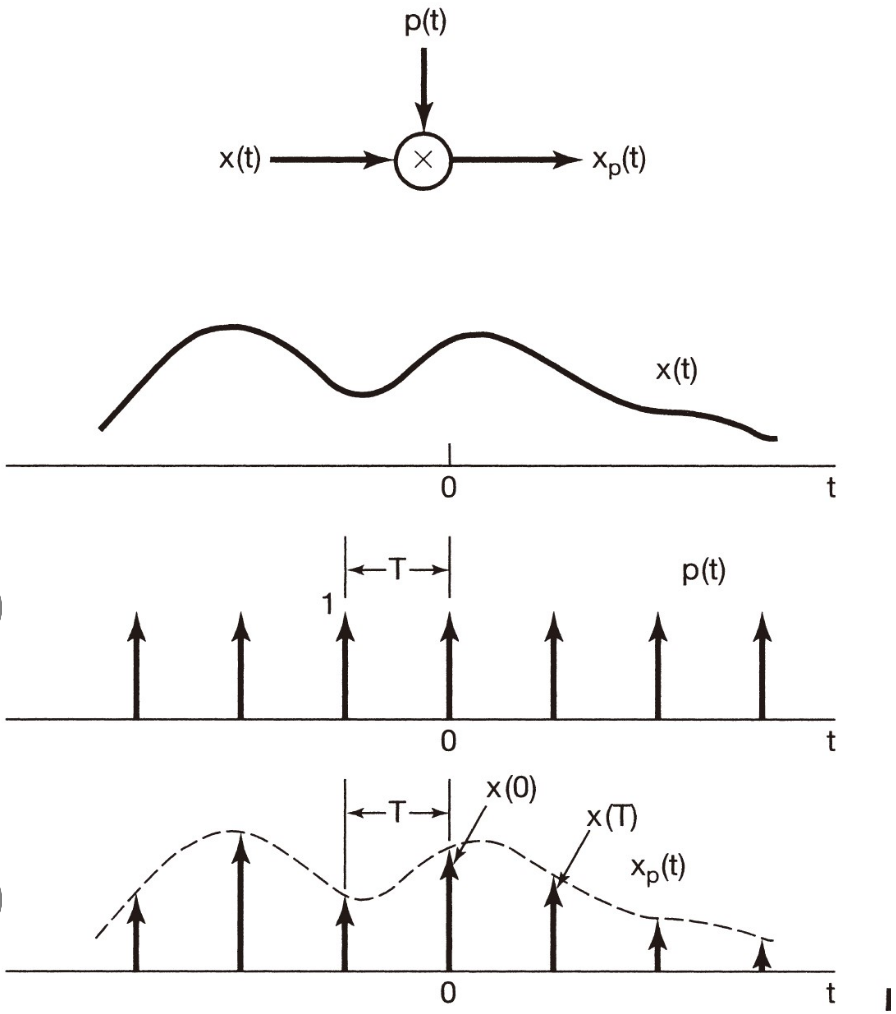
$x_d[n]=x(Tn)$

The periodic impulse train *p(t)* is referred to as the sampling function, the period *T* as the sampling period, and the fundamental frequency of *p(t)*, $\omega_s = 2\pi / T$, as the *sampling frequency*. In the time domain,

$$x_p(t) = x(t)p(t), \quad (7.1)$$

where

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT). \quad (7.2)$$



7.1.1 Impulse-Train Sampling

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT).$$

the impulses equal to the samples of *x(t)* at intervals spaced by *T*; that is,

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT). \qquad (7.3)$$

From the multiplication property (Section 4.5), we know that

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)P(j(\omega-\theta))d\theta. \qquad (7.4)$$

$\Updownarrow$

$$x_p(t) = x(t)p(t),$$

The FT of $p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT).$

- FS of $p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT).$

$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}p(t)e^{-jk\omega_s t}=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\omega_s t}=\frac{1}{T}$$

- In Example 4.6 we know

$$\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_s t}\leftrightarrow\sum_{k=-\infty}^{\infty}2\pi a_k\delta(j(\omega-k\omega_s))$$

- Therefore,

$$P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\big(j(\omega-k\omega_s)\big)(7.5)$$

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s)$$

$$X_p(j\omega) = \frac{1}{2\pi}X(j\omega) * P(j\omega)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}X(j\omega) * \delta(j(\omega - k\omega_s))$$

Since convolution with an impulse simply shifts a signal

$$X(j\omega) * \delta(j(\omega - \omega_s)) = X(j(\omega - \omega_s))$$

It follows that

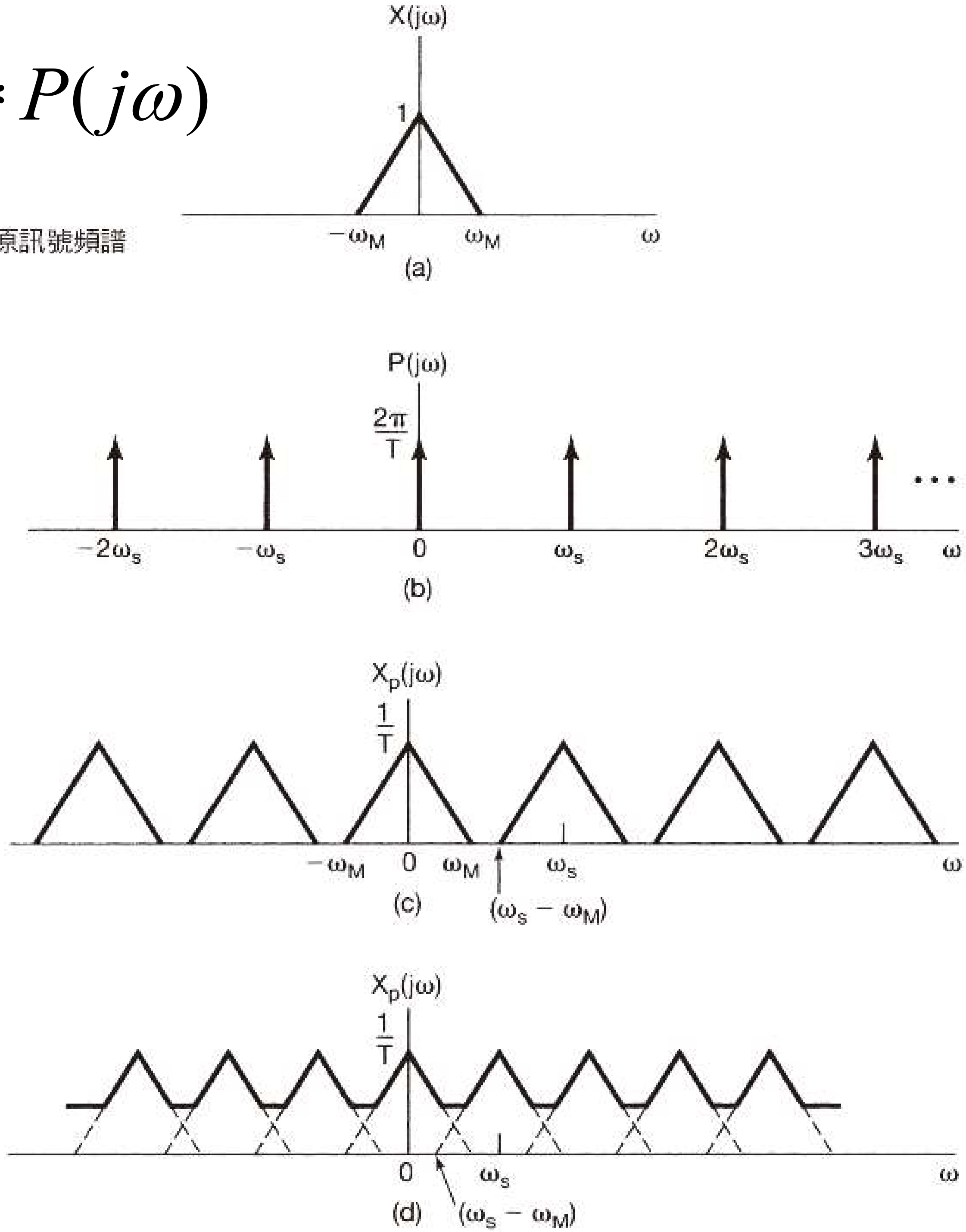
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty}X(j(\omega - k\omega_s)). \qquad (7.6)$$

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

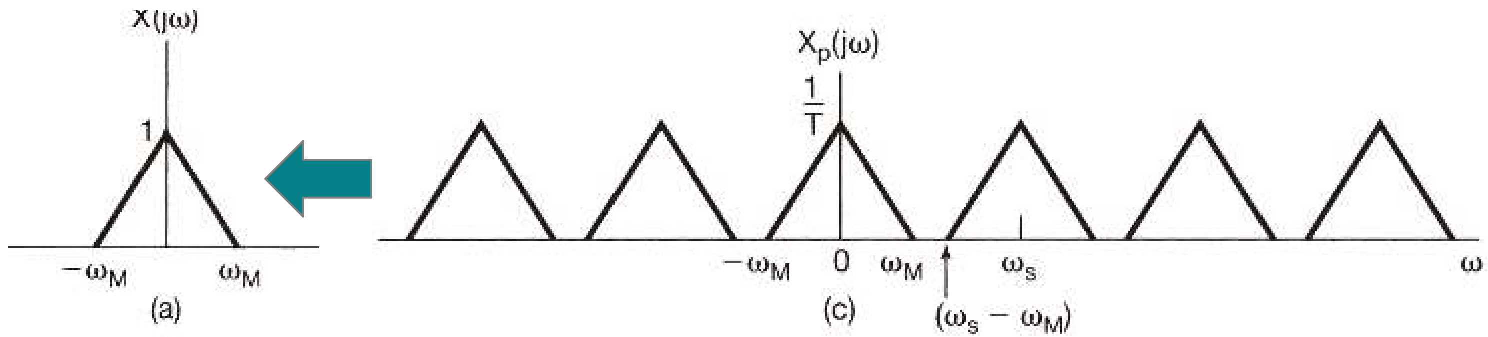
$$\frac{1}{T} \sum_{k=-\infty}^{+\infty} X\big(j(\omega - k\omega_s)\big).$$

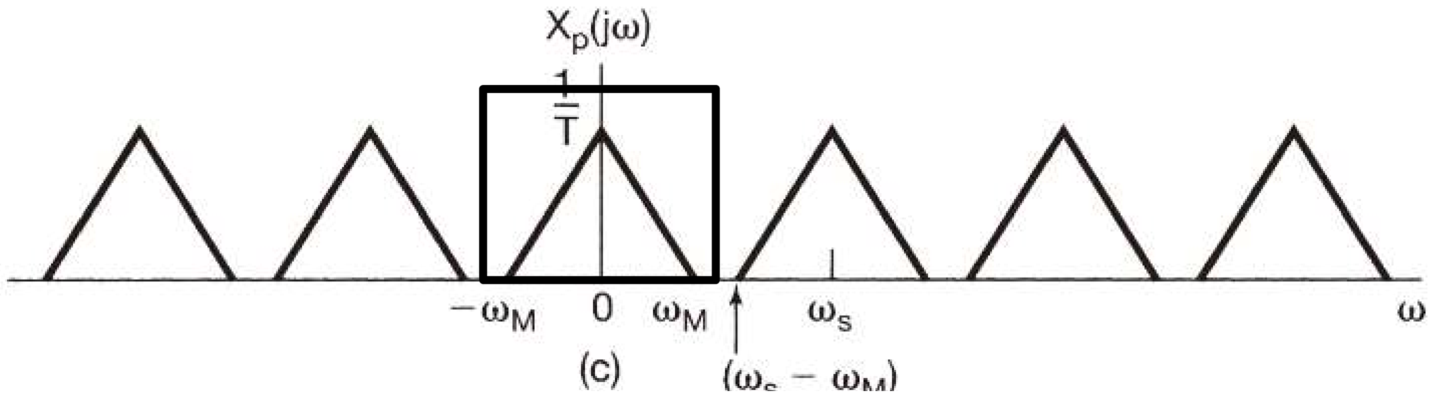
In Figure 7.3 (c), $\omega_M < (\omega_s - \omega_M)$ or equivalently, $\omega_s > 2\omega_M$, and thus there is no overlap between the shifted replicas of *X(jω),* whereas in Figure 7.3(d), with $\omega_s < 2\omega_M$, there is overlap.

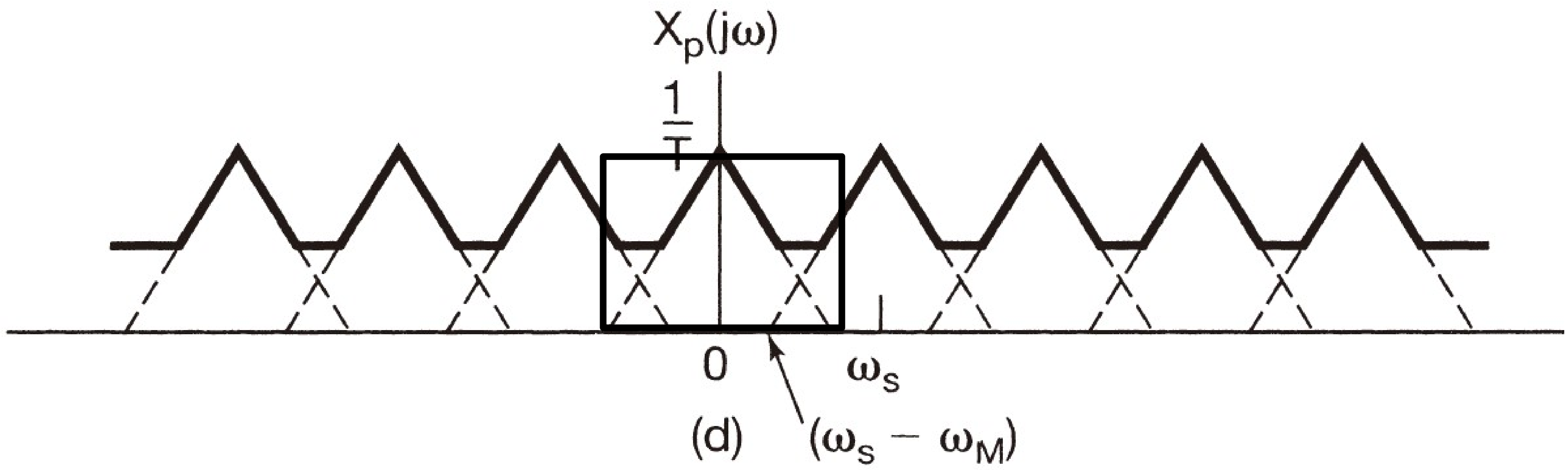
原訊號頻譜



How to Recover?







7.1 The Sampling Theorem

Sampling Theorem:

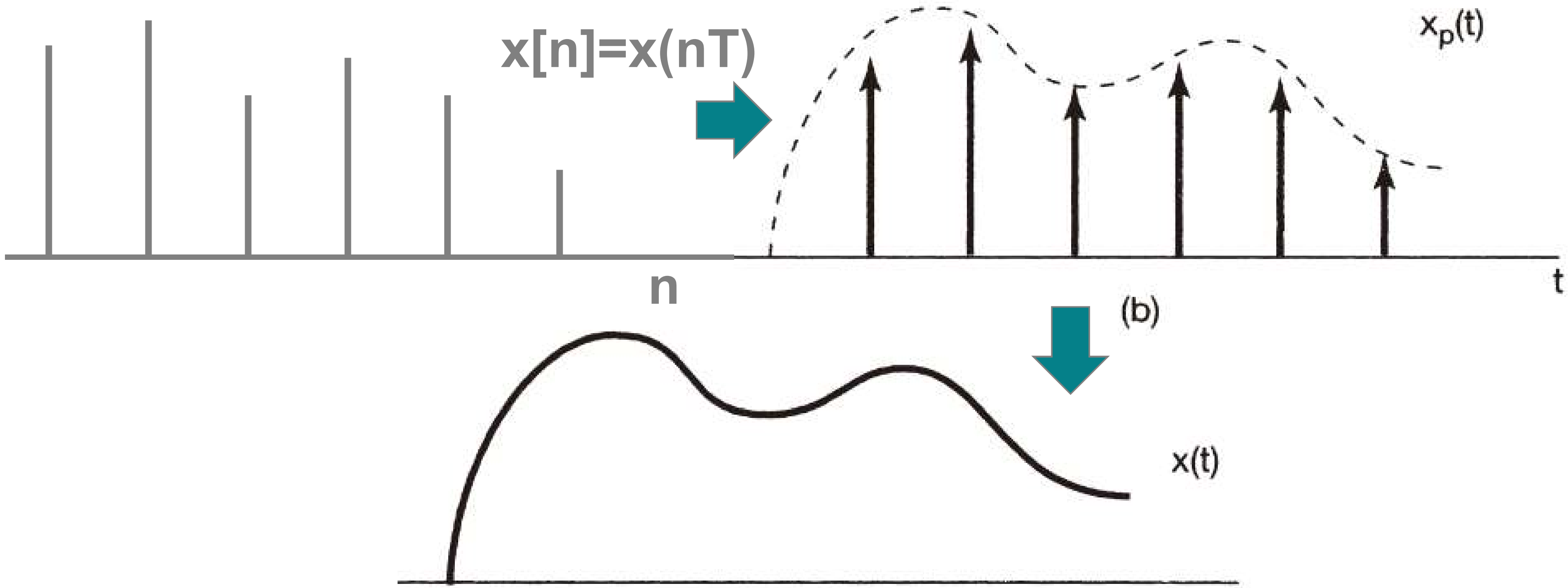
Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M,$$

where

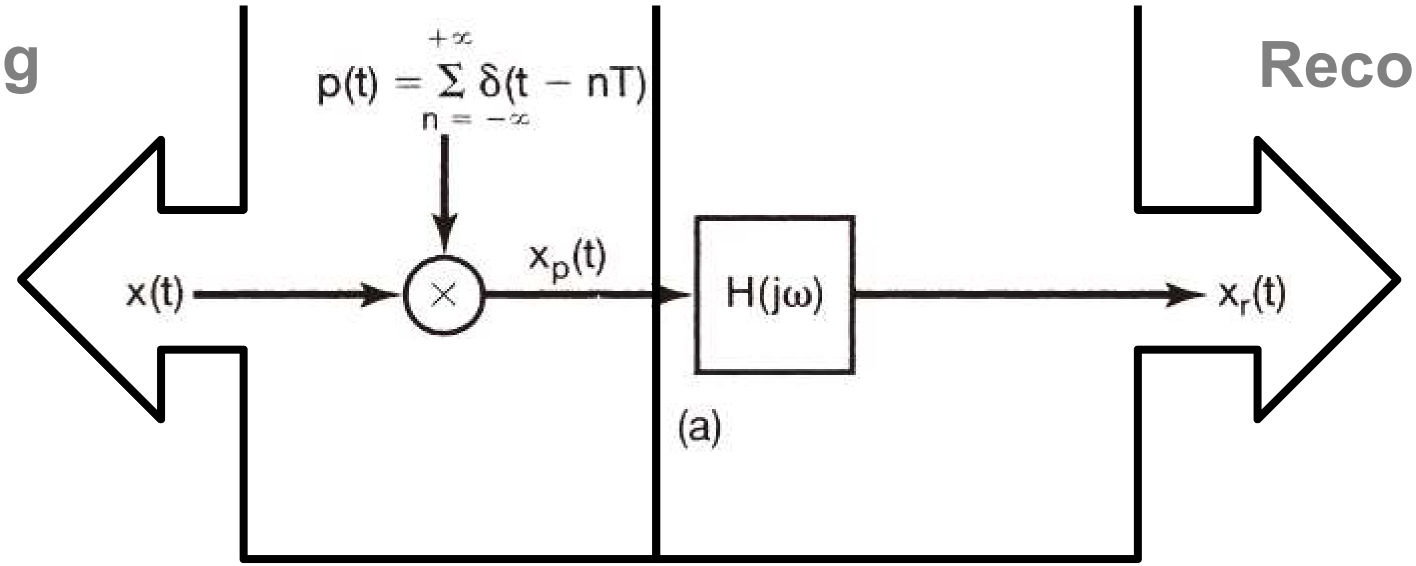
$$\omega_s = \frac{2\pi}{T}.$$

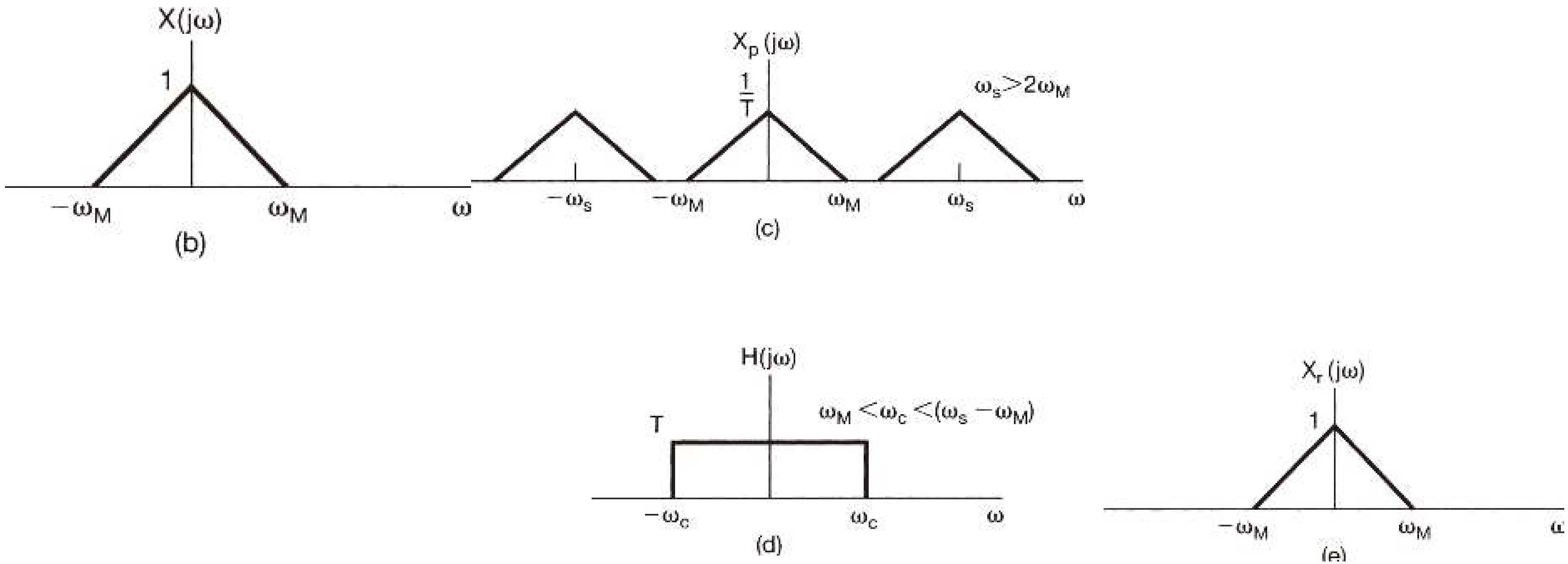
Given these samples, we can reconstruct *x(t)* by generating a periodic **impulse train** in which successive impulses have amplitudes that are successive **sample** values. This impulse train is then processed through an ideal lowpass filter with gain *T* and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal *x(t)*.



Sampling

Reconstruct





7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

The frequency $2\omega_M$, which, under the sampling theorem, must be exceeded by the sampling frequency, is commonly referred to as *Nyquist rate*.

Note that ideal filter will be approximated by a nonideal filter in practice. Obviously, any such approximation in the lowpass filtering stage will lead to some discrepancy between $x(t)$ and $x_r(t)$ in Figure 7.4 or, equivalently, between $X(j\omega)$ and $X_r(j\omega)$. We will ignore this in some of our discussion later.

7.1.2 Sampling with a Zero-order Hold

In practice, narrow & large-amplitude pulses, which approximate impulses, are hard to generate and transmit. It is much easier to generate "zero-order hold" signals as shown below.

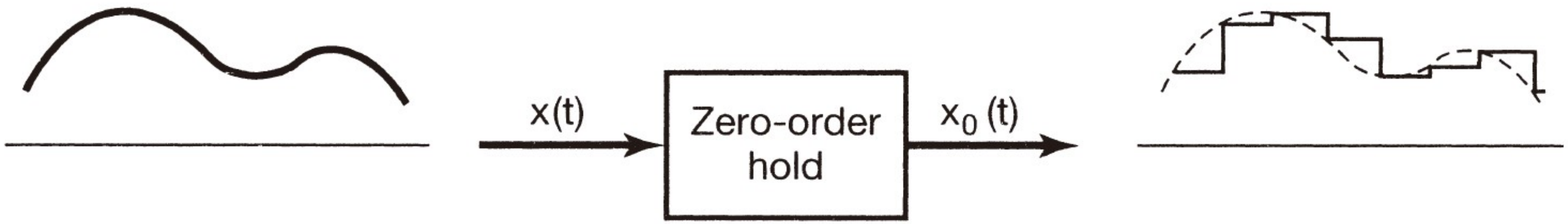
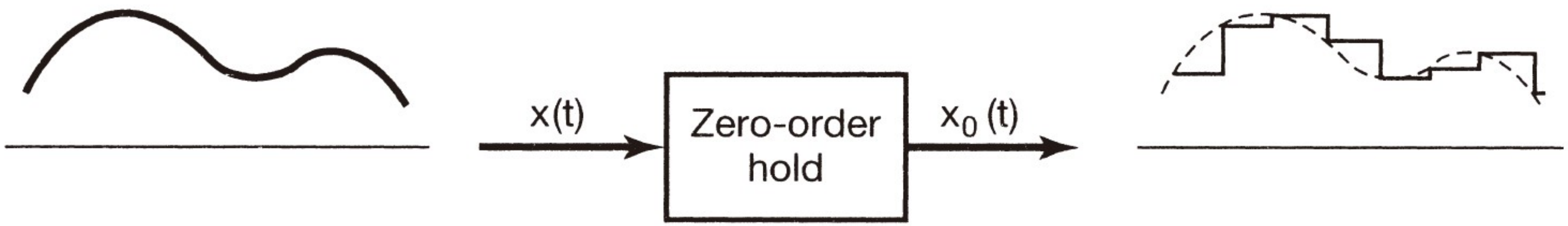
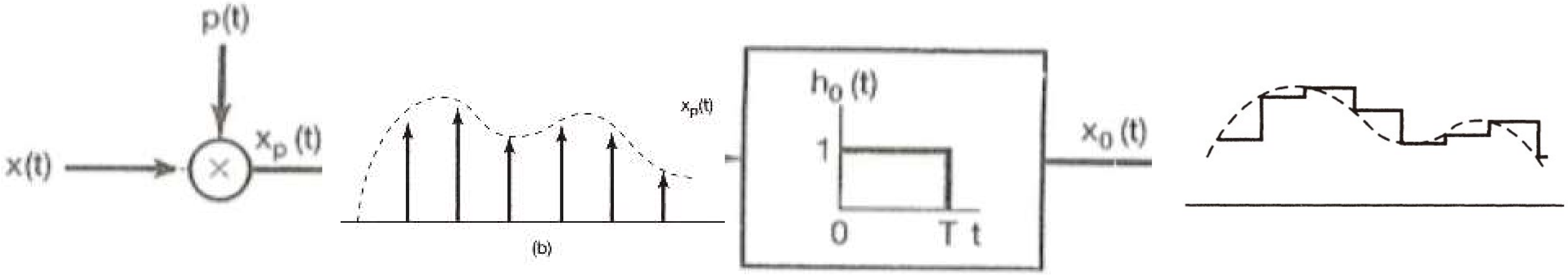
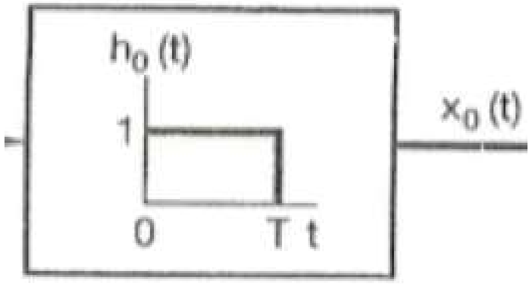


Figure 7.5 Sampling utilizing a zero-order hold.

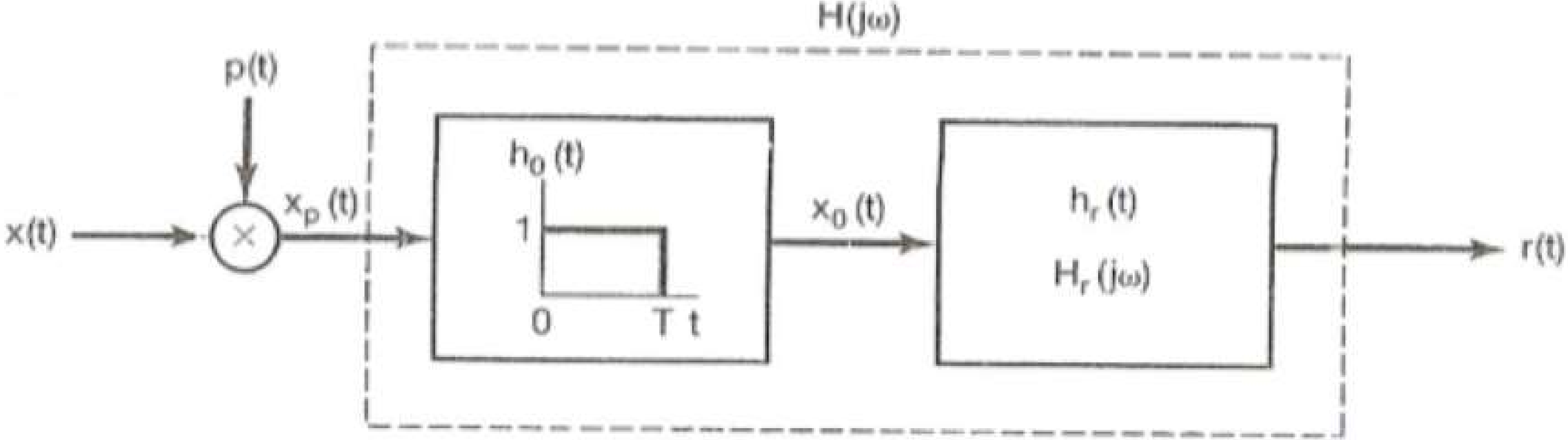
The reconstruction of *x(t)* from the output of a zero-order hold can again be carried out by lowpass filtering. However, the lowpass filter no longer requires a constant gain in the passband.







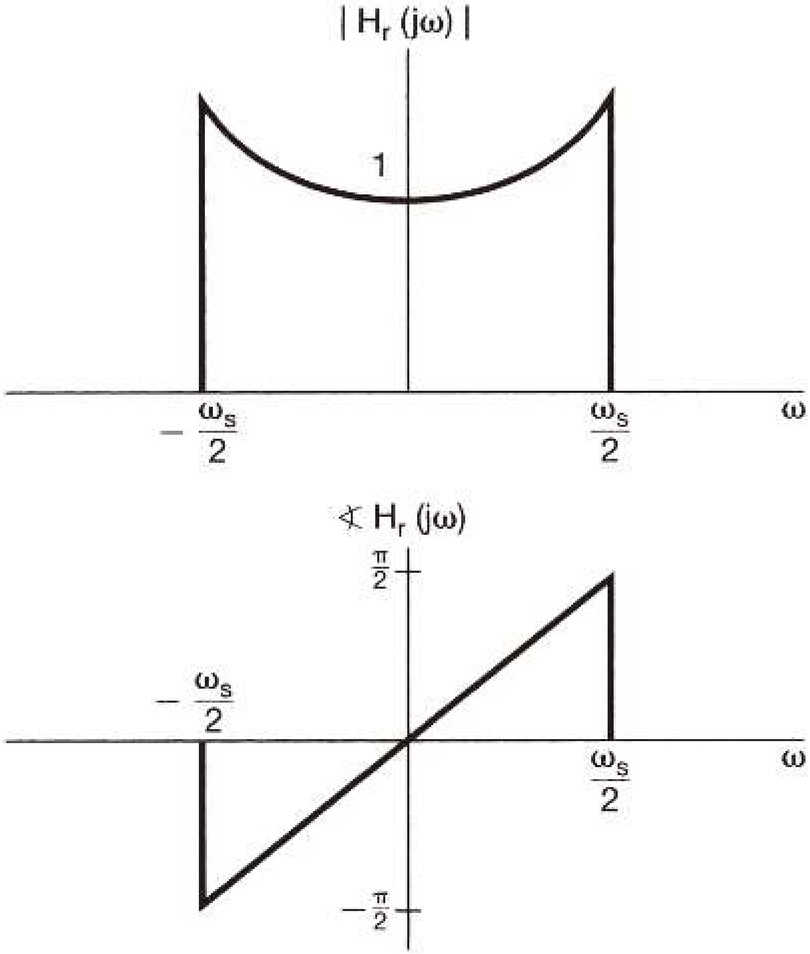
$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin\left(\omega T/2\right)}{\omega}\right] \quad (7.7)$$



this requires that

$$H_r(j\omega) = \frac{e^{j\omega T/2} H(j\omega)}{\dfrac{2\sin(\omega T/2)}{\omega}} \quad (7.8)$$

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

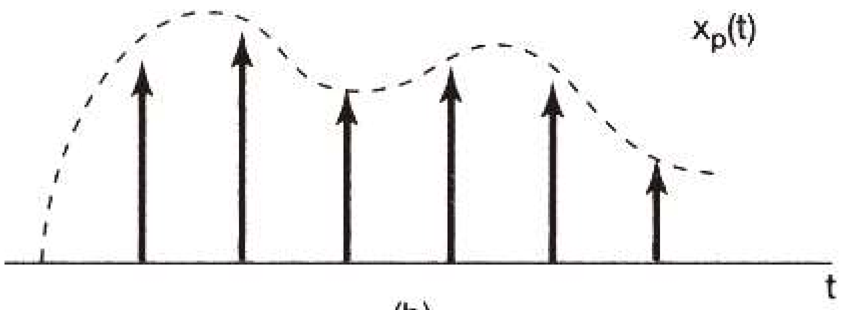


$$H_r(j\omega)=\frac{e^{j\omega T/2}H(j\omega)}{\frac{2\sin(\omega T/2)}{\omega}}$$

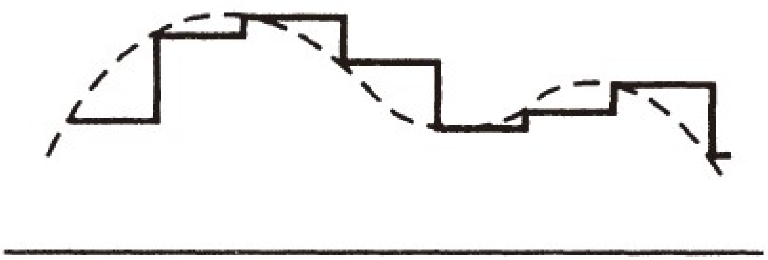
Figure 7.8 Magnitude and phase for the reconstruction filter for a zero-order hold.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

In fact, zero-order hold is an adequate approximation of the original signal by itself, without any additional lowpass filter. It can be consider as a very coarse interpolation between sample values. Next, we will discuss other forms of interpolations.

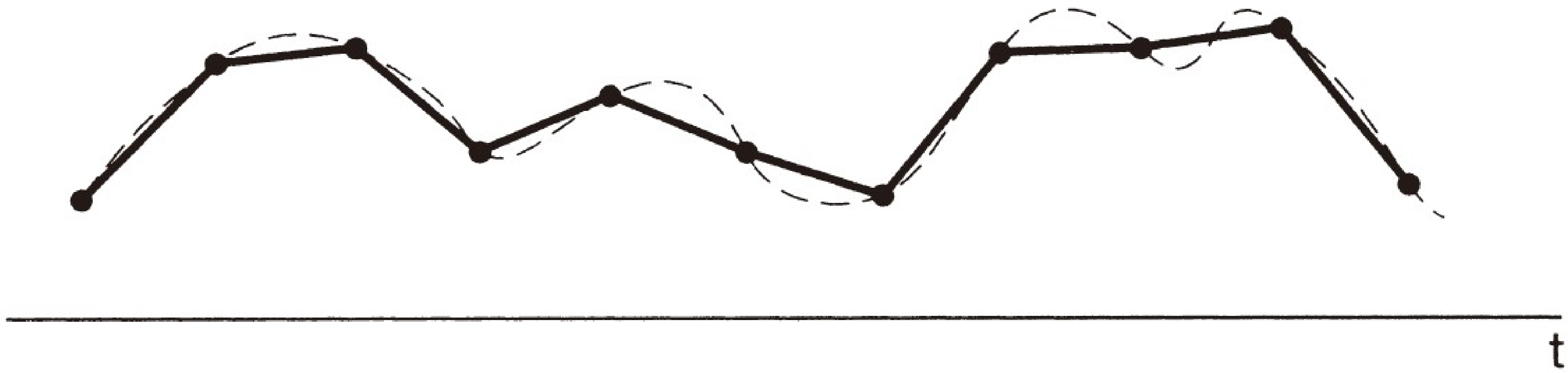


(b)



7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

Linear interpolation



7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT).$$

Sampling as interpolation.

The interpretation of the reconstruction of *x(t)* as a process of interpolation becomes evident when we consider the effect in the time domain of the lowpass filter in Figure 7.4. In particular, the output is

$$x_r(t) = x_p(t) * h(t)$$

or, with $x_p(t)$ given by eq. (7.3),

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT) \qquad (7.9)$$

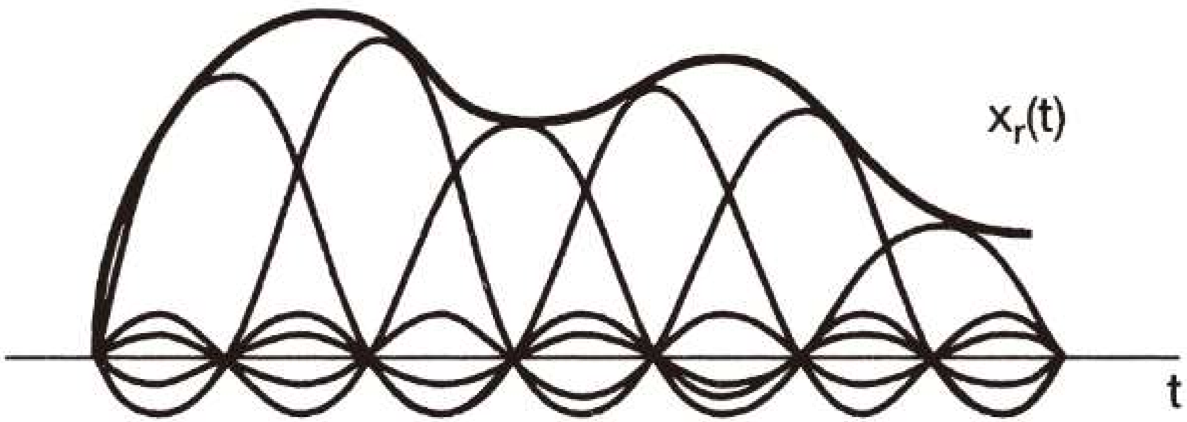
For the ideal lowpass filter *H(jω)* in Figure 7.4,

$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t}, \qquad (7.10)$$

so that

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{\omega_c T}{\pi} \frac{\sin\left(\omega_c \left(t - nT\right)\right)}{\omega_c \left(t - nT\right)}. \qquad (7.11)$$

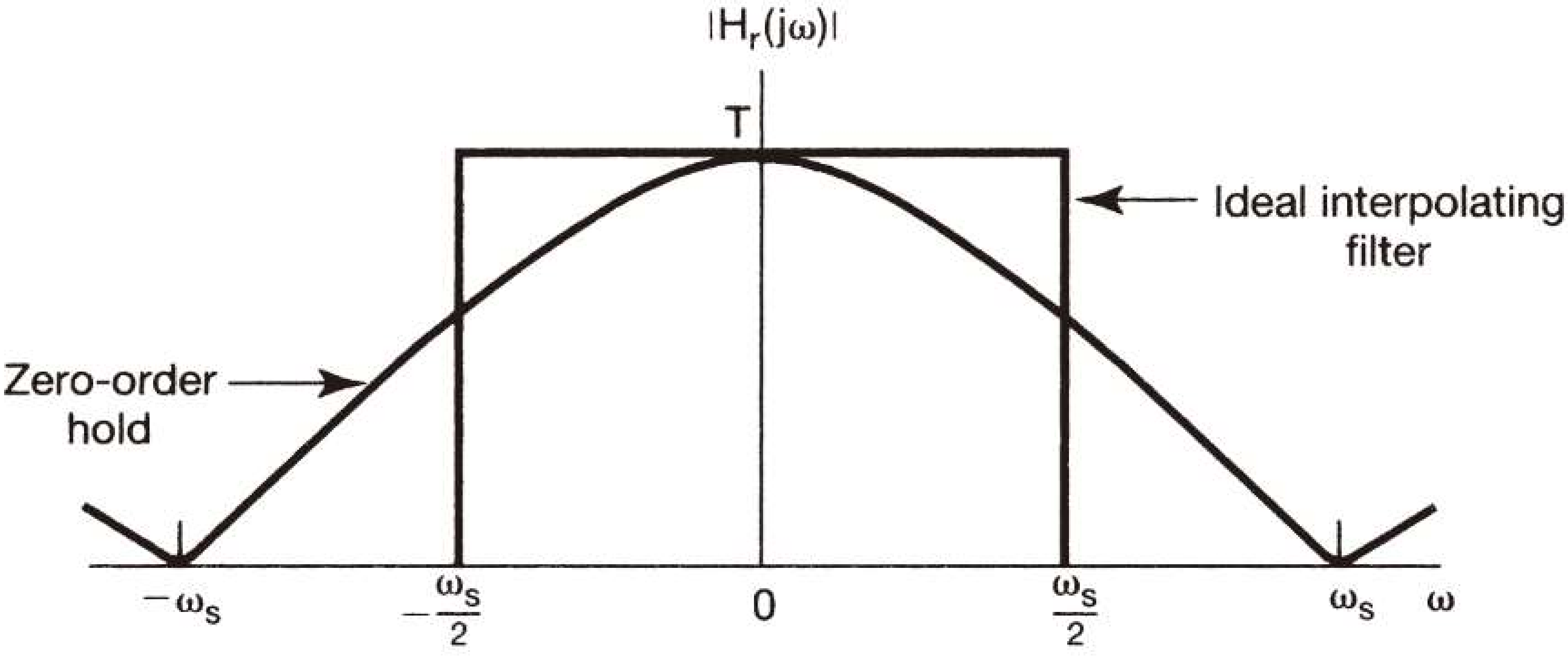
this is referred to band-limited interpolation.



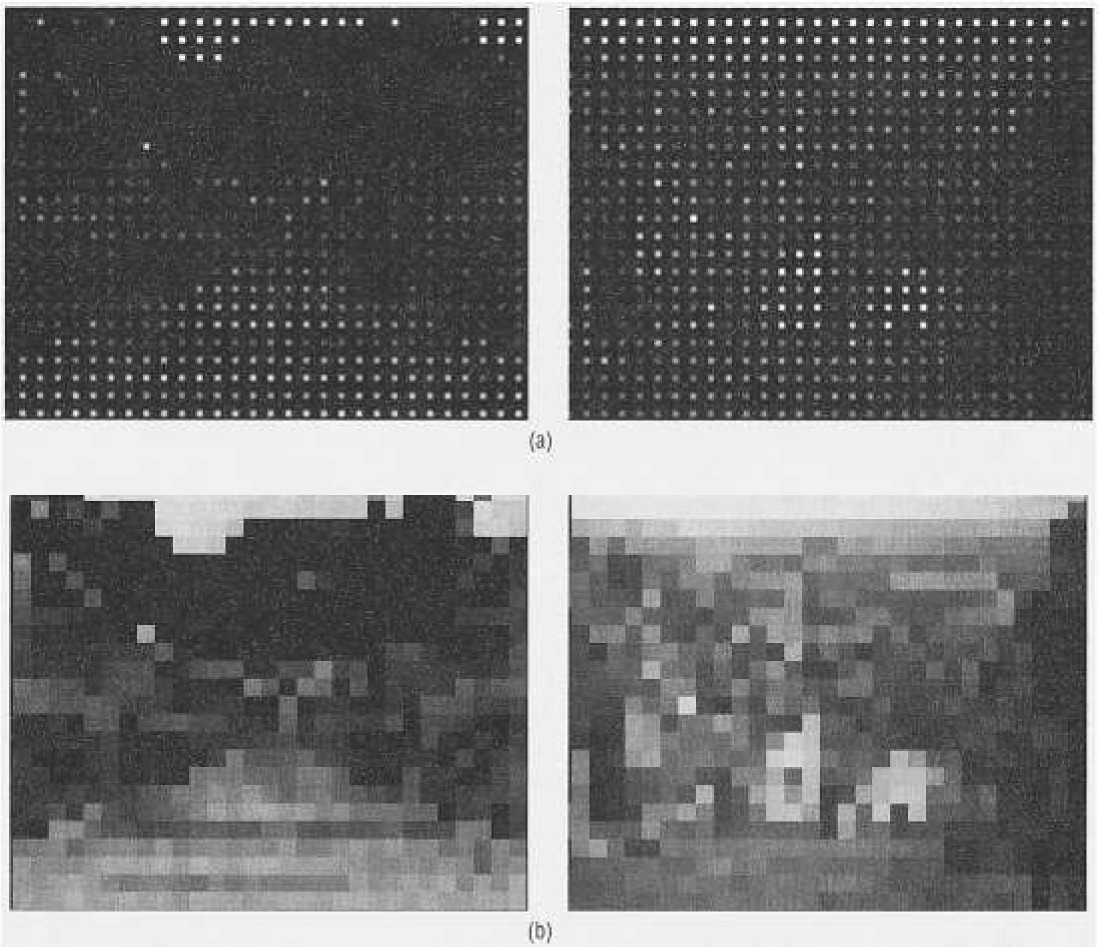
(c)

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right] \quad (7.7)$$



7.2 Reconstruction of A Signal From ITS Samples Using Interpolation



以脈衝取樣所得的影像。

加入零階保持處理的影像，呈現馬賽克效應（即產生如磁磚的拼貼效果）。因人類視覺系統有低通特性，馬賽克的不連續點會被平滑化。

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

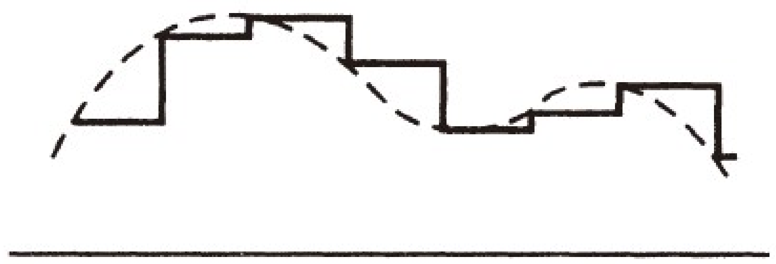
(c)

Figure 7.12 (a) The original pictures of Figures 6.2(a) and (g) with impulse sampling; (b) zero-order hold applied to the pictures in (a). The visual system naturally introduces lowpass filtering with a cutoff frequency that decreases with distance. Thus, when viewed at a distance, the discontinuities in the mosaic in Figure 7.12(b) are smoothed; (c) result of applying a zero-order hold after impulse sampling with one-fourth the horizontal and vertical spacing used in (a) and (b).

將脈衝取樣的水平及垂直間距改為 (b) 圖的 1/4 倍的結果。

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

If the crude interpolation provided by the zero-order hold is insufficient, we can use a variety of smoother interpolation strategies, some of which are known collectively as **higher order holds**. In particular, the zero-order hold produces an output signal, as in Figure 7.5, that is discontinuous.

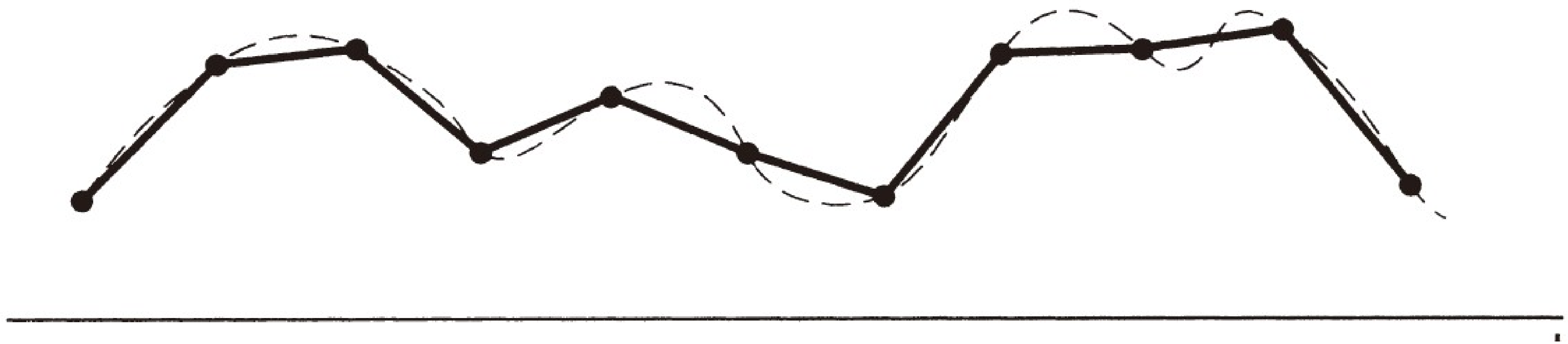


7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

$$H(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2. \qquad (7.12)$$

1st-order hold = linear interpolation.

The impulse response of a linear interpolation is a triangular function

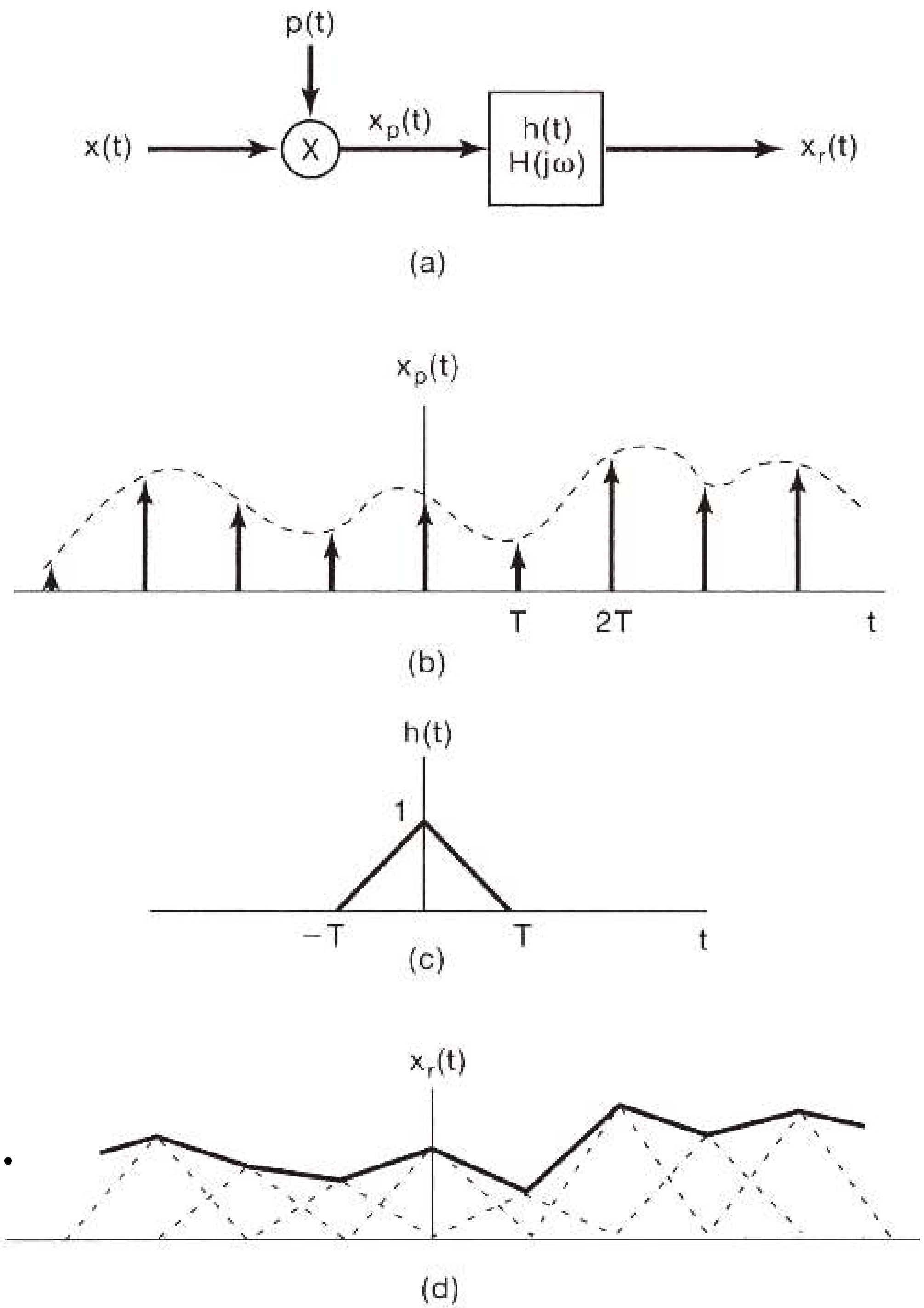


7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

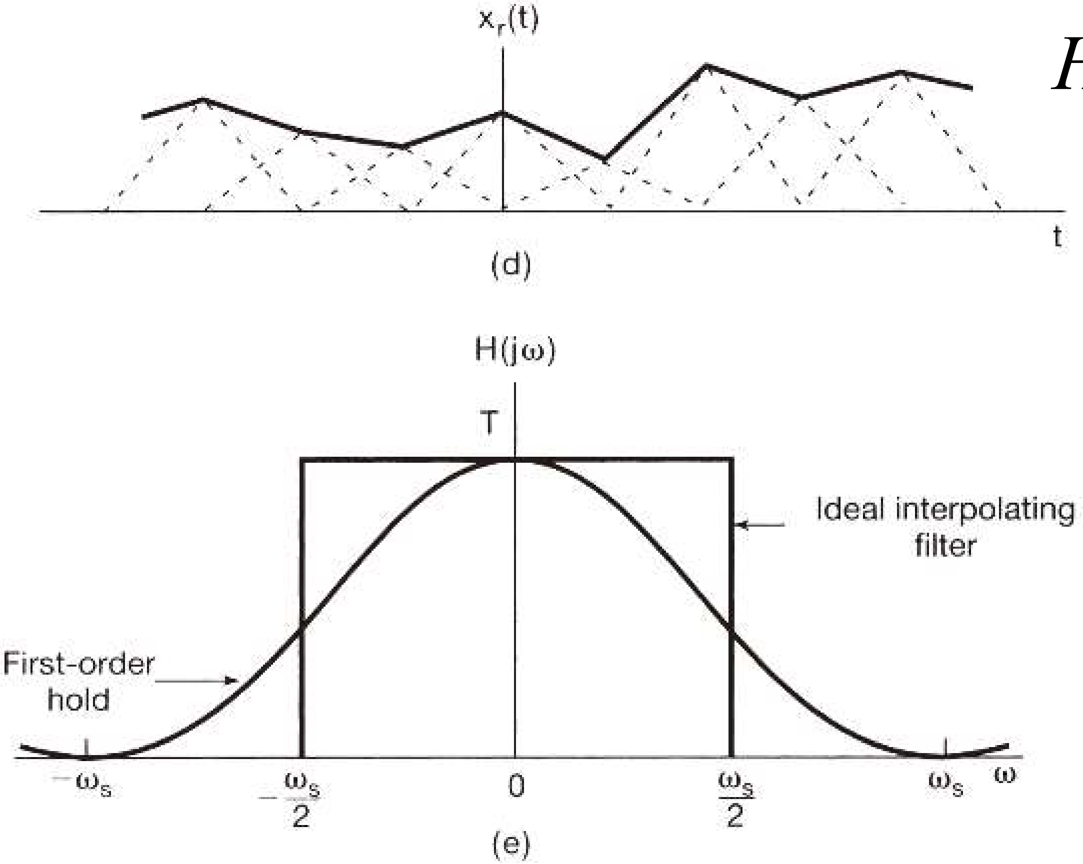
- 1^{st}-order hold = linear interpolation.

  The impulse response of a linear interpolation is a **triangular function** h(t).

$$H(j\omega)=\frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2.$$

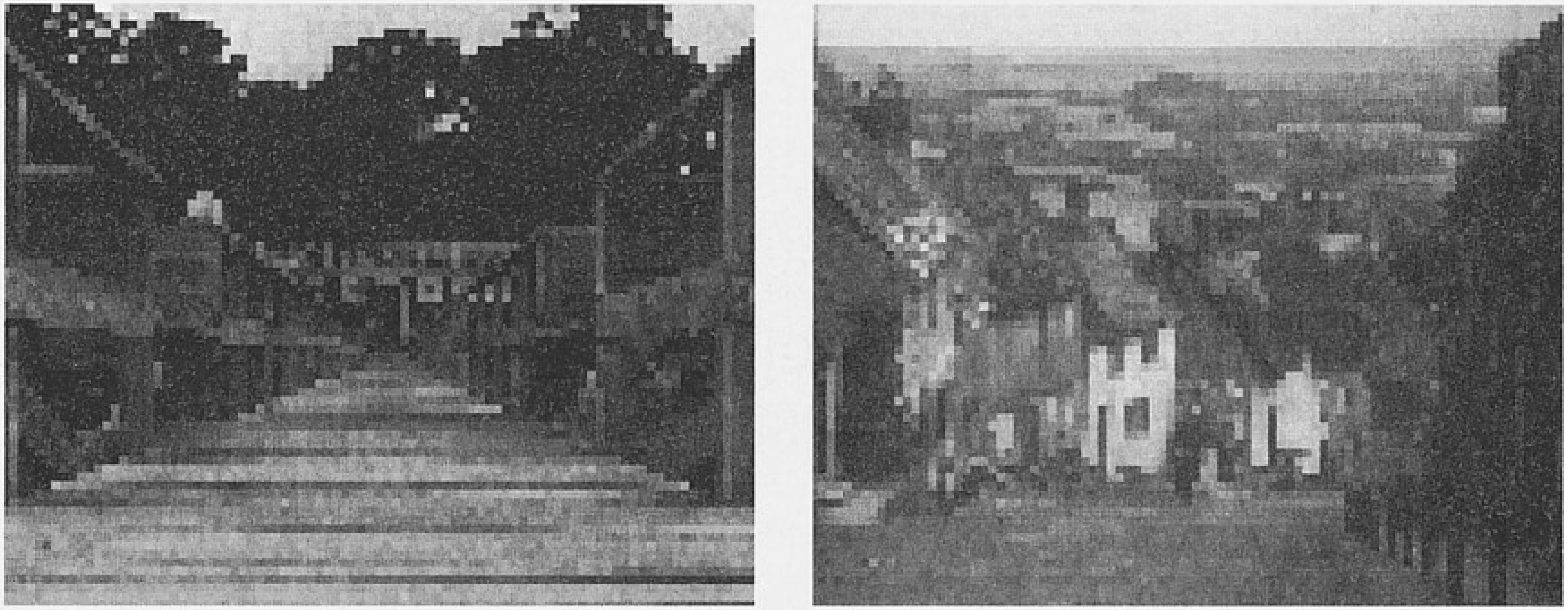


7.2 Reconstruction of A Signal From ITS Samples Using Interpolation



$$H(j\omega)=\frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2.$$

Figure 7.13 *Continued* (d) first-order hold applied to the sampled signal; (e) comparison of transfer function of ideal interpolating filter and first-order hold.



(c)

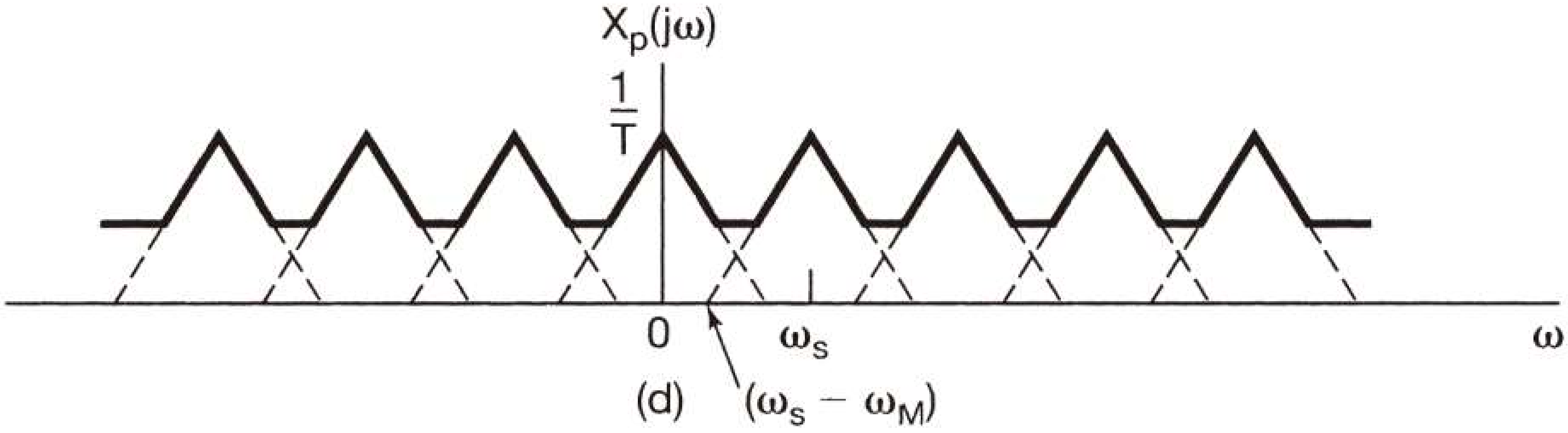
(a)

(b)

7.3 The Effect of Undersampling: Aliasing

When $\omega_s < 2\omega_M$, *X(jω),* the spectrum of *x(t),* is no longer replicated in $X_p(j\omega)$ and thus is no longer recoverable by lowpass filtering.

This effect, in which the individual terms overlap, is referred to as **aliasing**.



(d)

7.3 The Effect of Undersampling: Aliasing

Clearly, if the system of Figure 7.4 is applied to a signal with $\omega_s < 2\omega_M$, the reconstructed signal $x_r(t)$ will no longer be equal to *x(t)*.

However, as explored in Problem 7.25,

$$x_r(nT) = x(nT), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{7.13}$$

for any choice of ω_s.

7.3 The Effect of Undersampling: Aliasing

Some insight into the relationship between x(t) and $x_r(t)$ when $\omega_s < 2\omega_M$ is provided by considering in more detail the comparatively simple case of a sinusoidal signal. Thus, let

$$x(t) = \cos\omega_0 t, \qquad (7.14)$$

The lowpass filtered output $x_r(t)$ is given as follows:

$$(a) \qquad \omega_0 = \frac{\omega_s}{6}\ ; \qquad x_r(t) = \cos\omega_0 t = x(t)$$

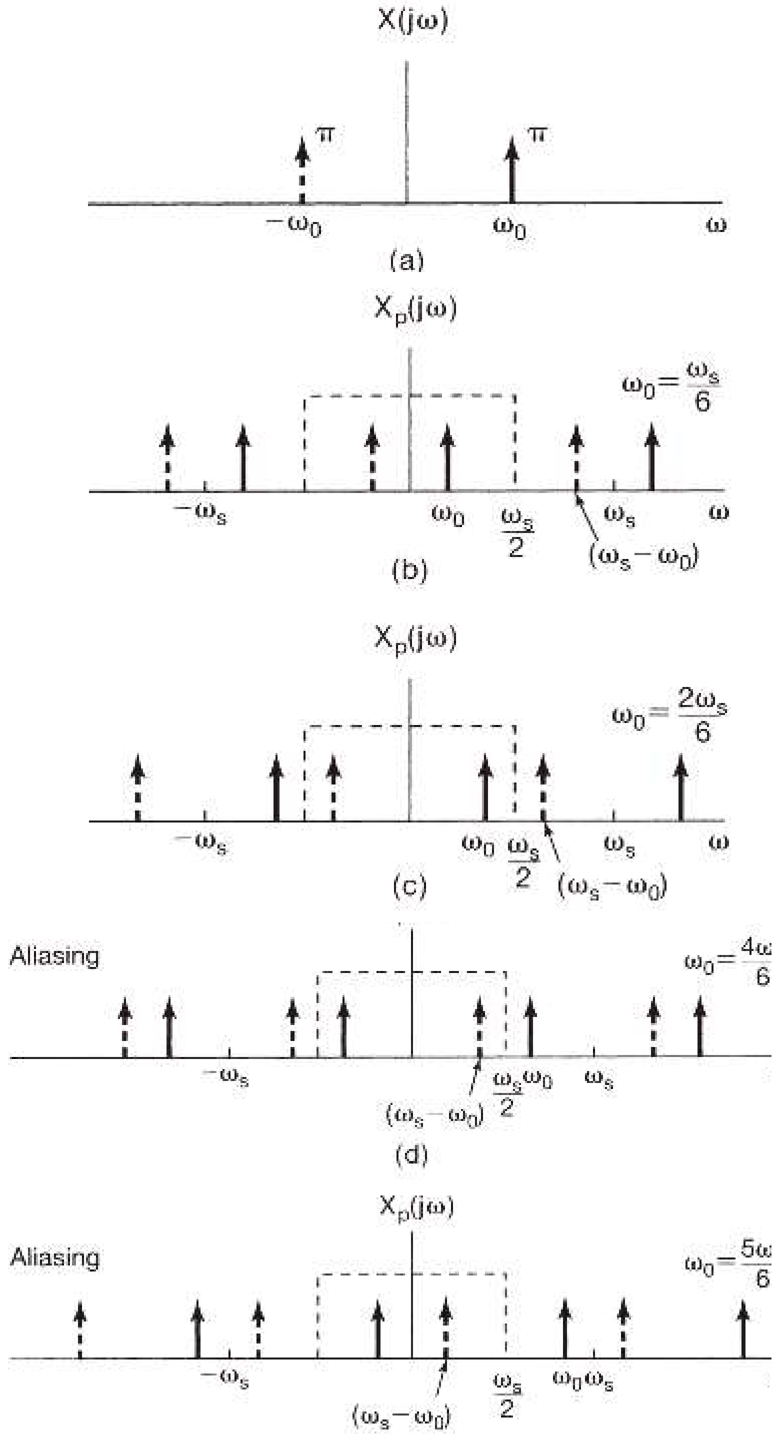
$$(b) \qquad \omega_0 = \frac{2\omega_s}{6}; \qquad x_r(t) = \cos\omega_0 t = x(t)$$

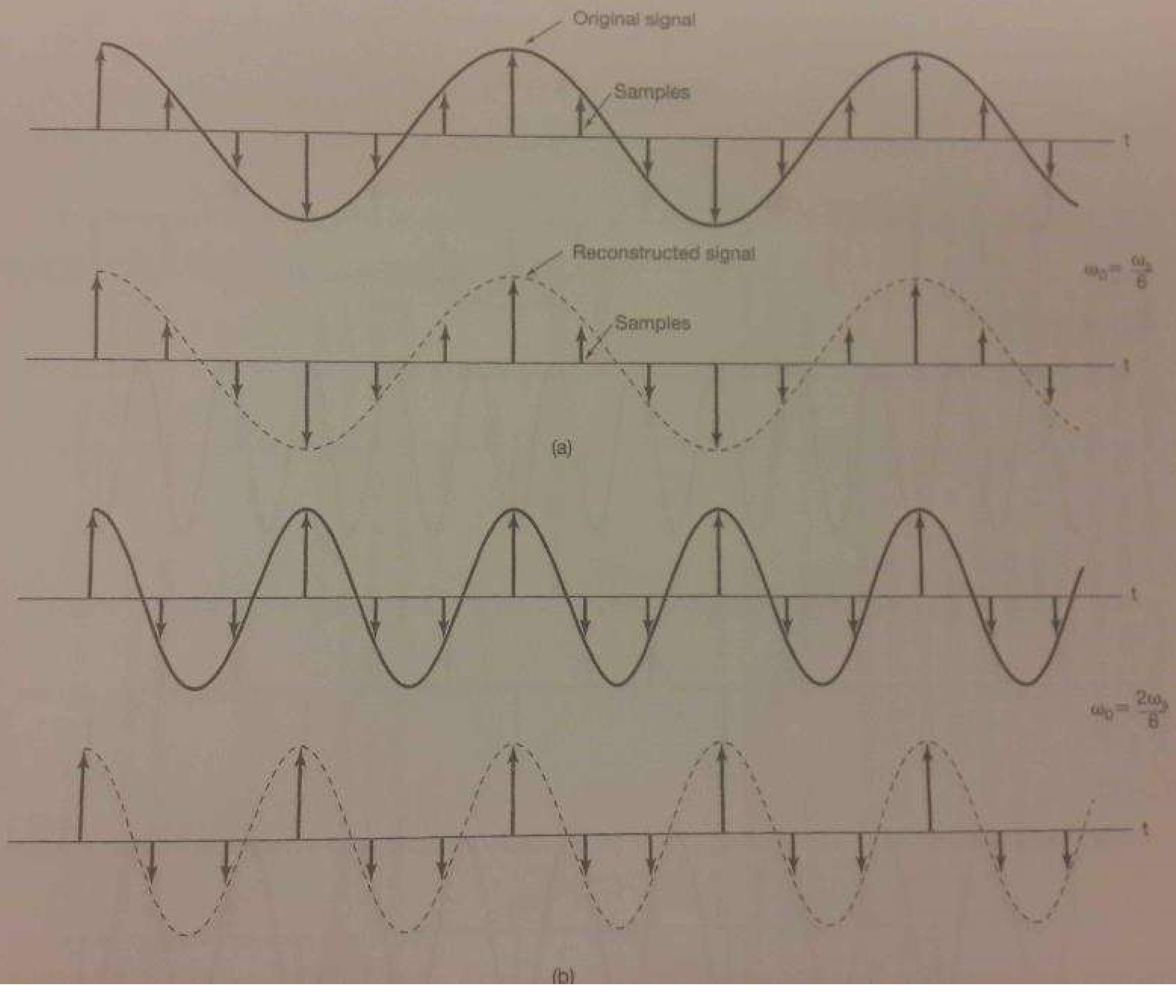
$$(c) \qquad \omega_0 = \frac{4\omega_s}{6}; \qquad x_r(t) = \cos(\omega_s - \omega_0)t$$

$$\neq x(t)$$

$$(d) \qquad \omega_0 = \frac{5\omega_s}{6}\ ; \qquad x_r(t) = \cos(\omega_s - \omega_0)t$$

$$\neq x(t)$$




Original signal
Samples
t
Reconstructed signal
Samples
t
$\omega_0 = \frac{\omega_s}{6}$
(a)
t
$\omega_0 = \frac{2\omega_s}{6}$
t
(b)

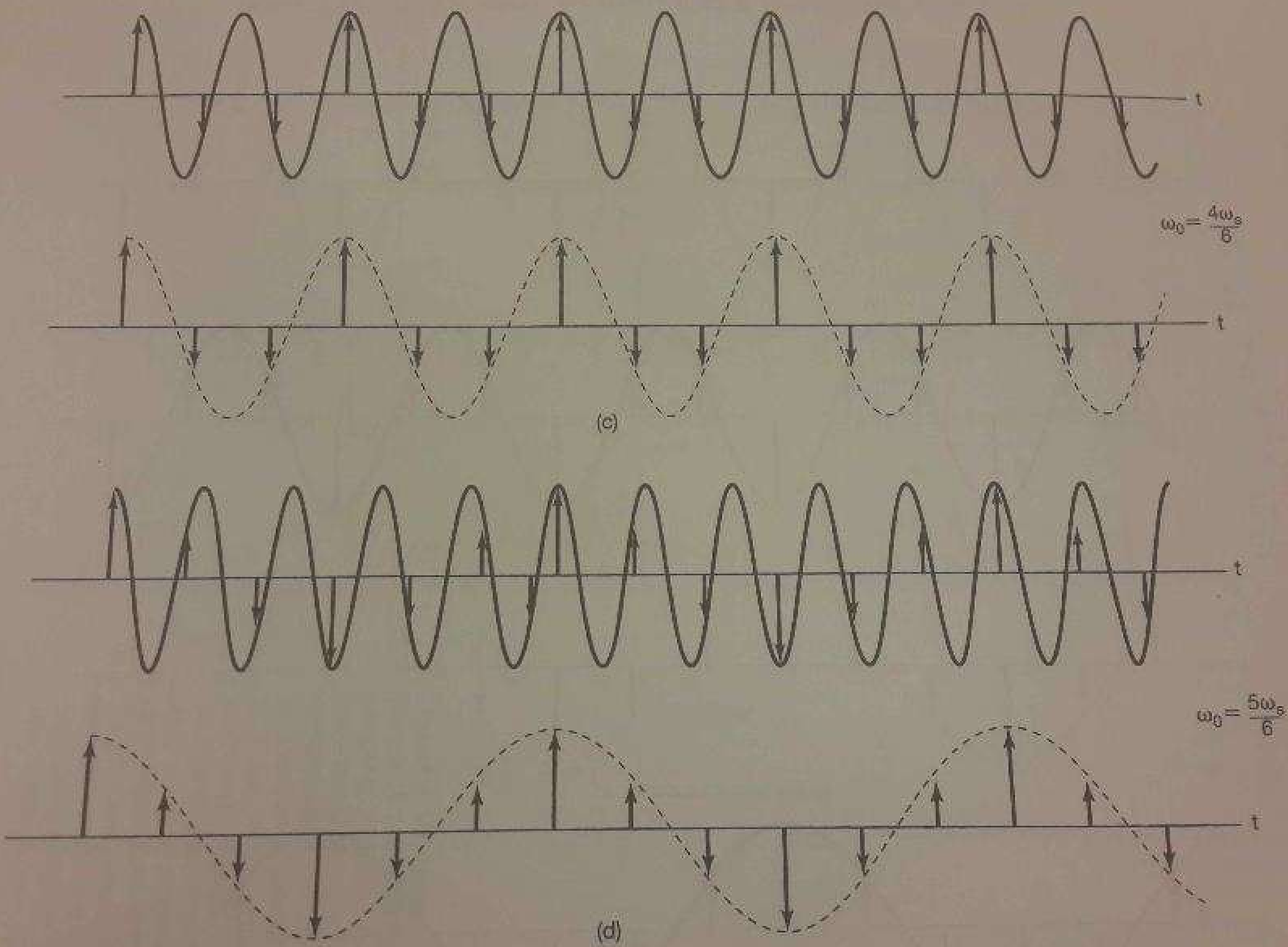
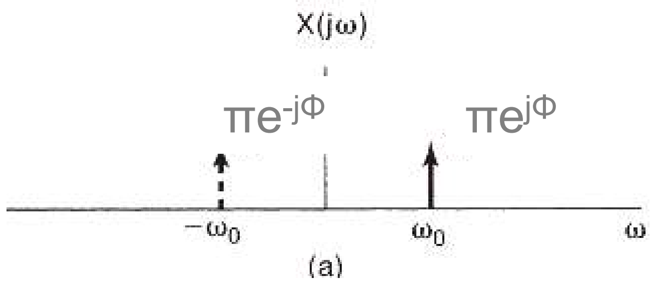


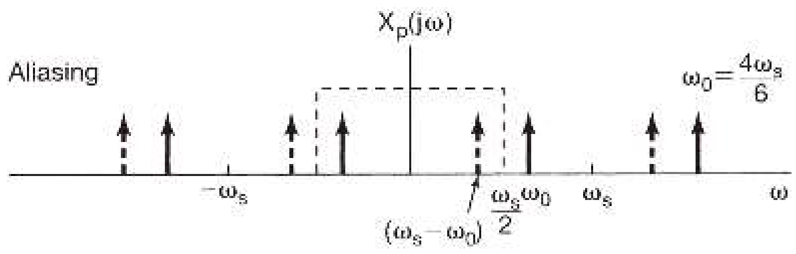
Figure 7.16 *Continued*

7.3 The Effect of Undersampling: Aliasing

As a variation on the preceding examples, consider the signal

$$x(t) = \cos(\omega_0 t + \phi). \qquad (7.15)$$





Phase reverse: $x_r(t) = \cos((\omega_s - \omega_0)t - \varphi).$

7.3 The Effect of Undersampling: Aliasing

It is important to note that the sampling theorem explicity requires that the sampling frequency be **greater** than twice the highest frequency in the signal, rather than **greater than or equal to** twice the highest frequency.

Example 7.1

Consider the sinusoidal signal

$$x(t) = \cos\left(\frac{\omega_s}{2}t + \varphi\right) = \frac{1}{2}(e^{j\left(\frac{\omega_s}{2}t+\varphi\right)} + e^{-j\left(\frac{\omega_s}{2}t+\varphi\right)}) = \frac{e^{j\varphi}}{2}e^{j\frac{\omega_s}{2}t} + \frac{e^{-j\varphi}}{2}e^{-j\frac{\omega_s}{2}t}$$

and suppose that this signal is sampled, using impulse sampling, at exactly twice the frequency of the sinusoid, i.e., at sampling frequency ω_s. As shown in Problem 7.39, if this impulse-sampled signal is applied as the input to an ideal lowpass filter with cutoff frequency $\omega_s/2$, the resulting output is

$$x_r(t) = (e^{j\varphi} + e^{-j\varphi})(\frac{e^{j\frac{\omega_s}{2}t}}{2} + \frac{e^{-j\frac{\omega_s}{2}t}}{2}) \propto (\cos\varphi)\cos\left(\frac{\omega_s}{2}t\right).$$

Example 7.1

As a consequence, we see that perfect reconstruction of *x(t)* occurs only in the case in which the phase Φ is zero (or an integer multiple of 2π). Otherwise, the signal $x_r(t)$ does not equal *x(t)*.

As an extreme example, consider the case in which $\Phi = -\pi/2$, so that

$$x(t) = \sin\left(\frac{\omega_s}{2} t\right).$$

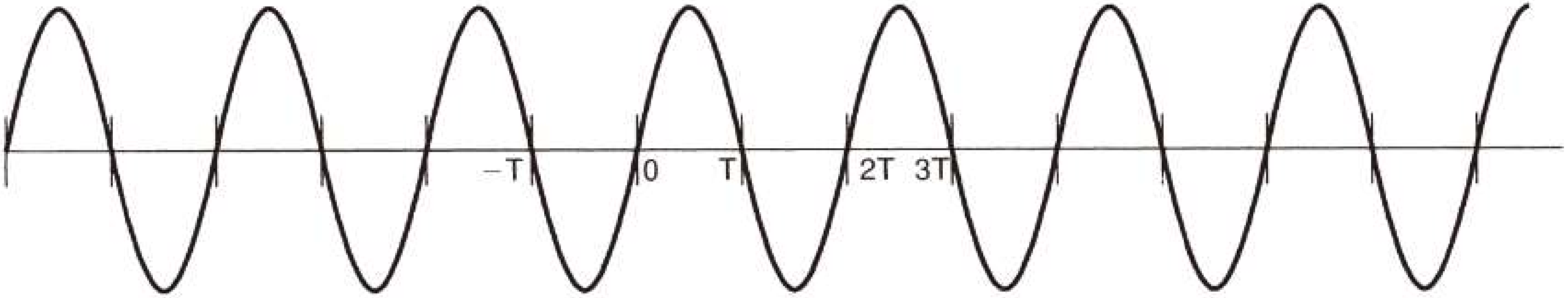


Figure 7.17 Sinusoidal signal for Example 7.1.

This signal is sketched in Figure 7.17. We observe that the values of the signal at integer multiples of the sampling period $2\pi / \omega_s$ are zero. Consequently, sampling at this rate produces a signal that is identically zero, and when this zero input is applied to the ideal lowpass filter, the resulting output $x_r(t)$ is also identically zero.

$$x_r(t) \propto (\cos \pi / 2) \cos\left(\frac{\omega_s}{2} t\right) = 0$$

7.3 The Effect of Undersampling: Aliasing

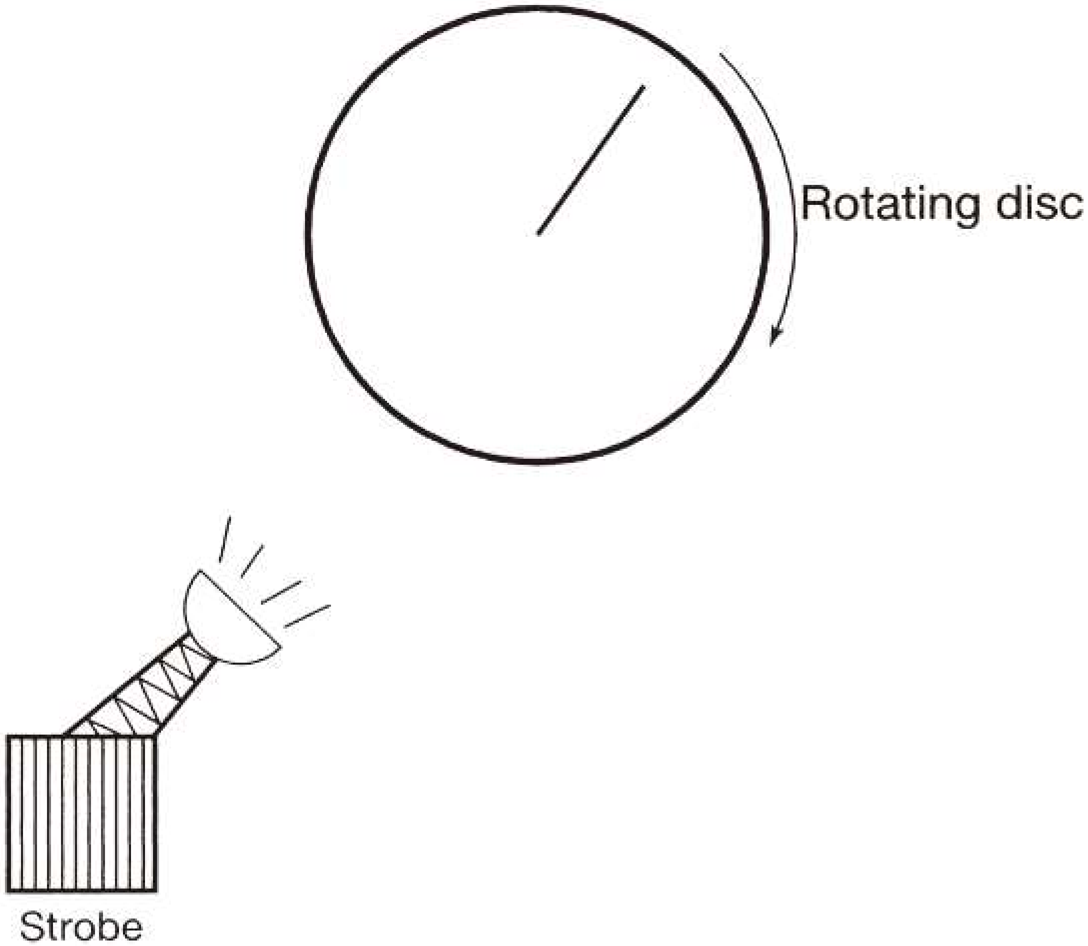
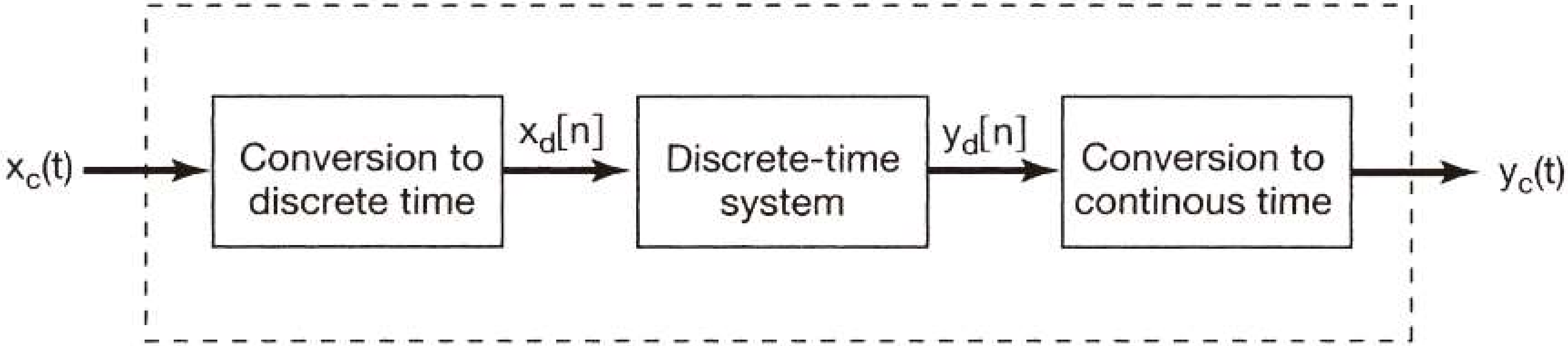


Figure 7.18 Strobe effect.

7.4 Discrete-Time Processing of Continuous-Time Signals

The continuous-time signal $x_c(t)$ is exactly represented by a sequence of instantaneous sample values $x_c(nT)$; that is, the discrete-time sequence is related to $x_d[n]$ by $x_c(t)$

$$x_d[n] = x_c(nT). \qquad (7.16)$$



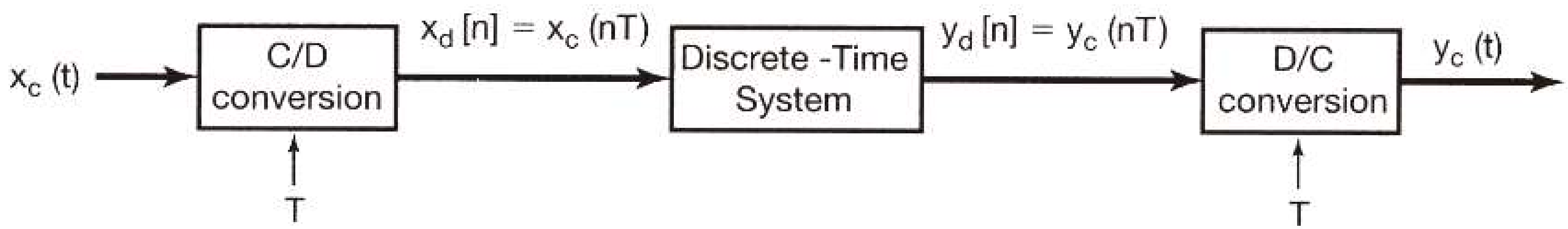
The transformation of $x_c(t)$ to $x_d[n]$ corresponding to the first system in Figure 7.19 will be referred to as *continuous-to-discrete-time conversion* and will be abbreviated C/D. The reverse operation corresponding to the third system in Figure 7.19 will be abbreviated D/C, representing *discrete-time to continuous-time conversion*.

$$x_d[n] = x_c(nT).$$

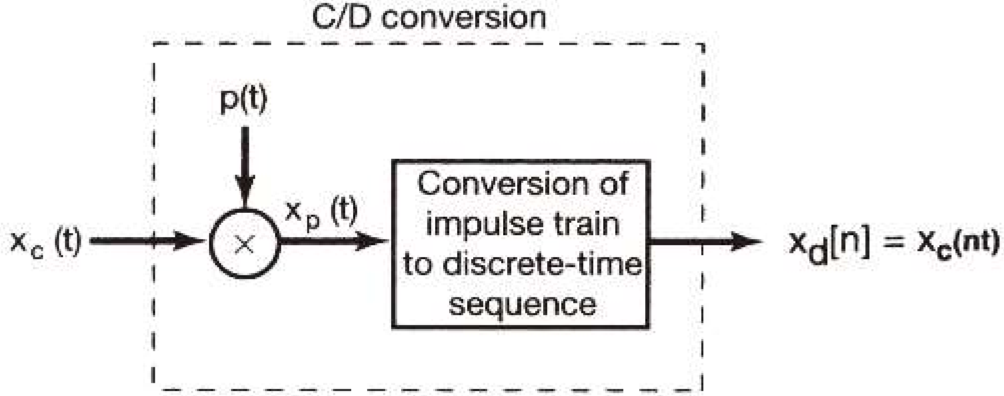
$$y_d[n] = y_c(nT).$$

7.4 Discrete-Time Processing of Continuous-Time Signals

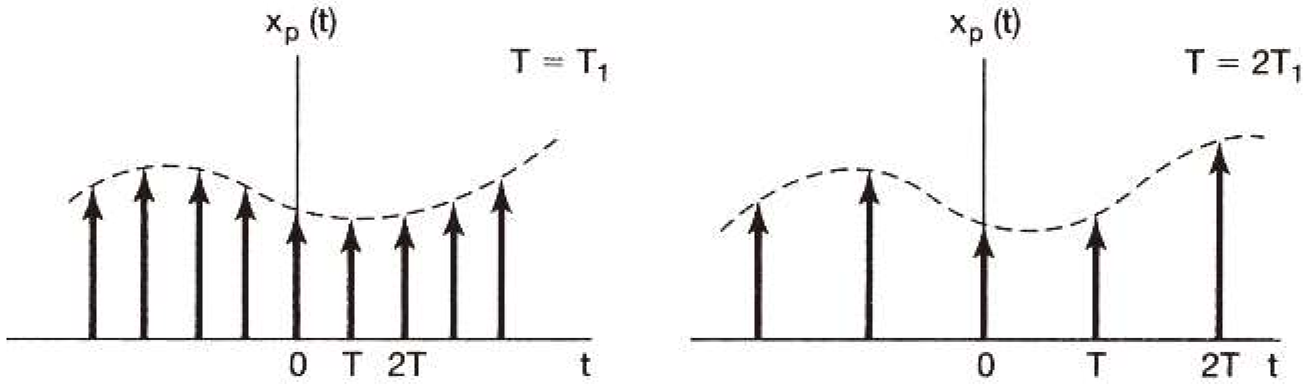
In systems such as digital computers and digital systems for which the discrete-time signal is represented in digital form, the **device** commonly used to implement the C/D conversion is referred to as an *analog-to-digital* (A-to-D) converter, and the device used to implement the D/C conversion is referred to as a *digital-to-analog* (D-to-A) converter.



7.4 Discrete-Time Processing of Continuous-Time Signals

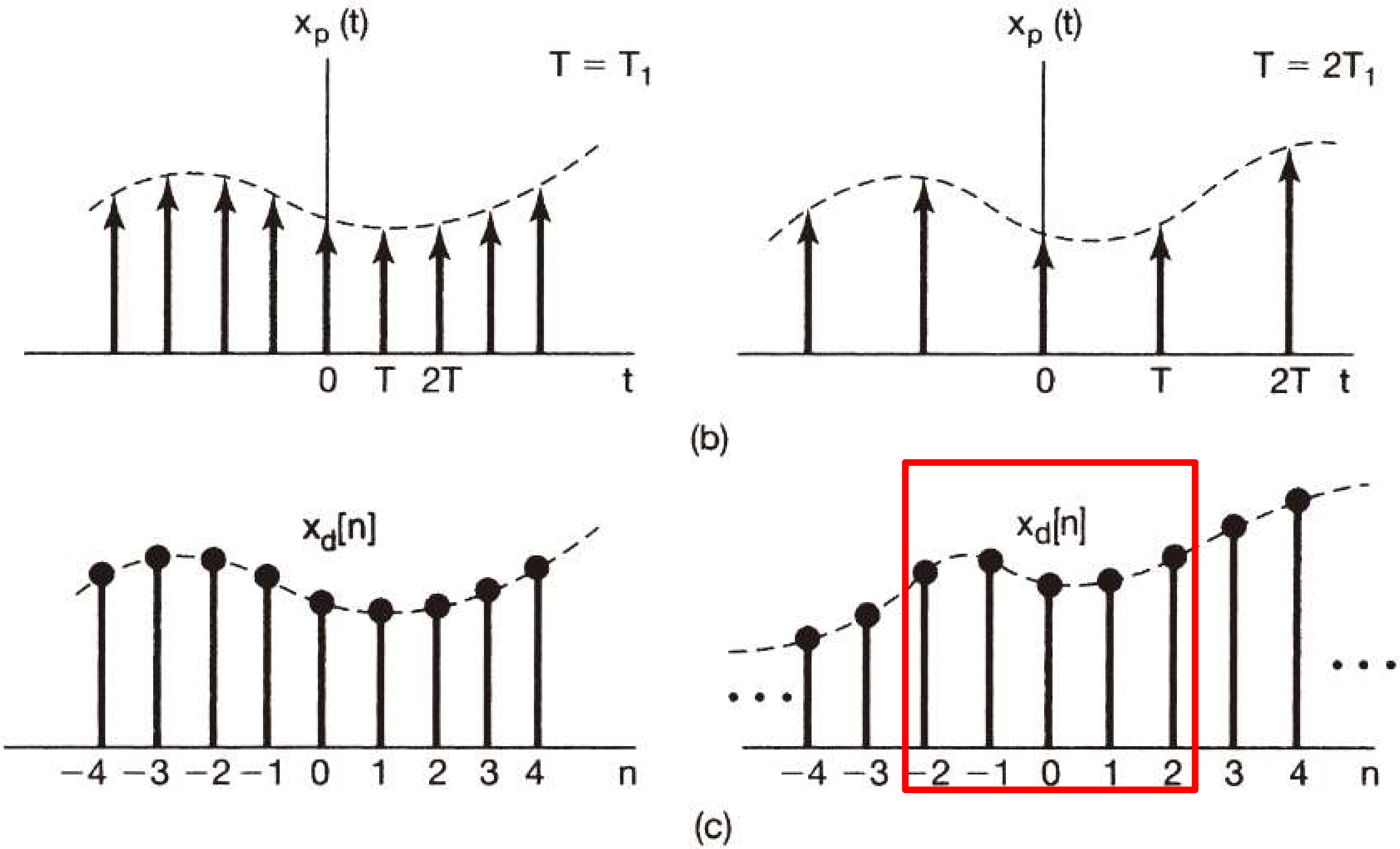


(a)



(b)

7.4 Discrete-Time Processing of Continuous-Time Signals



7.4 Relating $X_p(j\omega)$ and $X_d(e^{j\Omega})$

To begin let us express $X_p(j\omega)$, the continuous-time Fourier transform of $x_p(t)$, in terms of the sample values of $x_c(t)$ by applying the Fourier transform to eq. (7.3). Since

$$x_p(t) = \sum x_c(nT)\delta(t - nT), \qquad (7.17)$$

and since the transform of $\delta(t\text{-}nT)$ is $e^{-j\omega nT}$, it follows that

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\omega nT} \qquad (7.18)$$

7.4 Discrete-Time Processing of Continuous-Time Signals

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\omega nT}$$

Now consider the discrete-time Fourier transform of $x_d[n]$, that is,

$$X_d\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{+\infty} x_d[n] e^{-j\Omega n}, \qquad (7.19)$$

or, using eq. (7.16),

$$X_d\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\Omega n}. \qquad (7.20)$$

$$x_d[n] = x_c(nT) \quad (7.16)$$

$$X_p(j\omega)=\sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\omega nT} \quad (7.18) \qquad X_d\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{+\infty} x_c\left(nT\right)e^{-j\Omega n} \quad (7.20)$$

$$X_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(j\left(\omega-k\omega_s\right)\right) \quad (7.6)$$

Comparing eqs. (7.18) and (7.20), we see that $X_d\left(e^{j\Omega}\right)$ and $X_p(j\omega)$ are related through

$$X_d\left(e^{j\Omega}\right)=X_p\left(j\Omega/T\right).\omega=\Omega/T \qquad (7.21)$$

Also, recall that, as developed in eq. (7.6) and illustrated in Figure 7.3,

$$X_p\left(j\omega\right)=\frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(j\left(\omega-k\omega_s\right)\right). \qquad (7.22)$$

Consequently,

$$X_d\left(e^{j\Omega}\right)=\frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(j\left(\Omega-2\pi k\right)/T\right). \qquad (7.23)$$

$$X_d\left(e^{j\Omega}\right)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X_c\left(j(\Omega-2\pi k)/T\right).$$

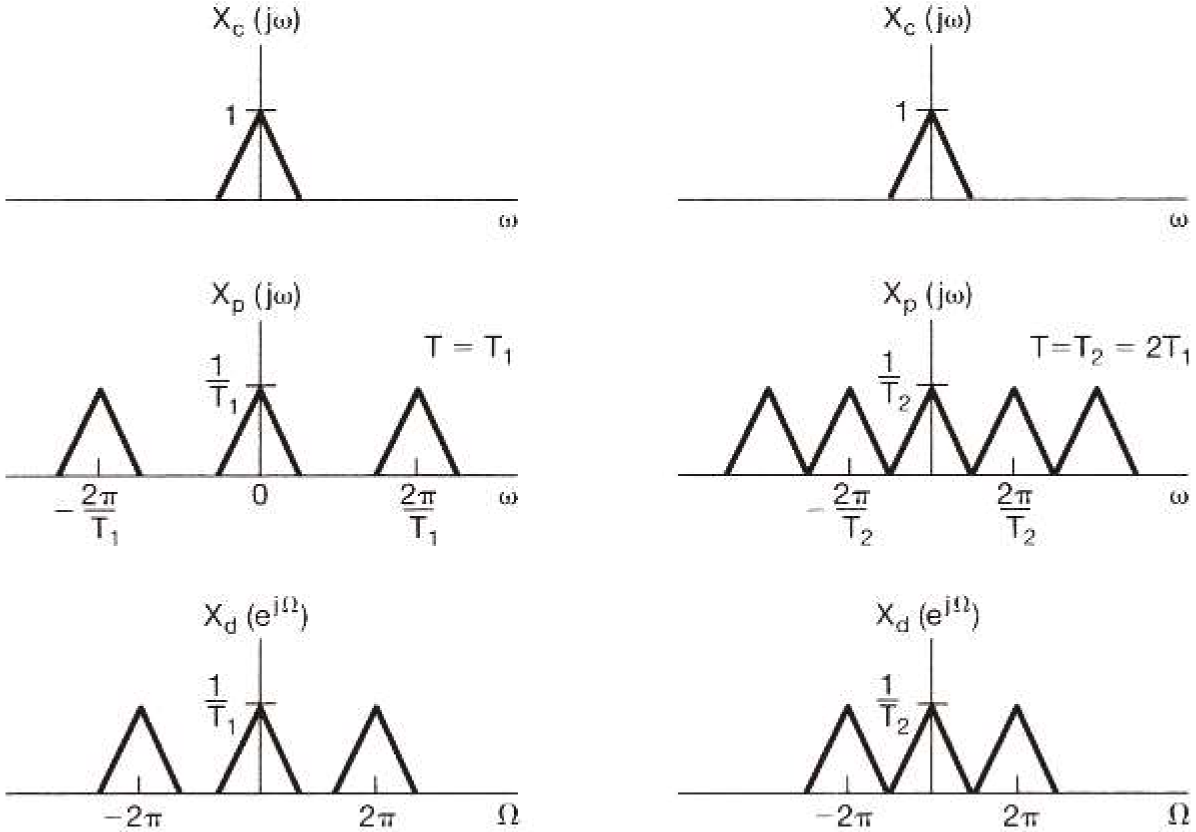


Figure 7.22 Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

7.4 Discrete-Time Processing of Continuous-Time Signals

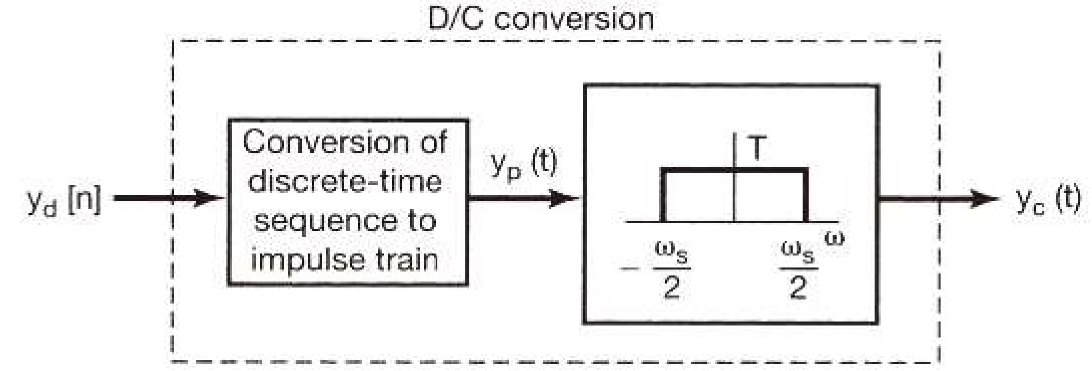


Figure 7.23 Conversion of a discrete-time sequence to a continuous-time signal.

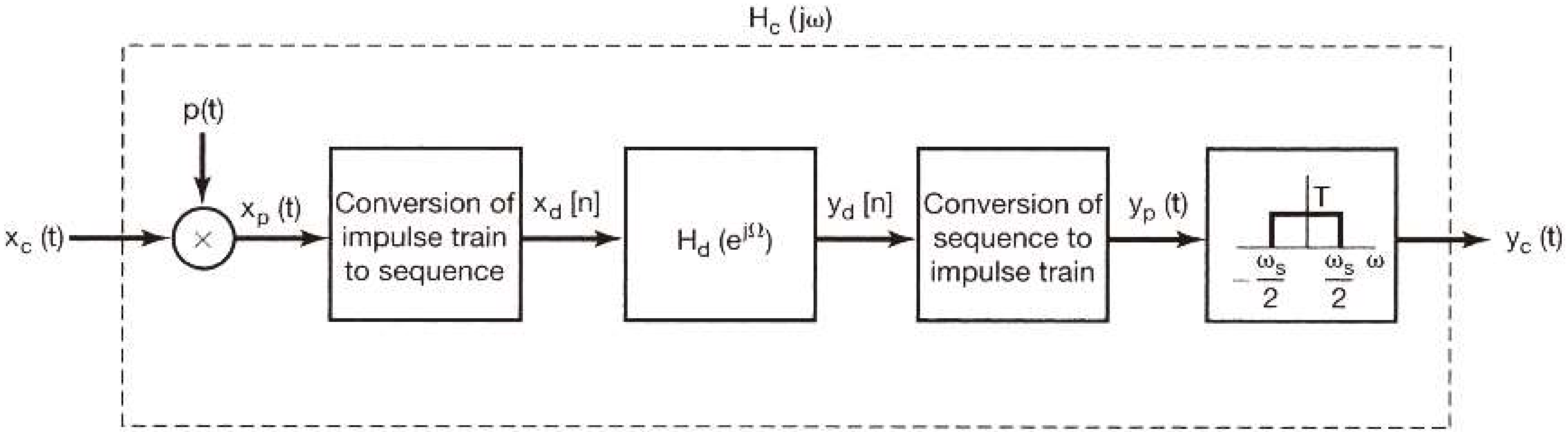
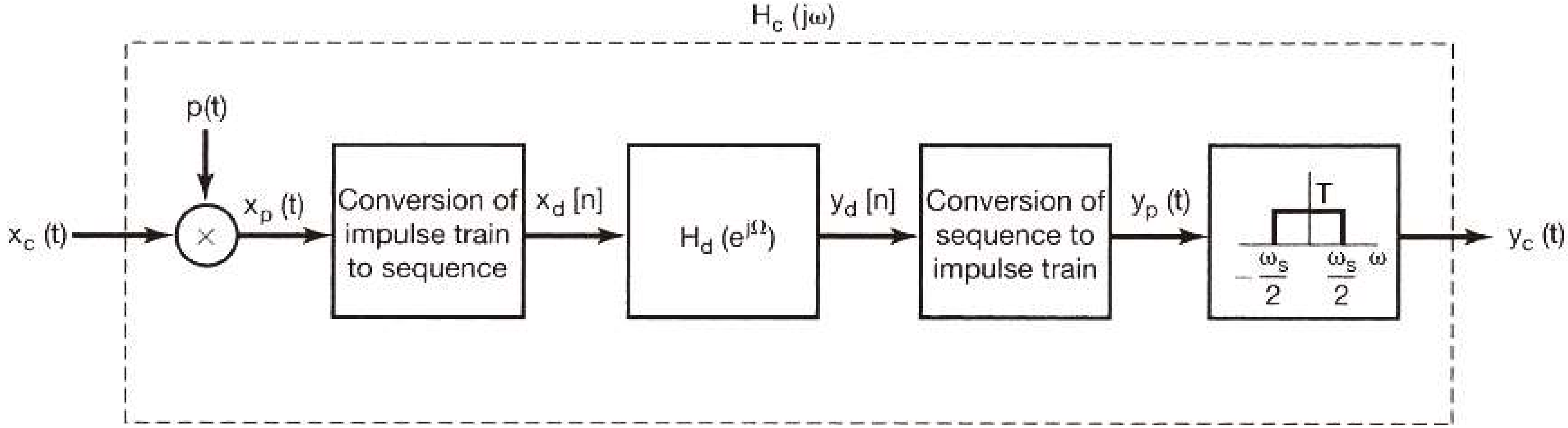
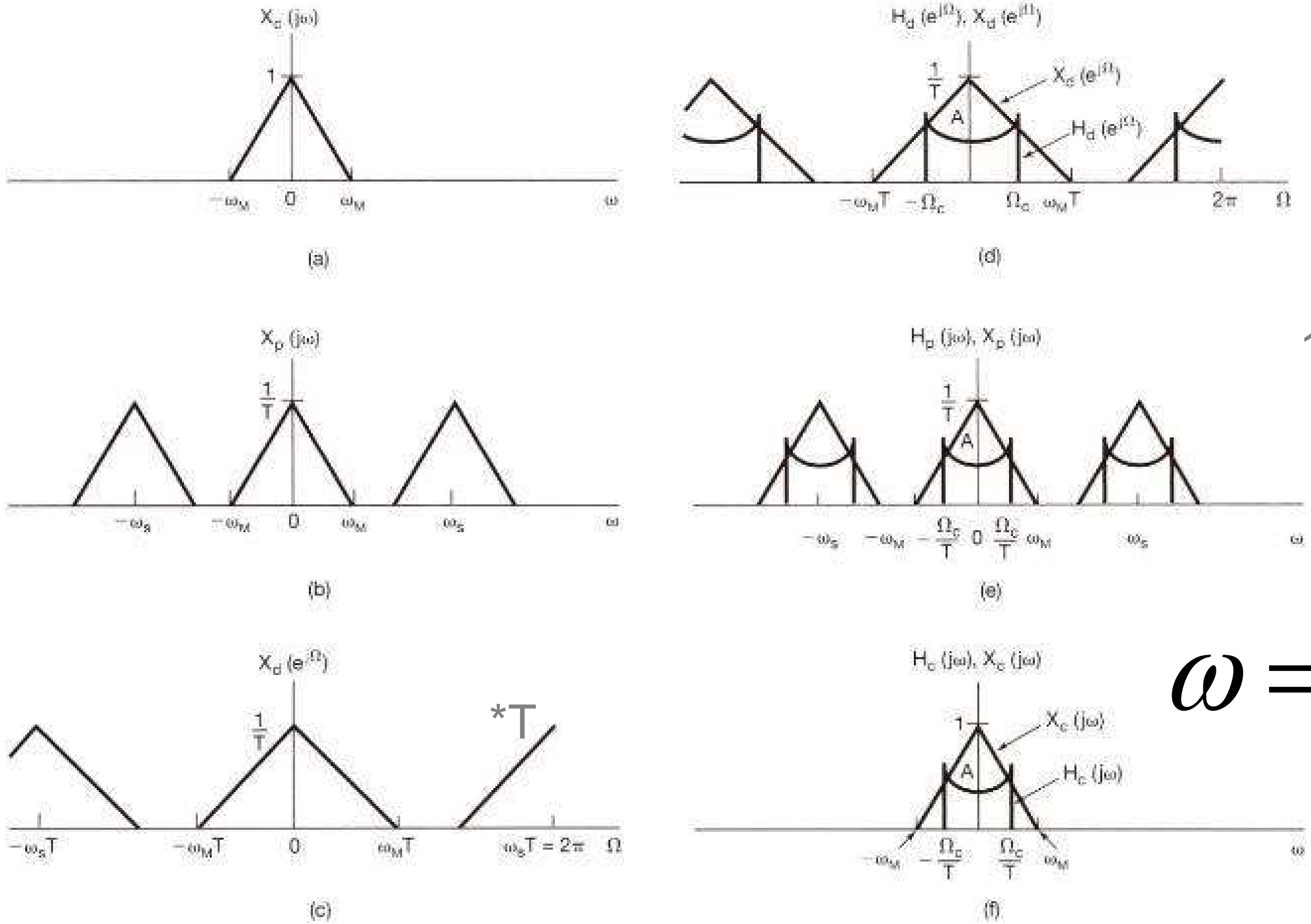


Figure 7.24 Overall system for filtering a continuous-time signal using a discrete-time filter.





1/T

$$\omega = \Omega / T$$

7.4 Discrete-Time Processing of Continuous-Time Signals

In comparing Figures 7.25(a) and (f), we see that

$$Y_c(j\omega) = X_c(j\omega) H_d(e^{j\omega T}).\Omega = \omega T \qquad (7.24)$$

equivalent to a continuous-time LTI system with frequency response $H_c(j\omega)$ which is related to the discrete-time frequency response $H_d\left(e^{j\Omega}\right)$ through

$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s / 2 \\ 0, & |\omega| > \omega_s / 2 \end{cases}. \qquad (7.25)$$

$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}.$$

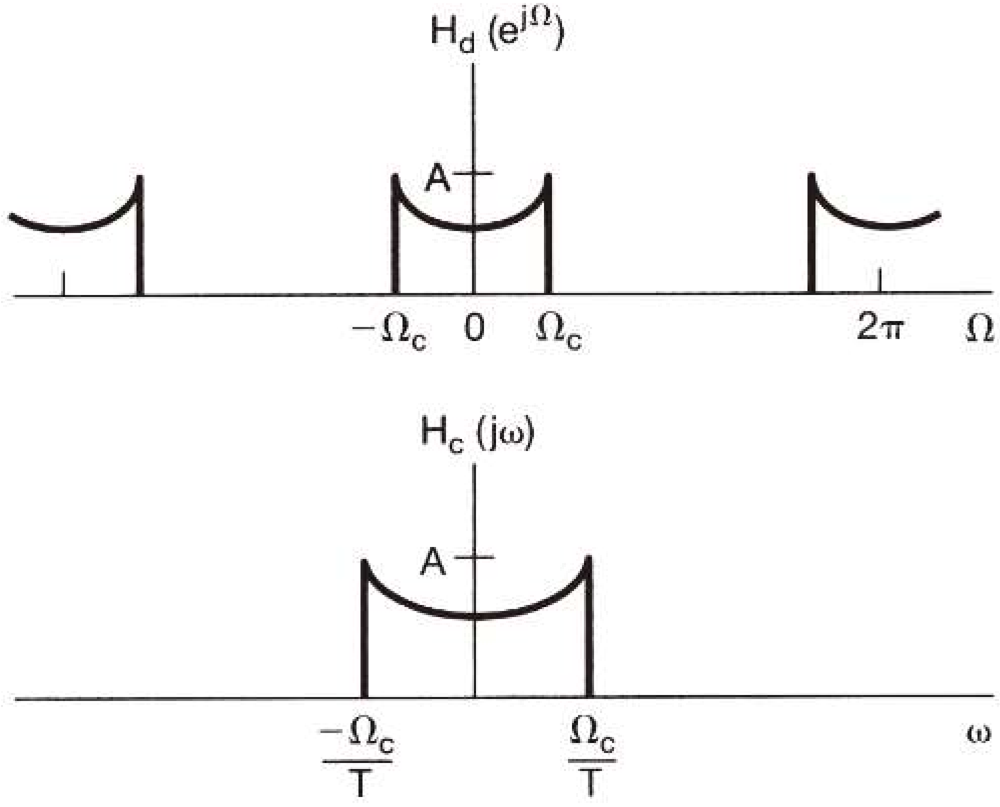


Figure 7.26 Discrete-time frequency response and the equivalent continuous-time frequency response for the system of Figure 7.24.

7.4.1 Digital Differentiator

Consider the discrete-time implementation of a continuous-time band-limited differentiating filter.

$$H_c(j\omega) = j\omega, \qquad (7.26)$$

連續時間微分濾波器的頻率響應

and that of a band-limited differentiator with cutoff frequency ω_c is

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \qquad (7.27)$$

截止頻率為 ω_c 的有限頻帶微分器

7.4.1 Digital Differentiator

$$\Omega = \omega T$$

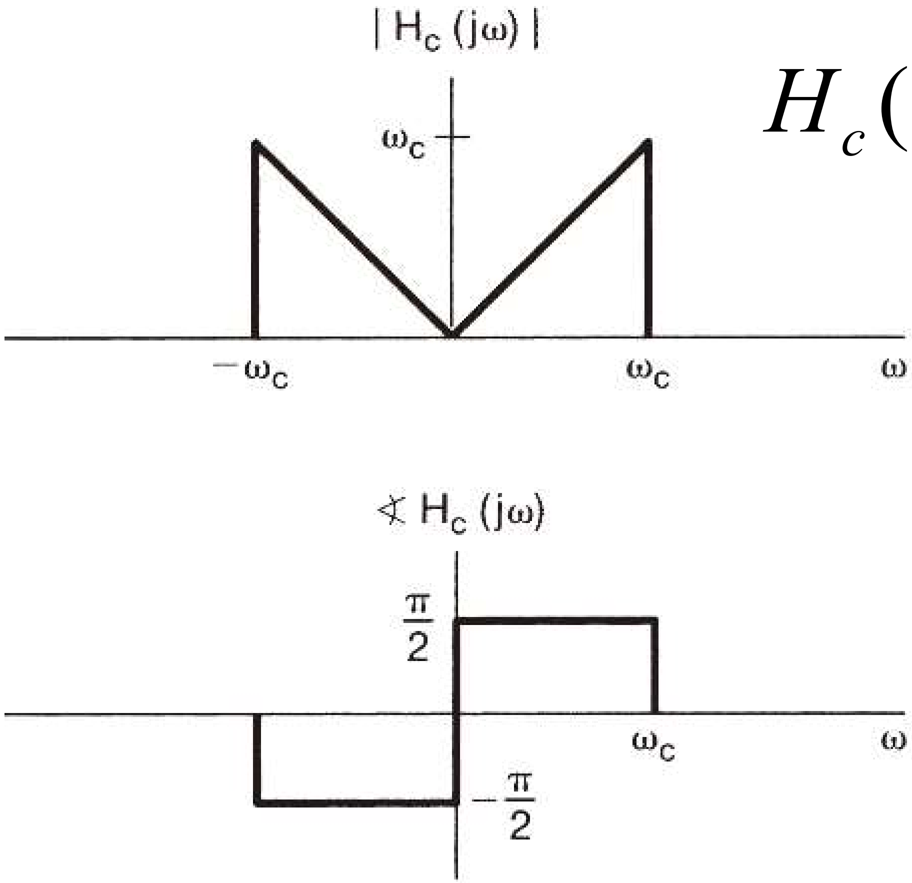
$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases},$$

As sketched in Figure 7.27. Using eq. (7.25) with a sampling frequency $\omega_s = 2\omega_c$, we see that the corresponding discrete-time transfer function is

$$H_d\left(e^{j\Omega}\right) = j\left(\frac{\Omega}{T}\right), \qquad \left|\frac{\Omega}{T}\right| < \omega_c, \rightarrow |\Omega| < T\frac{\omega_s}{2}$$

$$H_d\left(e^{j\Omega}\right) = j\left(\frac{\Omega}{T}\right), \qquad |\Omega| < \pi, \tag{7.28}$$

7.4.1 Digital Differentiator



$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases},$$

圖 7.27 爲 (7.27) 式的頻率響應圖。

Figure 7.27 Frequency response of a continuous-time ideal band-limited differentiator $H_c(j\omega) = j\omega,\ |\omega| < \omega_c$.

7.4.1 Digital Differentiator

$$H_d\left(e^{j\Omega}\right) = j\left(\frac{\Omega}{T}\right), \qquad |\Omega| < \pi,$$

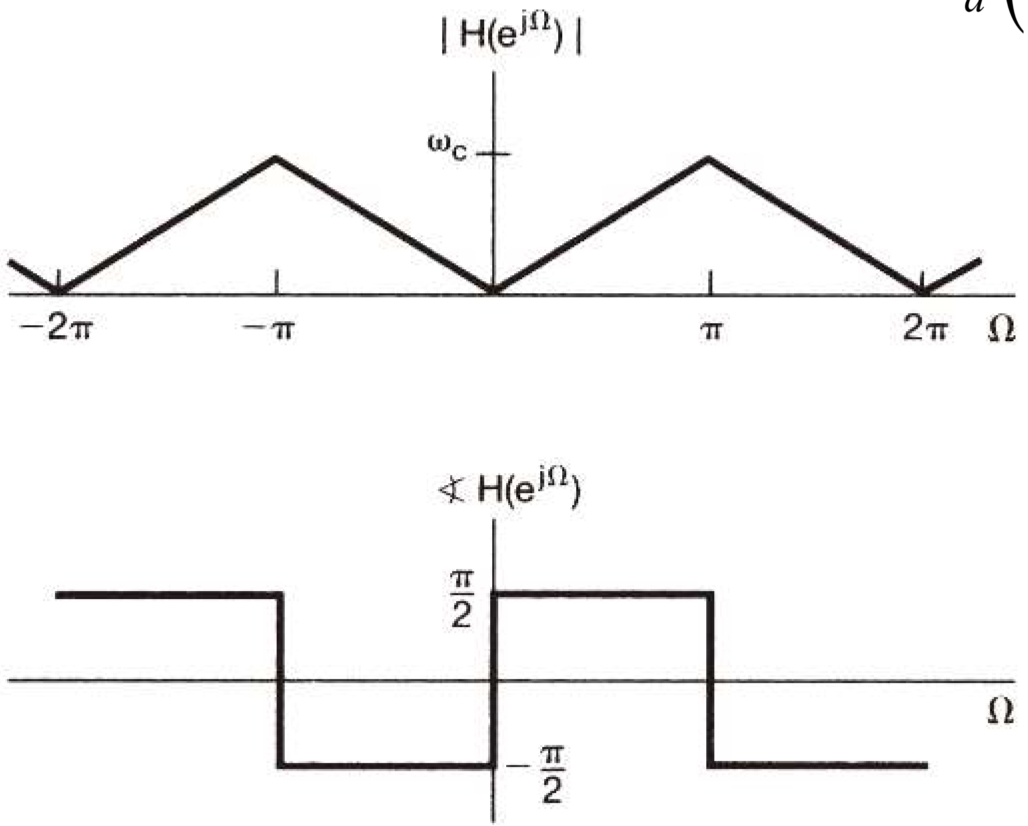


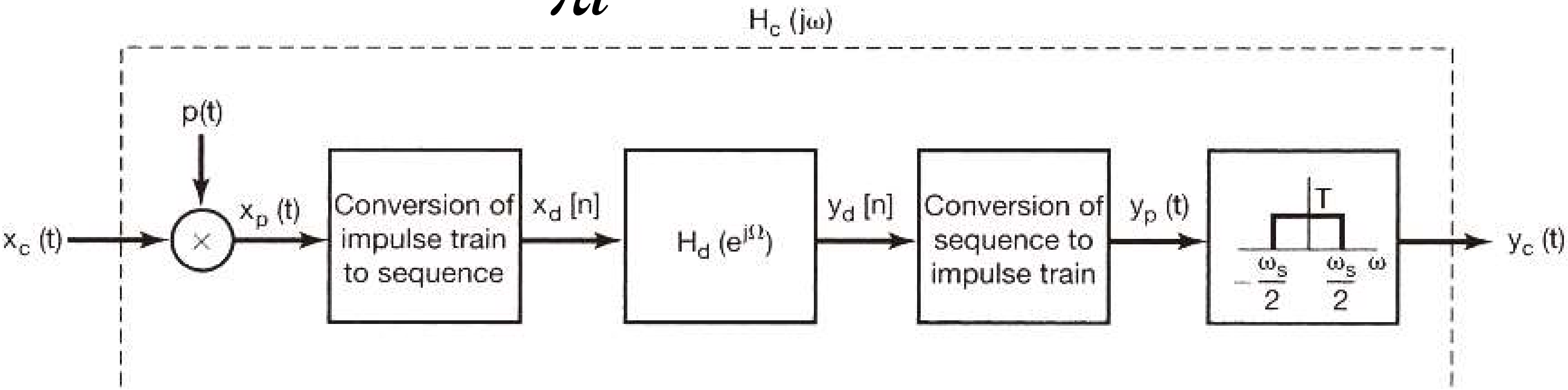
圖 7.28 爲 (7.28) 式的頻率響應。

Figure 7.28 Frequency response of discrete-time filter used to implement a continuous-time band-limited differentiator.

Example 7.2 $H_d\left(e^{j\Omega}\right)=j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi,$

By considering the output of the digital differentiator for a continuous-time sinc input, we may conveniently determine the **impulse response** $h_d[n]$ of the discrete-time filter in the implementation of the digital differentiator. With reference to Figure 7.24, let

$$x_c(t)=\frac{\sin(\pi t/T)}{\pi t}, \qquad (7.29)$$



Example 7.2

where T is the sampling period. Then

$$X_c(j\omega) = \begin{cases} 1, & |\omega| < \pi / T \\ 0, & otherwise \end{cases},$$

which is sufficiently band limited to ensure that sampling $x_c(t)$ at frequency $\omega_s = 2\pi / T$ does not give rise to any aliasing. It follows that the output of the digital differentiator is

$$y_c(t) = \frac{d}{dt} x_c(t) = \frac{\cos(\pi t / T)}{Tt} - \frac{\sin(\pi t / T)}{\pi t^2}. \qquad (7.30)$$

Example 7.2

$$x_c(t) = \frac{\sin(\pi t / T)}{\pi t} \quad (7.29)$$

For $x_c(t)$ as given by eq. (7.29), the corresponding signal $x_d[n]$ in Figure 7.24 may be expressed as

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]. \quad (7.31)$$

That is, for n ≠ 0, $x_c(nT) = 0$, while

$$x_d[0] = x_c(0) = \frac{1}{T}$$

Example 7.2

$$y_c(t) = \frac{\cos(\pi t / T)}{Tt} - \frac{\sin(\pi t / T)}{\pi t^2} \quad (7.30)$$

which can be verified by $\text{l'}H\hat{o}pital\text{'}s$ rule, We can similarly evaluate $y_d[n]$ in Figure 7.24 corresponding to $y_c(t)$ in eq. (7.30). Specifically

$$y_d[n] = y_c(nT) = \begin{cases} \dfrac{(-1)^n}{nT^2}, & n \neq 0 \\ 0 & n = 0 \end{cases}. \qquad (7.32)$$

which can be verified for $n \neq 0$ by direct substitution into eq. (7.30) and for $n = 0$ by application of $\text{l'}H\hat{o}pital\text{'}s$ rule.

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n].\ y_d[n] = y_c(nT) = \begin{cases} \frac{(-1)^n}{nT^2}, & n \neq 0 \\ 0 & n = 0 \end{cases}.$$

Thus when the input to the discrete-time filter given by eq. (7.28)is the scaled unit impulse in eq. (7.31), the resulting output is given by eq. (7.32). We then conclude that the impulse response of this filter is given by

$$h_d[n] = \begin{cases} \frac{(-1)^n}{nT}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

7.4.2 Half-Sample Delay

We require that the input and output of the overall system be related by

$$y_c(t) = x_c(t - \Delta) \tag{7.33}$$

From the time-shifting property derived in Section 4.3.2

$$Y_c(j\omega) = e^{-j\omega\Delta} X_c(j\omega).$$

7.4.2 Half-Sample Delay

From eq. (7.25), the equivalent continuous-time system to be implemented must be band limited. Therefore, we take

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & otherwise \end{cases}, \qquad (7.34)$$

有限頻寬連續時間延遲系統的頻率響應

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & otherwise \end{cases}, \qquad \Omega = \omega T$$

With the sampling frequency ω_s taken as $\omega_s = 2\omega_c$, the corresponding discrete-time frequency response is

$$H_d\left(e^{j\Omega}\right) = e^{-j\Omega\Delta/T}, \qquad |\Omega| < \pi, \qquad (7.35)$$

半取樣延遲系統的頻率響應

7.4.2 Half-Sample Delay

圖7.29(b)為 $\omega_s = 2\omega_c$ 之下的離散時間半取樣延遲系統。

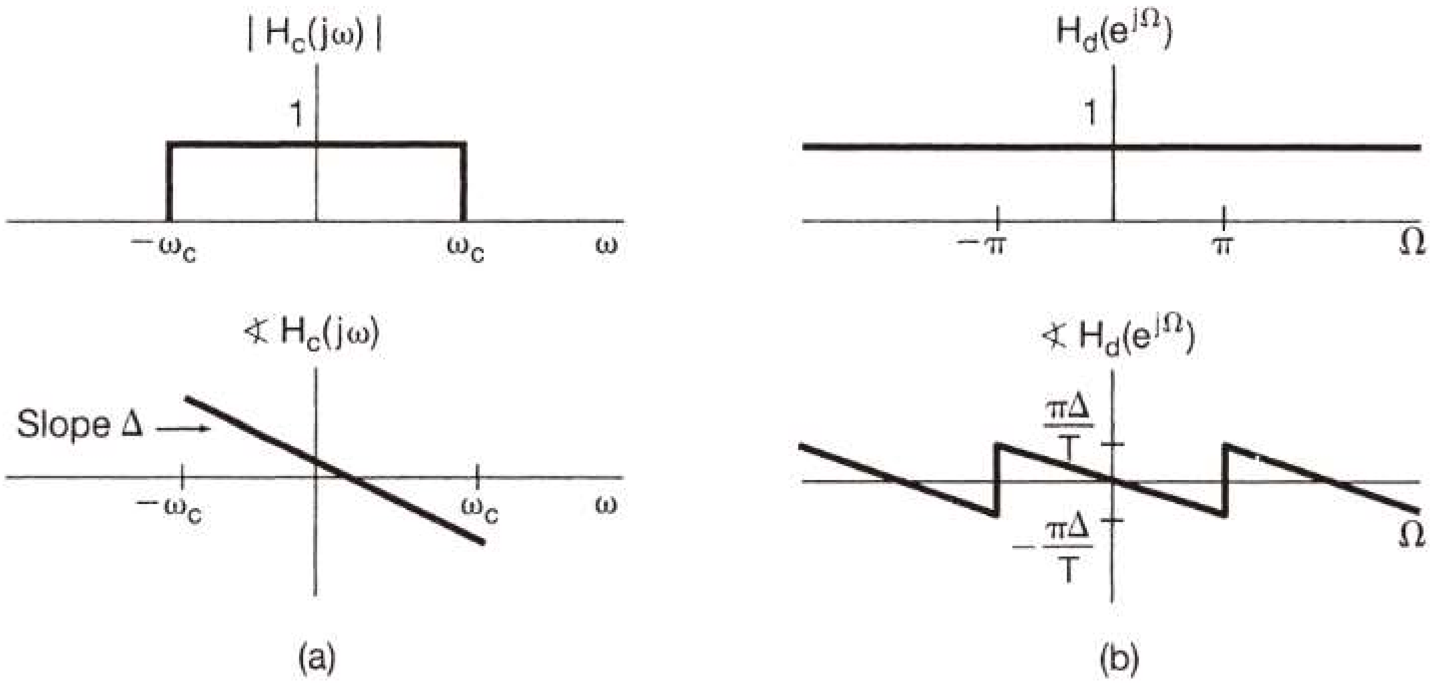


Figure 7.29 (a) Magnitude and phase of the frequency response for a continuous-time delay; (b) magnitude and phase of the frequency response for the corresponding discrete-time delay.

$$H_d\left(e^{j\Omega}\right) = e^{-j\Omega\Delta/T}, \qquad |\Omega| < \pi,$$

$\delta[n - n_0]$	$e^{-j\omega n_0}$

For Δ/T an integer, the sequence $y_d[n]$ is a delayed replica of $x_d[n]$; that is,

$$y_d[n] = x_d\left[n - \frac{\Delta}{T}\right]. \qquad (7.36)$$

What if Δ/T is not an integer?

Example 7.3

The approach in Example 7.2 is also applicable to determining the impulse response $h_d[n]$ of the discrete-time filter in the **half-sample delay** system. With reference to Figure 7.24, let

$$x_c(t) = \frac{\sin(\pi t / T)}{\pi t}. \qquad (7.37)$$

It follows form Example 7.2 that

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n].$$

Example 7.3 $y_c(t) = x_c(t - \Delta)$

Also, since there is no aliasing for the band-limited input in eq. (7.37), the output of the half-sample delay system is

$$y_c(t) = x_c\left(t - T/2\right) = \frac{\sin\left(\pi\left(t - T/2\right)/T\right)}{\pi\left(t - T/2\right)},$$

and the sequence $y_d[n]$ in Figure 7.24 is

$$y_d[n] = y_c(nT) = \frac{\sin\left(\pi(n - \frac{1}{2})\right)}{T\pi(n - \frac{1}{2})}. \qquad x_d[n] = x_c(nT) = \frac{1}{T}\delta[n].$$

We conclude that

$$h[n] = \frac{\sin\left(\pi(n - \frac{1}{2})\right)}{\pi(n - \frac{1}{2})}.$$

7.4.2 Half-Sample Delay

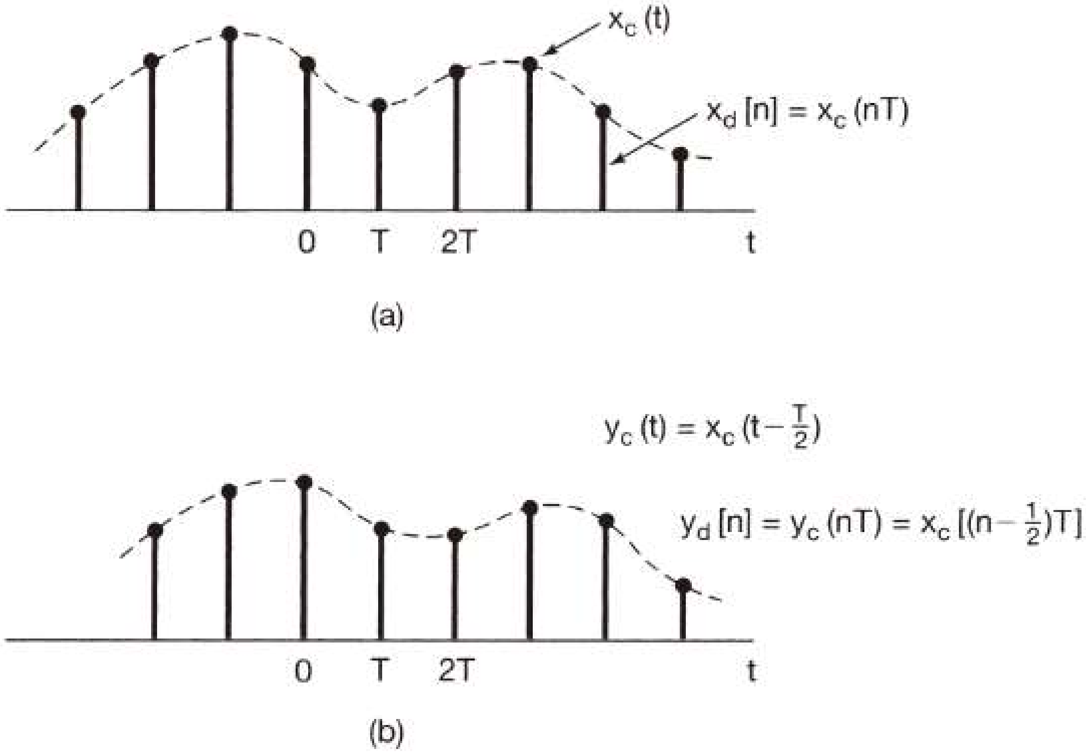
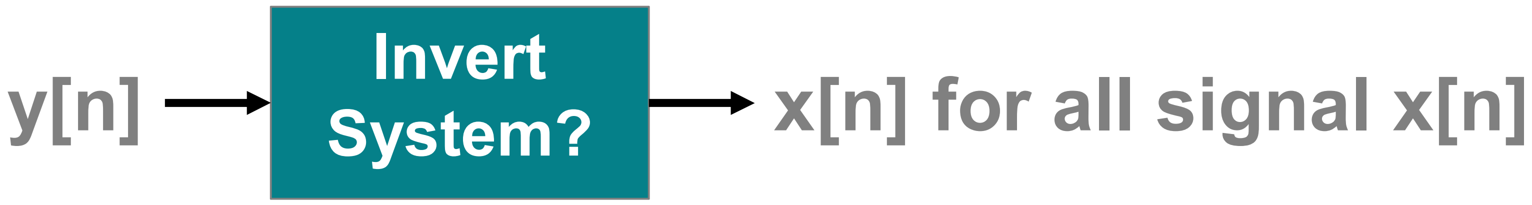


Figure 7.30 (a) Sequence of samples of a continuous-time signal $x_c(t)$; (b) sequence in (a) with a half-sample delay.

7.5 Sampling of Discrete-time Signals

- y[n]=x[2n] (homework 1)



7.5.1 Impulse-Train Sampling

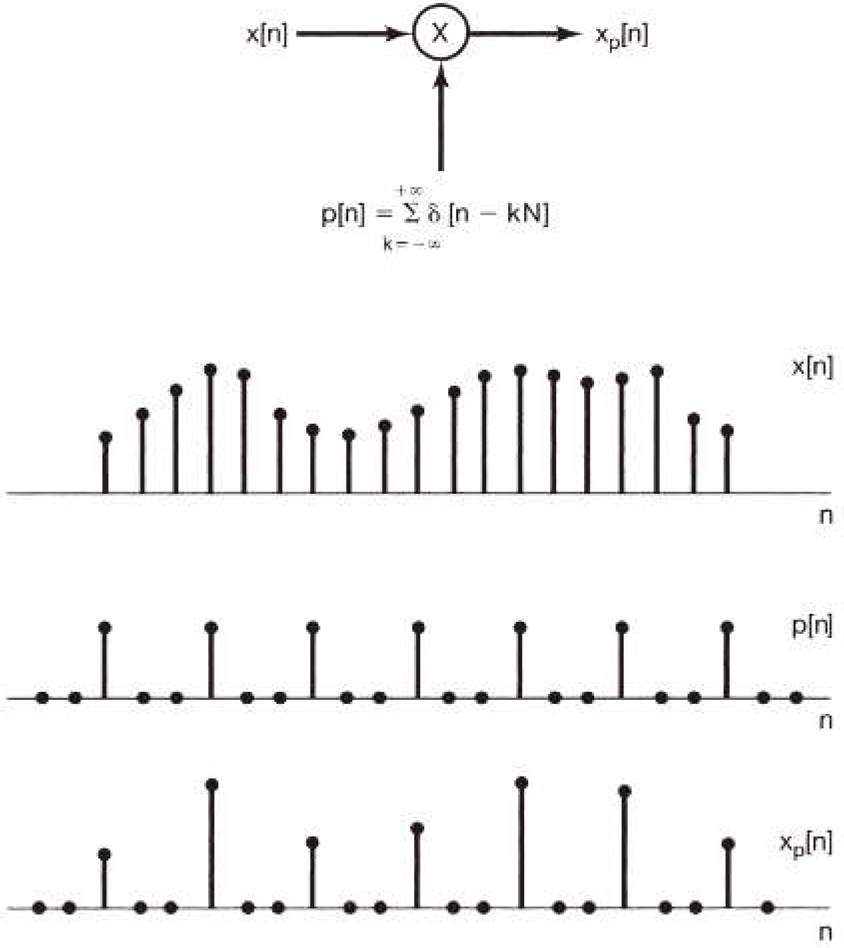


Figure 7.31 Discrete-time sampling.

7.5.1 Impulse-Train Sampling

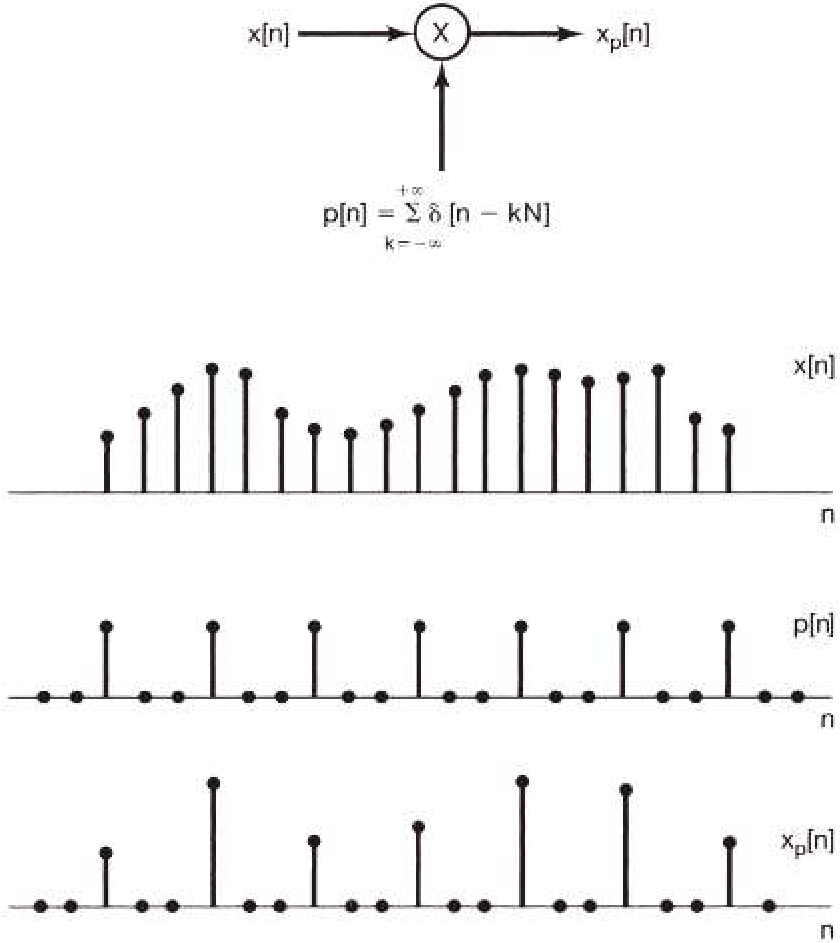
The new sequence $x_p[n]$ resulting from the sampling process is equal to the original sequence *x[n]* at integer multiples of the sampling period *N* and is zero at the intermediate samples;

$$x_p[n] = \begin{cases} x[n], & \text{if n = an integer multiple of N} \\ 0, & \text{otherwise} \end{cases} \qquad (7.38)$$

$x_p[n]$ 為*x[n]*以取樣週期*N*取值，而取樣點之間的訊號值均為0所得的序列。

7.5.1 Impulse-Train Sampling

Using the multiplication property developed in Section 5.5. Thus, with



(7.39)

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n-kN],$$

We have, in the frequency domain,

(7.40)

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right) X\left(e^{j(\omega-\theta)}\right) d\theta.$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right) X\left(e^{j(\omega-\theta)}\right) d\theta.$$

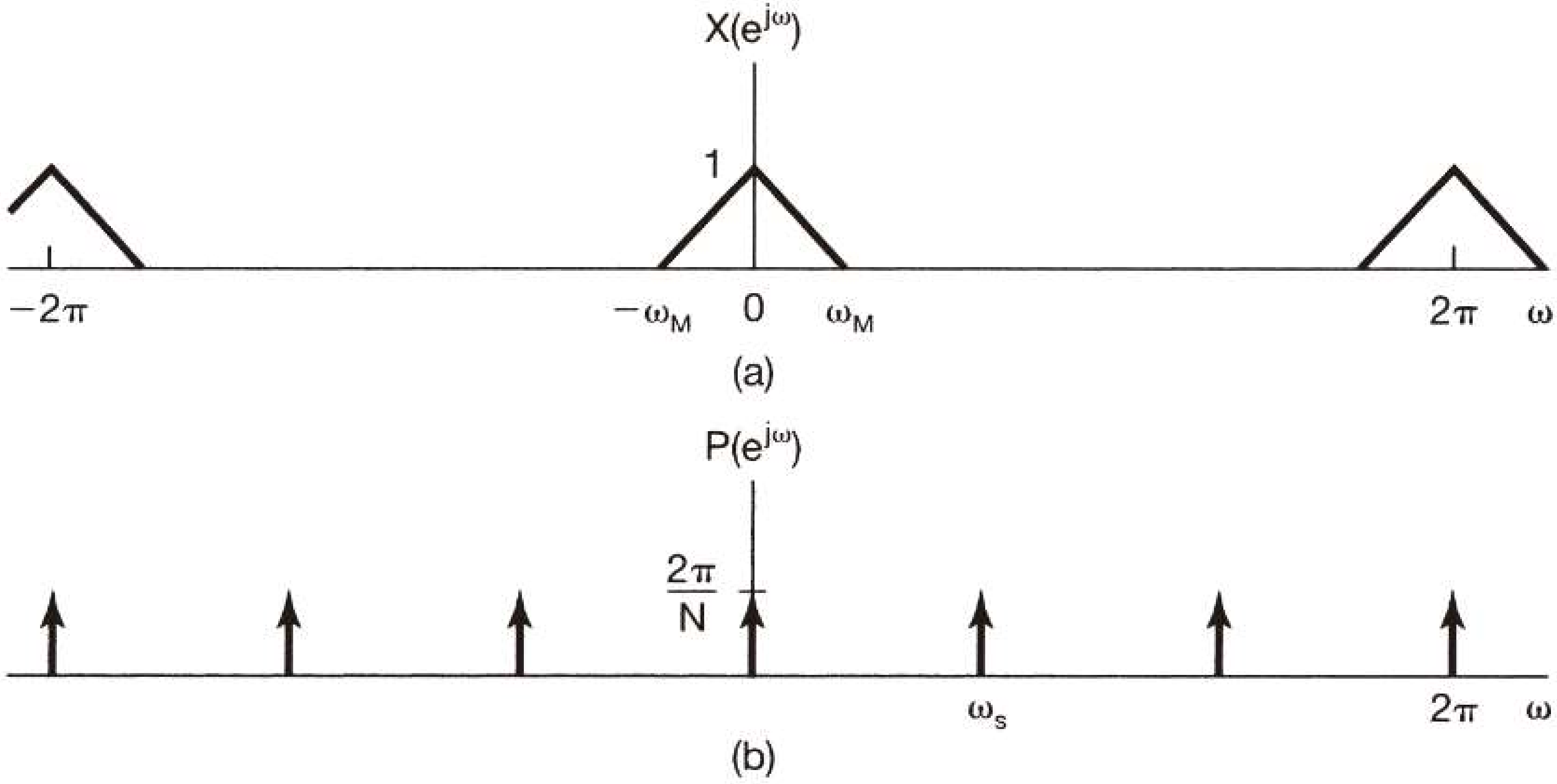
As in Example 5.6, the Fourier transform of the sampling sequence *p[n]* is

$$P\left(e^{j\omega}\right) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s), \qquad (7.41)$$

where ω_s, the sampling frequency, equals $2\pi/N$. Combining eqs. (7.40) and (7.41), we have

$$X_p\left(e^{j\omega}\right) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j(\omega-k\omega_s)}\right). \qquad (7.42)$$

$$P\left(e^{j\omega}\right)=\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s),$$



$$X_p\left(e^{j\omega}\right)=\frac{1}{N}\sum_{k=0}^{N-1}X\left(e^{j(\omega-k\omega_s)}\right). \qquad \omega_s=\frac{2\pi}{N}$$

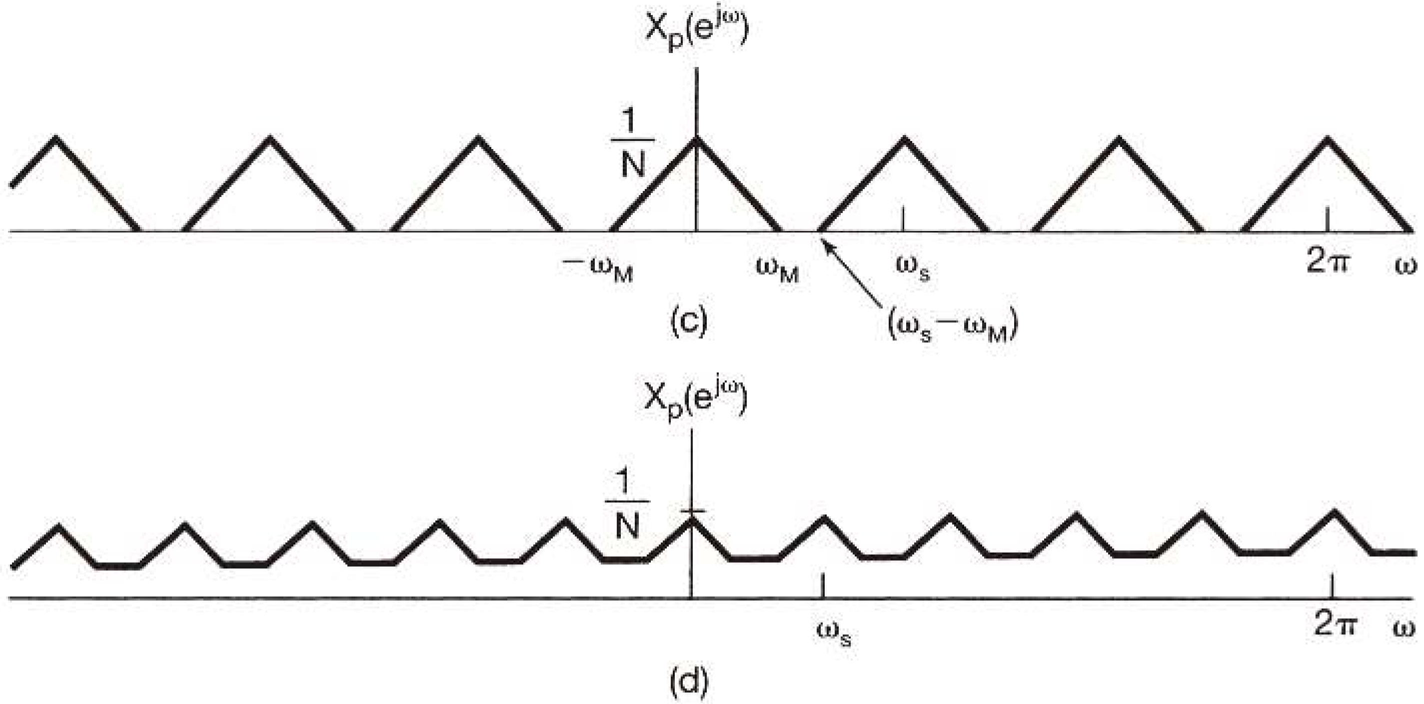


Figure 7.32 Effect in the frequency domain of impulse-train sampling of a discrete-time signal: (a) spectrum of original signal; (b) spectrum of sampling sequence; (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$. Note that aliasing occurs.

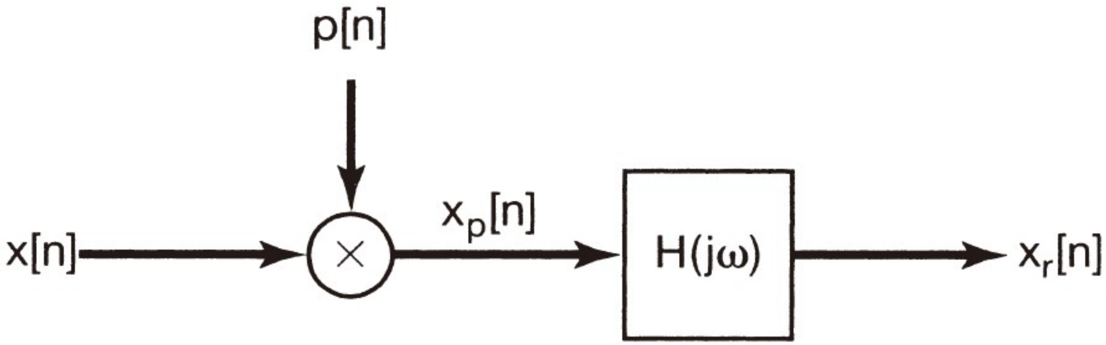
7.5.1 Impulse-Train Sampling

If the overall system of Figure 7.33(a) is applied to a sequence for which $\omega_s < 2\omega_M$, to that aliasing results, $x_r[n]$ will no longer be equal to *x[n]*. However, as with continuous-time sampling, the two sequences will be equal at multiples of the sampling period; that is, corresponding to eq. (7.13), we have

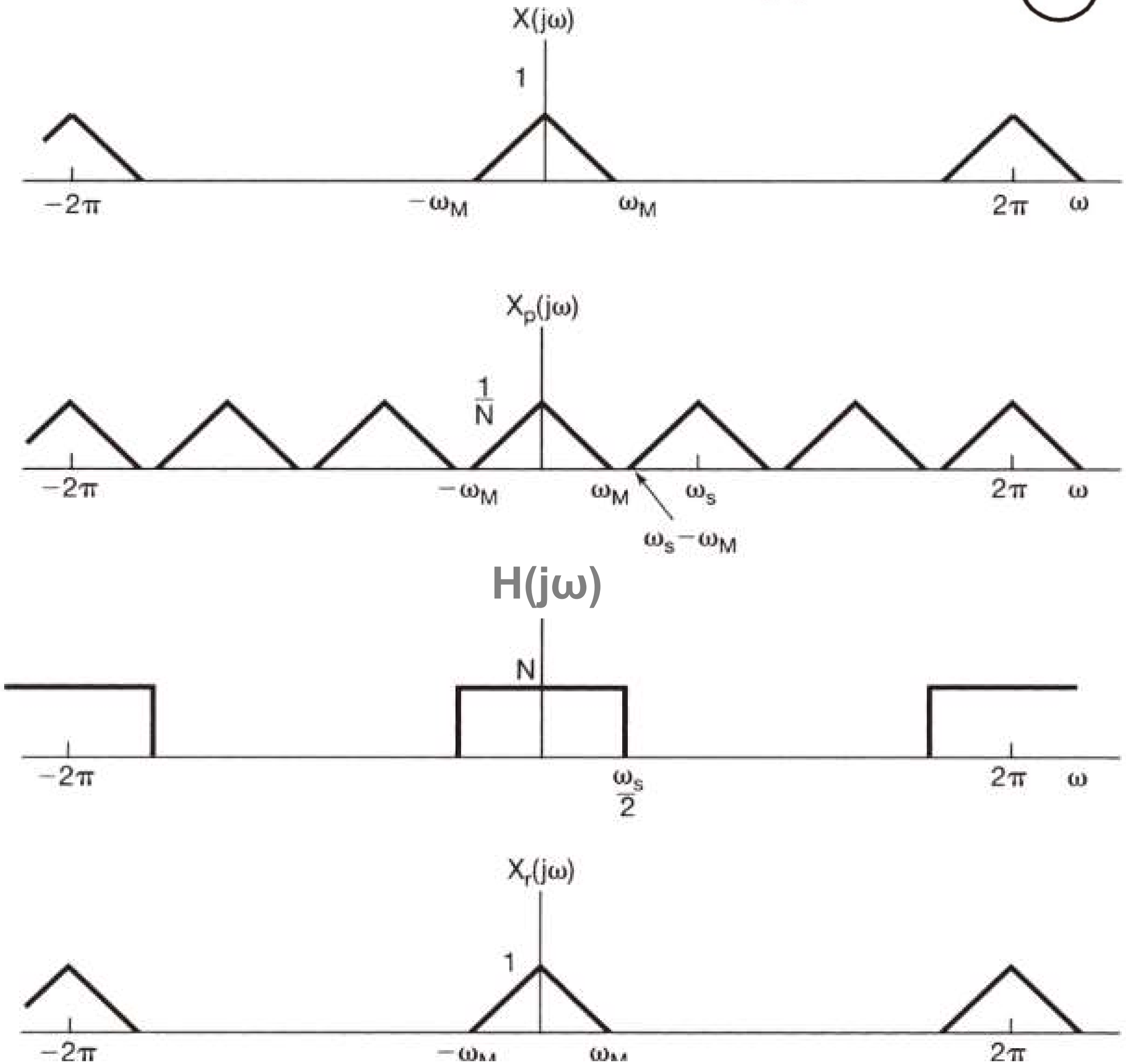
$$x_r[kN] = x[kN], \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{7.43}$$

若 $\omega_s < 2\omega_M$ ，則 $x_r[n]$ 無法還原成*x[n]*，但在取樣點處是相等的。

7.5.1 Recover



p[n]
x[n]
$x_p[n]$
H(jω)
$x_r[n]$



X(jω)
1
−2π
$-\omega_M$
ω_M
2π
ω
$X_p(j\omega)$
$\frac{1}{N}$
−2π
$-\omega_M$
ω_M
ω_s
2π
ω
$\omega_s-\omega_M$
H(jω)
N
−2π
$\frac{\omega_s}{2}$
2π
ω
$X_r(j\omega)$
1
−2π
2π

Example 7.4

Consider a sequence *x[n]* whose Fourier transform $X(e^{j\omega})$ has the property that

$$X(e^{j\omega}) = 0 \qquad for \qquad 2\pi/9 \le |\omega| \le \pi.$$

To determine the lowest rate at which *x[n]* may be sampled without the possibility of aliasing, we must find the largest *N* such that

$$\omega_s = \frac{2\pi}{N} \ge 2\left(\frac{2\pi}{9}\right) \Rightarrow N \le 9/2.$$

We conclude that $N_{\max} = 4$, and the corresponding sampling frequency is $2\pi/4 = \pi/2$.

$$\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wn}{\pi}\right)$$

$$0 < W < \pi$$

$$X(\omega) = \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}$$

$X(\omega)$ periodic with period 2π

With *h[n]* denoting the impulse response of the lowpass filter, we have

$$h[n] = \frac{N\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n}. \qquad (7.44)$$

The reconstructed sequence is then

$$x_r[n] = x_p[n] * h[n], \qquad (7.45)$$

or equivalently,

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]\frac{N\omega_c}{\pi}\frac{\sin \omega_c(n-kN)}{\omega_c(n-kN)}. \qquad (7.46)$$

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]h_r[n-kN], \qquad (7.47)$$

7.5.2 Discrete-Time **Decimation** and Interpolation

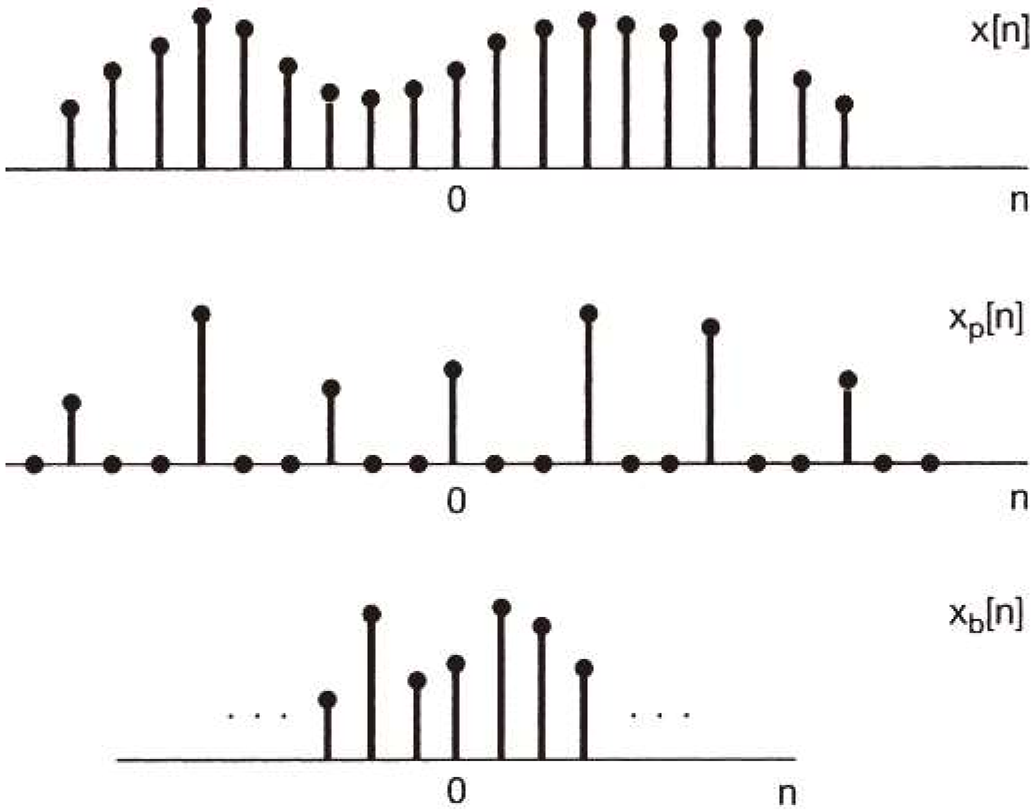
The sampled sequence is typically replaced by a new sequence $x_b[n]$, which is simply every Nth value of $x_p[n]$;that is,

let $x_b[n] = x_p[nN]$.

$$x_b[n] = x_p[nN] \quad (7.48)$$

Also, equivalently,

$$x_b[n] = x[nN], \quad (7.49)$$



7.5.2 Discrete-Time Decimation and Interpolation

To determine the effect in the frequency domain of decimation, we wish to determine the relationship between $X_b(e^{j\omega})$ —the Fourier transform of $x_b[n]$ — and $X(e^{j\omega})$. To this end, we note that

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b[k] e^{-j\omega k}, \tag{7.50}$$

or, using eq. (7.48),

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_p[kN] e^{-j\omega k}. \tag{7.51}$$

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_p[kN]e^{-j\omega k}.$$

If we let *n* = *kN*, or equivalently *k* = *n/N*, we can write

$$X_b(e^{j\omega}) = \sum_{\substack{n = \text{integer} \\ \text{Multiple of N}}} x_p[n]e^{-j\omega n/N}$$

and since $x_p[n] = 0$ when *n* is not an integer multiple of *N*, we can also write

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_p[n]e^{-j\omega n/N}. \qquad (7.52)$$

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_p[n] e^{-j\omega n/N}.$$

Furthermore, we recognize the right-hand side of eq.(7.52) as the Fourier transform of $x_p[n]$; that is,

$$\sum_{n=-\infty}^{+\infty} x_p[n] e^{-j\omega n/N} = X_p(e^{j\omega/N}). \qquad (7.53)$$

Thus, from eqs. (7.52) and (7.53), we conclude that

$$X_b(e^{j\omega}) = X_p(e^{j\omega/N}). \qquad (7.54)$$

抽離訊號與離散時間取樣訊號的傅立葉轉換關係式

7.5.2 Discrete-Time Decimation and Interpolation

$$X_b(e^{j\omega}) = X_p(e^{j\omega/N}).$$

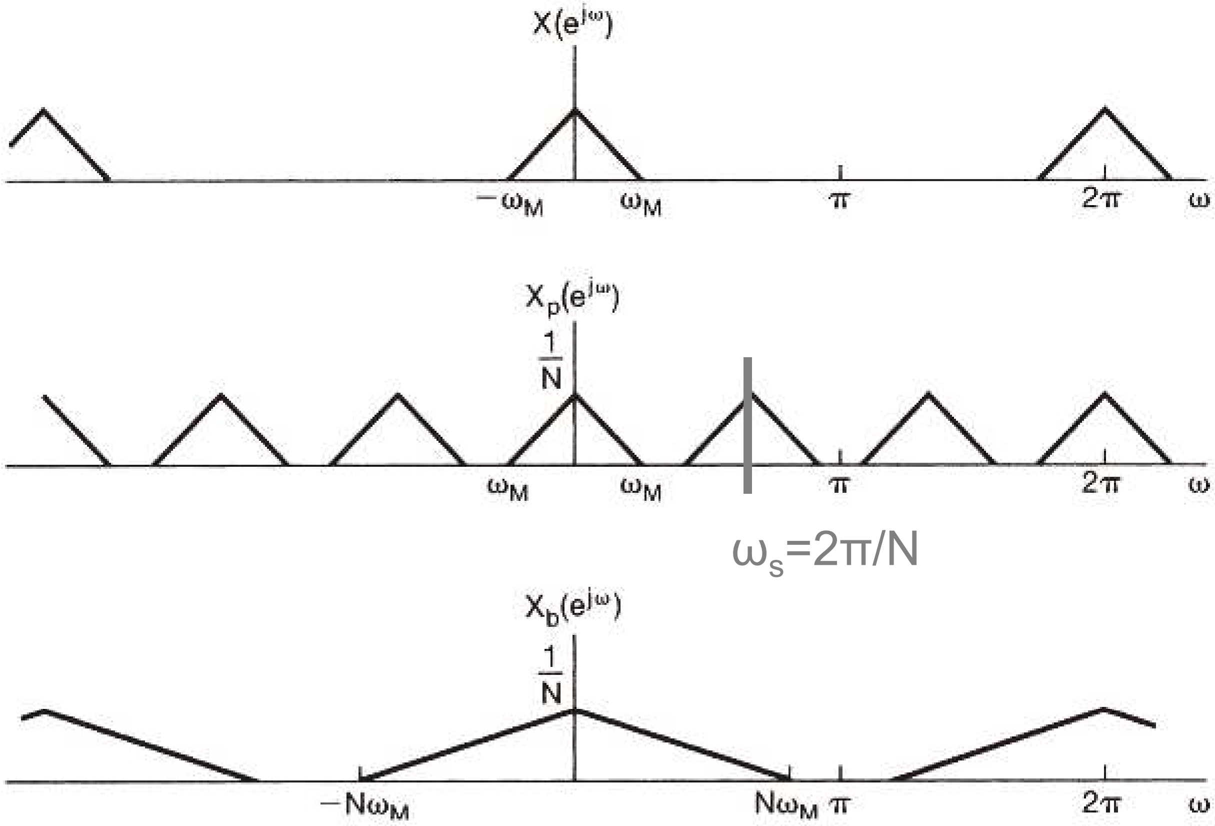


Figure 7.35 Frequency-domain illustration of the relationship between sampling and decimation.

7.5.2 Decimation (downsampling)

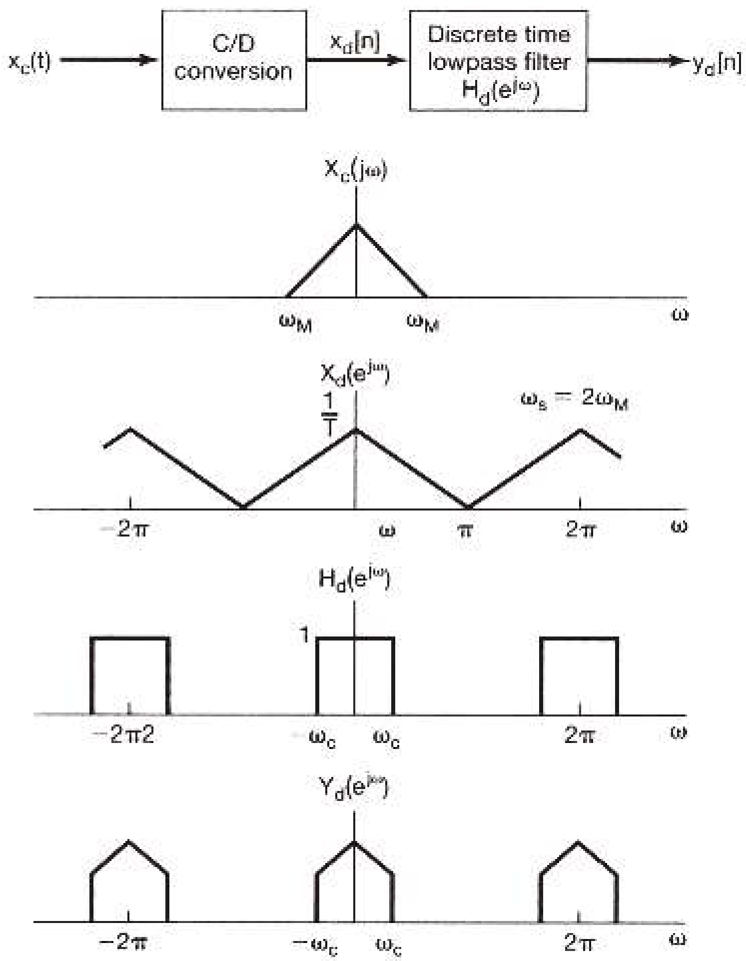


Figure 7.36 Continuous-time signal that was originally sampled at the Nyquist rate. After discrete-time filtering, the resulting sequence can be further downsampled. Here $X_c(j\omega)$ is the continuous-time Fourier transform of $x_c(t)$, $X_d(e^{j\omega})$ and $Y_d(e^{j\omega})$ are the discrete-time Fourier transforms of $x_d[n]$ and $y_d[n]$ respectively, and $H_d(e^{j\omega})$ is the frequency response of the discrete-time lowpass filter depicted in the block diagram.

7.5.2 Interpolation (upsampling)

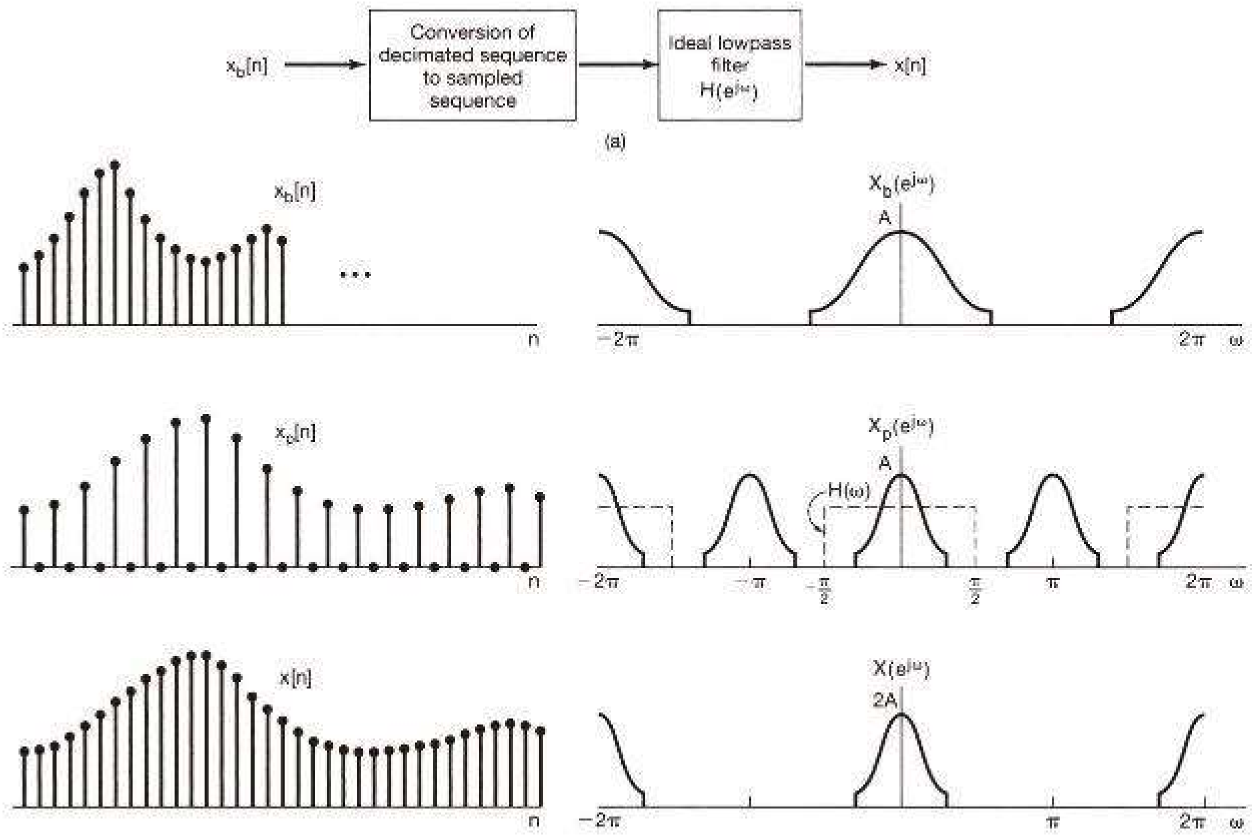


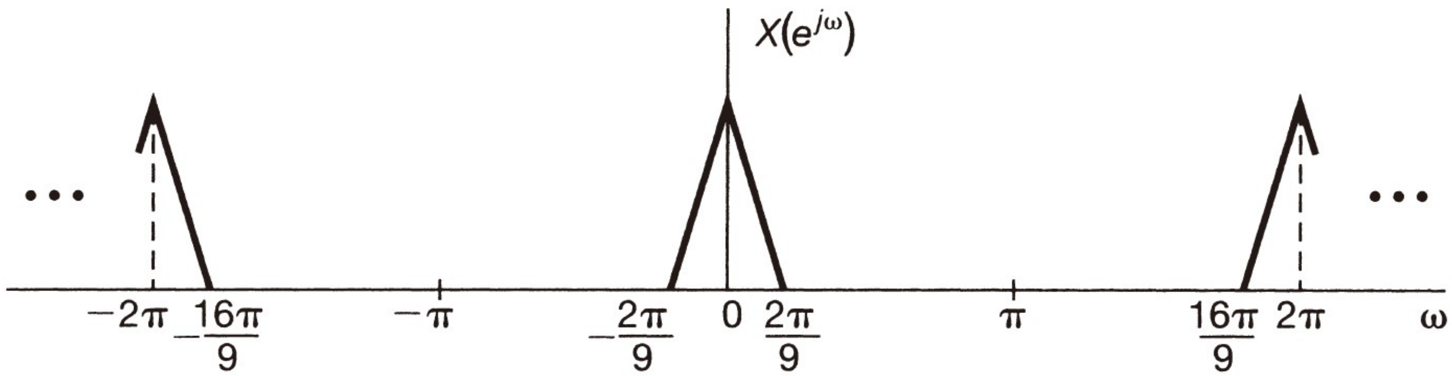
Figure 7.37 Upsampling: (a) overall system; (b) associated sequences and spectra for upsampling by a factor of 2.

Example 7.5

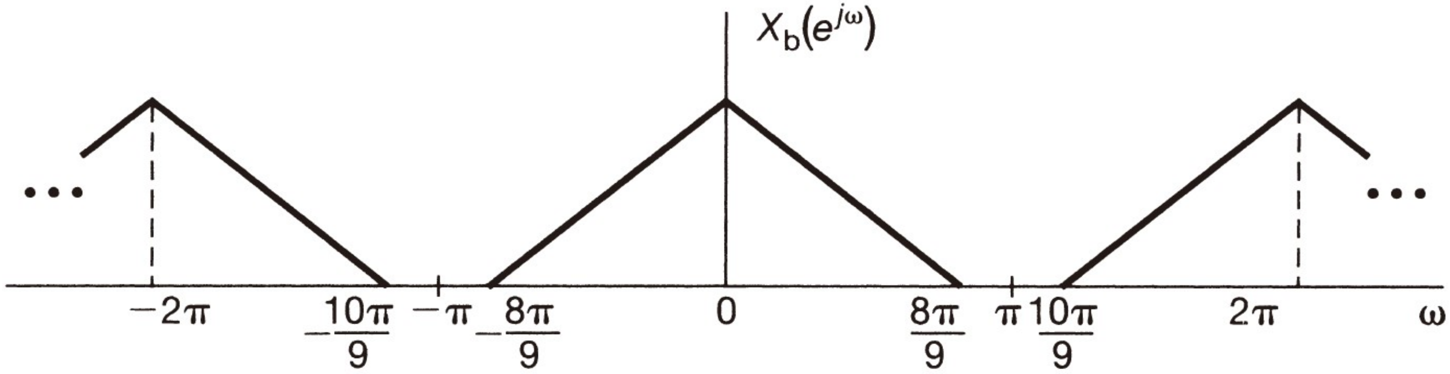
In this example, we illustrate how a combination of interpolation and decimation may be used to further **downsample a sequence without incurring aliasing**. It should be noted that maximum possible downsampling is achieved once the non-zero portion of one period of the discrete-time spectrum has expanded to fill the entire band from $-\pi$ to π.

Example 7.5 downsample

$$X_b(e^{j\omega}) = X_p(e^{j\omega/N}).$$



(a)



(b)

Example 7.5

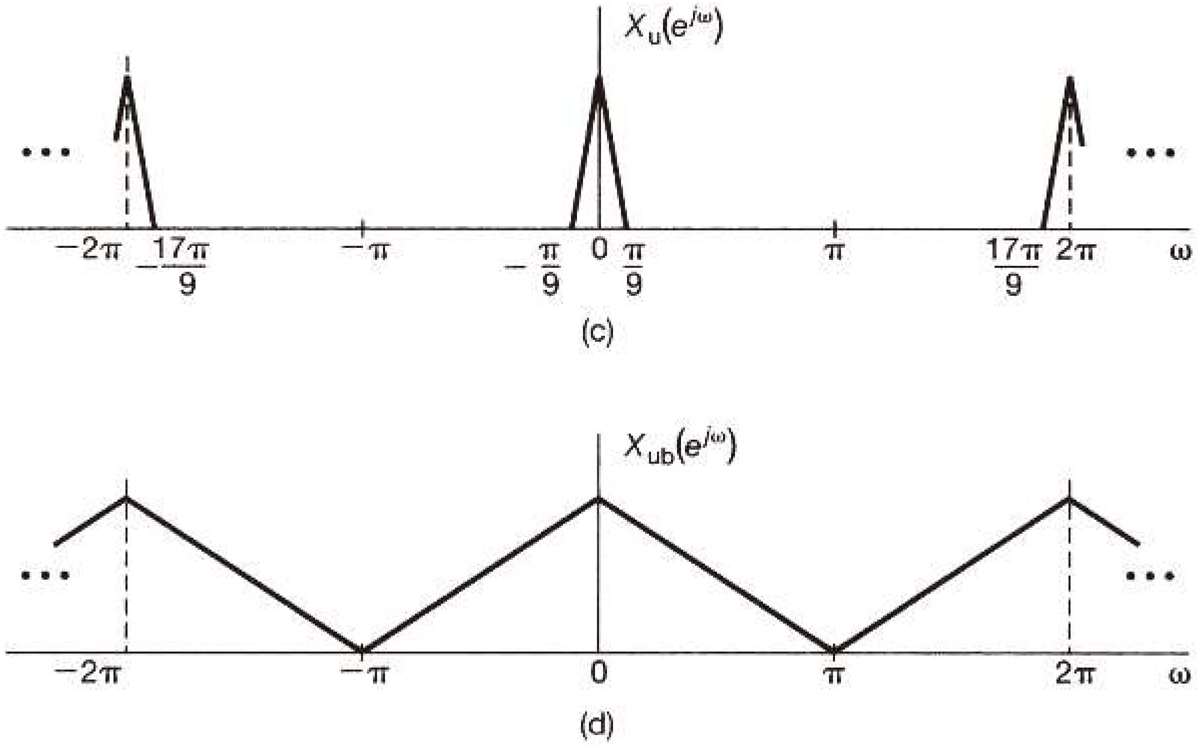
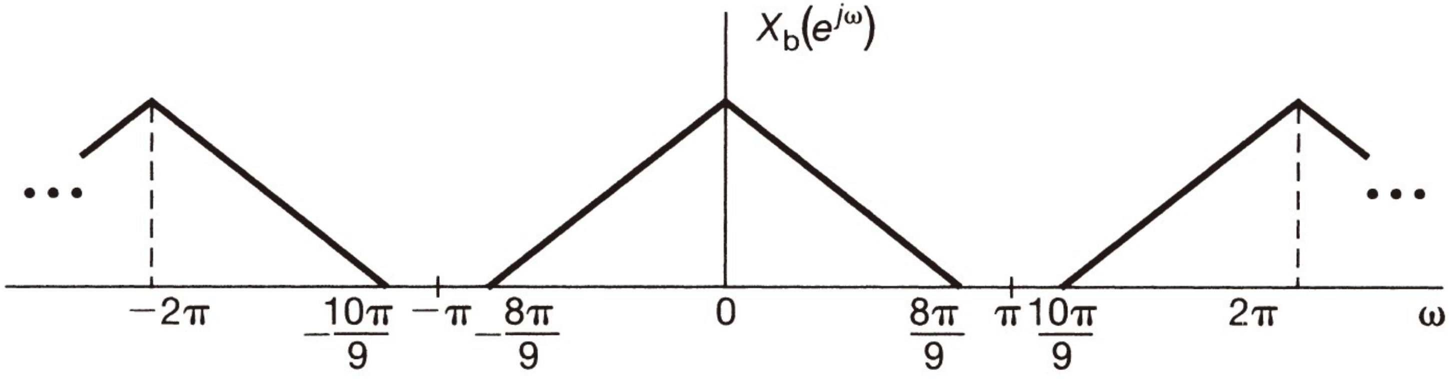
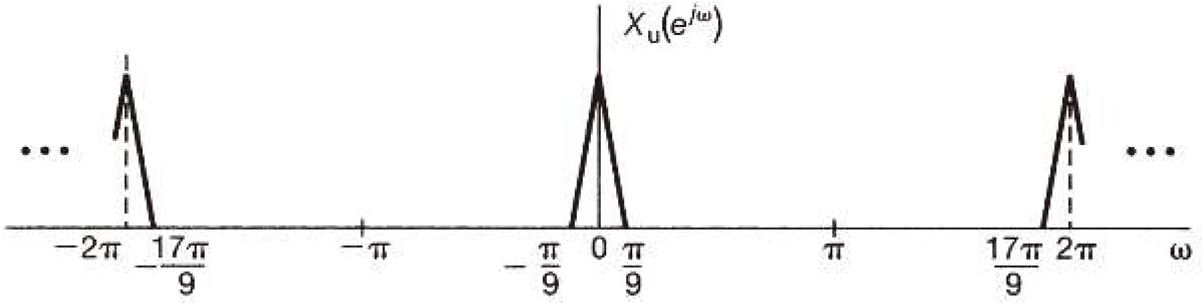


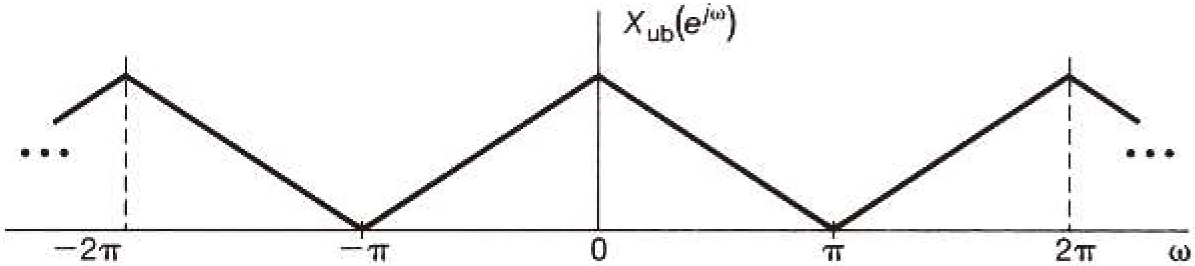
Figure 7.38 Spectra associated with Example 7.5. (a) Spectrum of $x[n]$; (b) spectrum after downsampling by 4; (c) spectrum after upsampling $x[n]$ by a factor of 2; (d) spectrum after upsampling $x[n]$ by 2 and then downsampling by 9.



Consider the sequence *x[n]* whose Fourier transform $X(e^{j\omega})$ is illustrated in Figure 7.38(a). As discussed in Example 7.4, the lowest rate at which impulse-train sampling may be used on this sequence without incurring aliasing is $2\pi/4$. This corresponds to sampling every 4th value of *x[n]*. If the result of such sampling is decimated by a factor of 4, we obtain a sequence $x_b[n]$ whose spectrum is shown in Figure 7.38(b). Clearly, there is still no aliasing of the original spectrum. However, this spectrum is zero for $8\pi/9 \le |\omega| \le \pi$, which suggests there is room for further downsampling.



Specifically, examining Figure 7.38(a) we see that if we could scale frequency by a factor of 9/2, the resulting spectrum would have nonzero values over the entire frequency interval from $-\pi$ to π. However, since 9/2 is not an integer, we can't achieve this purely by **downsampling**. Rather we must first **upsample** *x[n]* by factor of 2 and then **downsampled** by a factor of 9, the spectrum of the signal $x_u[n]$ obtained when *x[n]* is upsampled by a factor of 2, is displayed in Figure 7.38(c).



when $x_u[n]$ is then downsampled by a factor of 9, the spectrum of the resulting sequence $x_{ub}[n]$ is as shown in Figure 7.38(d). This combined result effectively corresponds to downsampling *x[n]* by a noninteger amount, 9/2. Assuming that *x[n]* represents unaliased samples of a continuous-time signal $x_c(t)$, our interpolated and decimated sequence represents the maximum possible (aliasing-free) downsampling of $x_c(t)$.

7.6 Summary

- Sampling using Impulse-Train
- Sampling Theorem
- Reconstruction as Interpolation (zero-order, 1^{st}-order, etc.)
- Aliasing
- Discrete-time system approximate Continuous-time system
- Discrete-time Sampling
- Decimation (Downsampling) and Interpolation (Upsampling)